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Original Article

A novel analytical approach for advection diffusion equation for radionuclide release from an area source

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ABSTRACT

The method of the Laplace transform has been used to obtain an analytical solution of the three-dimensional steady state advection diffusion equation for the airborne radionuclide release from any nuclear installation such as the power reactor in an area source. The present treatment takes into account the removal of the pollutants through the nuclear reaction. We assume that the pollutants are emitted as a constant rate from the area source. This physical consideration is achieved by assuming that the vertical eddy diffusivity coefficient should be a constant. The prevailing wind speed is a constant in $x-$ direction and a linear function of the vertical height z. The present model calculations are compared with the other models and the available data of the atmospheric dispersion experiments that were carried out in the nuclear power plant of Angra dos Reis (Brazil). The results show that the present treatment performs well as the analytical dispersion model and there is a good agreement between the values computed by our model and the observed data.

1. Introduction

The atmospheric dispersion has been performed for a hypothetical accidental airborne radionuclide release from any nuclear installation such as the power reactor, the research reactor or any facility which uses the radioisotopes as its tools. The radioactive materials which released from the nuclear power plant through an experiment or eventually in an accidental events could be dispersed in the atmosphere and result in the radiation exposure of the human direct or by the plants or the animals. Thus, the evaluation of the airborne radioactive materials transport in the atmosphere is one of the requirements for safety measuring around the nuclear power plant in the environment.

In order to analyze the possible consequences of the radioactive release, using the atmospheric dispersion model, we need to use the specific meteorological parameters and the conditions in the region. The mathematical models are the basis for prediction of the mean concentrations of the contaminants for a given emission source distribution. The analytical solutions of the equations are of the fundamental importance in understanding and describing a physical phenomena, since they allow us to take into account all the parameters of the problem and investigate their influence such as [19–21]. Unfortunately, no general solution is known for the equations describing the air pollution transport and the dispersion advection diffusion equation (*ADE*), but with some assumptions for the pollutant dispersion mechanism, there are many solutions for (*ADE*) for the radioactive pollutants such as Morier et al. [10,12] who found a semi-analytical solution using multilayered method for the time dependent *ADE* with the eddy diffusivity profiles and the wind speed as function of the height z and solved *ADE* by the general integral laplace transform approach. Busk et al. [6] used the integral transform method for solving the time dependent general *ADE*. Weymar et al. [17] found the solution for the time-dependent three-dimensional advection-diffusion equation.

In the present study, we consider the *ADE* model which describes the ambient air concentration of the pollutants emitted from an area source. The removal of The pollutants is taken into account through the nuclear reaction from both a radioactive decay

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and a dry deposition. We assume that the pollutants are emitted in a constant rate from the area source. We solved the three-dimensional steady state *ADE* for the radioactive pollutants analytically using the Laplace transform method.

The paper is organized in 5 sections. Section 1 is an introduction, section 2 presents the mathematical formulation of the solution of the advection-diffusion equation. Section 3 presents the application of this solution through the experimental data of the Angra campaign using the associated meteorological conditions [3], section 4 presents the statistical analysis of the predicted model. Finally, in section 5 we present the conclusion extracted from the present study.

2. Mathematical treatment

The concentration turbulent fluxes are often assumed to be proportional to the mean concentration gradient. This assumption, along with the equation of continuity, leads to the *ADE*. The *ADE* can be written as [2]:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\frac{\partial}{\partial x}\left(k_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial C}{\partial z}\right)-\lambda C \quad (1)$$

where $C(x,y,z)$ denotes the concentration, k_x, k_y, k_z are the cartesian components of the eddy diffusivity, u, v, w are the cartesian components of a wind speed, x, y are the cartesian coordinates and z is the height above the ground surface and λ is the decay constant.

In order to solve equation (1) we included the following assumptions: u is constant, the lateral v components of the mean flow are assumed to be zero, $w=\beta z$, assume that eddy diffusivity profile along x-direction is negligible compared to the wind velocity, k_y and k_z are constants. The mean horizontal flow is horizontally homogeneous and steady state, then, equation (1) is simplified in the form:

$$u\frac{\partial C}{\partial x}+\beta z\frac{\partial C}{\partial z}=k_y\frac{\partial^2 C}{\partial y^2}+k_z\frac{\partial^2 C}{\partial z^2}-\lambda C \quad (2)$$

The physical description of this model shown in Fig. 1 and the

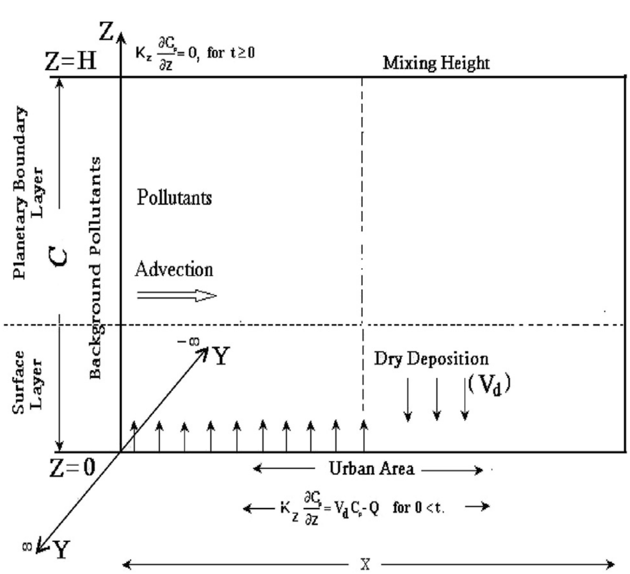

Fig. 1. Diagram showing the physical description of the boundary conditions.

mathematical boundary conditions of the dispersion problem (2) are given as

$$C(x,y,z)=0, \quad at \quad x=0 \quad (3)$$

$$C(x,y,z)=0, \quad at \quad x,y,z\to\pm\infty \quad (4)$$

$$k_z\frac{\partial C}{\partial z}=0, \quad at \quad z=h \quad (5)$$

We assume that the chemically reactive air pollutants are being emitted at a steady rate from the ground level and they are removed from the atmosphere by ground adsorption. Hence the corresponding boundary condition takes the form:

$$k_z\frac{\partial C}{\partial z}=v_d C-Q, \quad at \quad z=0 \quad (6)$$

where v_d is the dry deposition velocity, $\delta()$ is Dirac delta function, h is the inversion height of the planetary boundary layer and Q is the emission rate.

In the first, we obtain the cross-wind concentration by taking integration for equation (2) with respect to y as follows [10]:

$$u\int_{-\infty}^{\infty}\frac{\partial C}{\partial x}dy+\beta z\int_{-\infty}^{\infty}\frac{\partial C}{\partial z}dy=k_y\int_{-\infty}^{\infty}\frac{\partial^2 C}{\partial y^2}dy+k_z\int_{-\infty}^{\infty}\frac{\partial^2 C}{\partial z^2}dy-\lambda\int_{-\infty}^{\infty}C dy \quad (7)$$

From boundary condition (4) the term $\int_{-\infty}^{\infty}\frac{\partial^2 C}{\partial y^2}dy=\frac{\partial C}{\partial y}\Big|_{-\infty}^{\infty}=0$, equation (7) has the form:

$$u\frac{\partial C_y}{\partial x}+\beta z\frac{\partial C_y}{\partial z}=k_z\frac{\partial^2 C_y}{\partial z^2}-\lambda C_y \quad (8)$$

where $C_y=\int_{-\infty}^{\infty}C(x,y,z)dy$ is the cross-wind concentration. Applying the Laplace transform (*LT*) for equation (8) with respect to x and use the boundary condition (3), we obtain:

$$k_z\frac{d^2\tilde{C}_y(s,z)}{dz^2}-\beta z\frac{d\tilde{C}_y(s,z)}{dz}-(\lambda+us)\tilde{C}_y=0 \quad (9)$$

$\tilde{C}_y=0(s,z)$ is the Laplace transform of the cross-wind concentration w.r.to x.

equation (9) is a hypergeometric differential equation which has the solution in the form:

$$\tilde{C}_y(s,z)=c_1\,{}_1F_1\left(\frac{\nu}{2},\frac{1}{2},\frac{\beta z^2}{2k_z}\right)+c_2 H_{-\nu}\left(\sqrt{\frac{\beta}{2k_z}}z\right) \quad (10)$$

where: ${}_1F_1$ is the first kind hypergeometric function, $H_{-\nu}$ is the Hermite function which is defined in Refs. [14,15], and

$$\nu=\frac{\lambda+us}{\beta} \quad (10a)$$

From Ref. [4], the hermite polynomial function H_n has the properties:

$$e^{xt-t^2}=\sum_{0}^{\infty}\frac{t^n}{n!}H_n(x)$$

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \left(\frac{d}{dx}\right)^n \left(e^{\frac{-x^2}{2}}\right)$$

From Ref. [1], the Hermite-Kampé de Fériet polynomials is defined as:

$$H_n(x,y) = n!\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^r x^{n-2r}}{(n-2r)!r!}$$

$$\frac{\partial}{\partial x}H_n(x,y) = nH_{n-1}(x,y)$$

$$\frac{\partial}{\partial y}H_n(x,y) = n(n-1)H_{n-2}(x,y)$$

$$H_n(x,0) = nx^n$$

and the higher order Kampé de Fériet polynomials is:

$$H_n^{(m)}(x,y) = n!\sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{y^r x^{n-mr}}{(n-mr)!r!}$$

$$\frac{\partial}{\partial x}H_n^{(m)}(x,y) = nH_n^{(m)}(x,y)$$

$$\frac{\partial}{\partial y}H_n^{(m)}(x,y) = \frac{n!}{(n-m)!}H_{n-m}^{(m)}(x,y)$$

$$H_n^{(m)}(x,0) = x^n$$

From Ref. [4], the hermite function h_n

$$h_n(x) = \left(\frac{1}{\sqrt{2\pi}n!}\right)^{\frac{1}{2}} H_n(x) e^{\frac{-x^2}{2}}$$

$$2\frac{d}{dx}h_n(x) = \sqrt{n}h_{n-1}(x) - \sqrt{n+1}h_{n+1}(x)$$

$$xh_n(x) = \sqrt{n}h_{n-1}(x) - \sqrt{n+1}h_{n+1}(x)$$

From the boundary condition (4), we find that:

$$c_1 \lim_{z\to\infty} {}_1F_1\left(\frac{\nu}{2},\frac{1}{2},\frac{\beta z^2}{2k_z}\right) + c_2 \lim_{z\to\infty} H_{-\nu}\left(\sqrt{\frac{\beta}{2k_z}}z\right) = 0$$

from the properties of hypergeometric function of the first kind that: $\lim_{z\to\infty}F_1\left(\frac{\nu}{2},\frac{1}{2},\frac{\beta z^2}{2k_z}\right) = 0$, so to get the definite solution taking $c_2 = 0$ and hence the solution becomes:

$$\tilde{C}_y(s,z) = c_1\ {}_1F_1\left(\frac{\nu}{2},\frac{1}{2},\frac{\beta z^2}{2k_z}\right) \tag{11}$$

Taking Laplace transform for the boundary condition (5) after integration w.r.to.y, we get:

$$k_z\frac{d\tilde{C}_y}{dz} = \nu_d\tilde{C}_y - \frac{Qy}{s}, \qquad \text{at} \qquad z = 0 \tag{12}$$

Substituting equation (11) into equation (12), we obtain:

$$\tilde{C}_y(s,z) = \frac{Qy}{s\nu_d}\ {}_1F_1\left(\frac{\nu}{2},\frac{1}{2},\frac{\beta z^2}{2k_z}\right) \tag{13}$$

To get the closed form of the solution of our problem, we should find the inverse of LT using the definition of the first hypergeometric function [5] in the form:

$$ {}_1F_1(a,b,x) = \sum_{i=0}^{\infty}\frac{(a)_i}{i!(b)_i}x^i \tag{13a}$$

$(a)_i$ is Pochhammer's symbol which is defined as:

$$(a)_i = \frac{\Gamma(a+i)}{\Gamma(a)} \tag{14}$$

From equation (13a) and equation (14), equation (13) becomes in the form:

$$\tilde{C}_y(s,z) = \frac{Qy}{s\nu_d}\sum_{i=0}^{\infty}\frac{\left(\frac{\nu}{2}\right)_i}{i!\left(\frac{1}{2}\right)_i}\left(\frac{\beta z^2}{2k_z}\right)^i \tag{15}$$

$$\tilde{C}_y(s,z) = \frac{Qy}{s\nu_d}\sum_{i=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)}{i!\Gamma\left(\frac{1}{2}+i\right)}\left(\frac{\beta z^2}{2k_z}\right)^i\frac{\Gamma\left(\frac{\nu}{2}+i\right)}{\Gamma\left(\frac{\nu}{2}\right)} \tag{16}$$

Substituting from equation (10a) into equation (16), we obtain:

$$\tilde{C}_y(s,z) = \frac{Qy\sqrt{\pi}}{s\nu_d}\sum_{i=0}^{\infty}\frac{1}{i!\Gamma\left(\frac{1}{2}+i\right)}\left(\frac{\beta z^2}{2k_z}\right)^i\frac{\Gamma\left(\frac{\lambda+us}{2\beta}+i\right)}{\Gamma\left(\frac{\lambda+us}{2\beta}\right)} \tag{17}$$

Taking inverse LT for equation (17), we get:

$$C_y(x,z) = \frac{Qy\sqrt{\pi}}{\nu_d}\sum_{i=0}^{\infty}\frac{1}{i!\Gamma\left(\frac{1}{2}+i\right)}\left(\frac{\beta z^2}{2k_z}\right)^i \times L^{-1}\left[\frac{\Gamma\left(\frac{\lambda+us}{2\beta}+i\right)}{s\Gamma\left(\frac{\lambda+us}{2\beta}\right)}\right] \tag{18}$$

Since $\frac{1}{s} = (s+1)_{-1}$, see Ref. [5], we obtain:

$$L^{-1}\left[\frac{\Gamma\left(\frac{\lambda+us}{2\beta}+i\right)}{s\Gamma\left(\frac{\lambda+us}{2\beta}\right)}\right] = L^{-1}\left[\frac{\Gamma\left(\frac{\lambda+us}{2\beta}+i\right)}{\Gamma\left(\frac{\lambda+us}{2\beta}\right)}(s+1)_{-1}\right] \tag{19}$$

From [7]:

$$L^{-1}\left[\frac{\Gamma(p+\alpha)}{\Gamma(p+\gamma)}(p+\zeta)_l\right] = \frac{e^{-\alpha t}\left(1-e^{-t}\right)^{\gamma-\alpha-l-1}}{\Gamma(\gamma-\alpha-l)} \times\ {}_2F_1\left(-l,\gamma-\zeta-l,\gamma-\alpha-l,1-e^{-t}\right) \tag{20}$$

Substituting equation (19) into equation (20), we obtain:

$$L^{-1}\left[\frac{\Gamma\left(\frac{\lambda+us}{2\beta}+i\right)}{\Gamma\left(\frac{\lambda+us}{2\beta}\right)}(s+1)_{-1}\right]=\frac{e^{-\frac{2\beta}{u}\left(\frac{\lambda}{2\beta}+i\right)x}\left(1-e^{-\frac{2\beta}{u}x}\right)^{-i}}{\Gamma(1-i)}$$

$$\times{}_2F_1\left(1,\frac{\lambda}{2\beta},1-i,1-e^{-\frac{2\beta}{u}x}\right) \tag{21}$$

From equation (21) into equation (18), the cross-wind concentration is:

$$C_y(x,z)=\frac{\sqrt{\pi}Qy}{v_d}\sum_{j=1}^{\infty}\frac{e^{-\frac{2\beta}{u}\left(\frac{\lambda}{2\beta}+1-j\right)x}\left(1-e^{-\frac{2\beta}{u}x}\right)^{j-1}}{(j-1)\,!\Gamma\left(\frac{3}{2}-j\right)\Gamma(j)}\times\left(\frac{\beta z^2}{2k_z}\right)^{j-1}$$

$$\times\ {}_2F_1\left(1,\frac{\lambda}{2\beta},j,1-e^{-\frac{2\beta}{u}x}\right) \tag{22}$$

Since

$$\Gamma\left(\frac{3}{2}-j\right)=(-1)^j\frac{\Gamma\left(\frac{3}{2}\right)}{\left(1-\frac{3}{2}\right)_j}=(-1)^j\frac{\pi}{2\Gamma\left(j+\frac{1}{2}\right)}$$

The cross-wind concentration becomes in the form:

$$C_y(x,z)=\frac{2\sqrt{\pi}Qy}{\pi v_d}\sum_{j=1}^{\infty}(-1)^j\frac{\Gamma\left(\frac{1}{2}+j\right)\left(\frac{\beta z^2}{2k_z}\right)^{j-1}}{(j-1)!\Gamma(j)}$$

$$\times e^{-\frac{2\beta}{u}\left(\frac{\lambda}{2\beta}+1-j\right)x}\left(1-e^{-\frac{2\beta}{u}x}\right)^{j-1}\times\ {}_2F_1\left(1,\frac{\lambda}{2\beta},j,1-e^{-\frac{2\beta}{u}x}\right) \tag{23}$$

The concentration of the pollutants is defined as:

$$C(x,y,z)=C_y(x,z)e^{-\frac{y^2}{2\sigma_y^2}} \tag{24}$$

The general solution of the predicted model is:

$$C(x,y,z)=\frac{2\sqrt{\pi}Qye^{-\frac{y^2}{2\sigma_y^2}}}{\pi v_d}\sum_{j=1}^{\infty}(-1)^j\frac{\Gamma\left(\frac{1}{2}+j\right)\left(\frac{\beta z^2}{2k_z}\right)^{j-1}}{(j-1)!\Gamma(j)}$$

$$\times e^{-\frac{2\beta}{u}\left(\frac{\lambda}{2\beta}+1-j\right)x}\left(1-e^{-\frac{2\beta}{u}x}\right)^{j-1}\times\ {}_1F_1\left(1,\frac{\lambda}{2\beta},j,1-e^{-\frac{2\beta}{u}x}\right) \tag{25}$$

where σ_y is the lateral dispersion parameter.

2.1. The point source concentration

To perform our predicted model with the experiment model for a point source, the predicted concentration should be obtained for a point source. The concentration profile at (x,y) due to a point source of unit source strength may be obtained from the profile for an area source of unit strength from Ref. [18] as follows:

$$C_p(x,y,z)=\frac{\partial^2 C_A(x,y,z)}{\partial y\partial x} \tag{26}$$

where C_p is the point source concentration and C_A is the area source concentration which is obtained in equation (25). By substituting from equation (25) into equation (26), we get:

$$C_p(x,y,z)=\frac{2\sqrt{\pi}Q}{\pi v_d}e^{-\frac{y^2}{2\sigma_y^2}}\left(\frac{\partial\chi(x,z)}{\partial x}-\chi\frac{\partial}{\partial x}\left(\frac{y^2}{\sigma_y^2}\right)\right)$$

$$-\frac{2\sqrt{\pi}Qy}{\pi v_d}\left(\frac{ye^{-\frac{y^2}{2\sigma_y^2}}}{\sigma_y^2}\frac{\partial\chi(x,z)}{\partial x}+\chi\left(\frac{\partial}{\partial y}\left(e^{-\frac{y^2}{2\sigma_y^2}}\frac{\partial}{\partial x}\left(\frac{y^2}{\sigma_y^2}\right)\right)\right)\right) \tag{27}$$

where

$$\chi(x,z)=\sum_{j=1}^{\infty}(-1)^j\frac{\Gamma\left(\frac{1}{2}+j\right)\left(\frac{\beta z^2}{2k_z}\right)^{j-1}}{(j-1)!\Gamma(j)}e^{-\frac{2\beta}{u}\left(\frac{\lambda}{2\beta}+1-j\right)x}\left(1-e^{-\frac{2\beta}{u}x}\right)^{j-1}$$

$${}_2F_1\left(1,\frac{\lambda}{2\beta},j,1-e^{-\frac{2\beta}{u}x}\right) \tag{28}$$

The centerline ground-level concentration is defined as [10]:

$$C(x,0,0)=\frac{C(x,0)}{\sqrt{2\pi}\sigma_y} \tag{29}$$

The predicted centerline ground-level concentration becomes in the form:

$$C_p(x,z)=\frac{2\sqrt{2}Q\beta}{\pi u v_d\sigma_y}\sum_{j=1}^{\infty}\frac{\Gamma\left(\frac{1}{2}+j\right)\left(\frac{\beta z^2}{2k_z}\right)^{j-1}}{(j-1)!\Gamma(j)}$$

$$e^{-\frac{2\beta}{u}\left(\frac{\lambda}{2\beta}+1-j\right)x}\left(1-e^{-\frac{2\beta}{u}x}\right)^{j-1}$$

$$\times\left\{\left(\frac{\lambda}{2\beta}+\frac{(j-1)e^{-\frac{2\beta}{u}x}}{1-e^{-\frac{2\beta}{u}x}}\right){}_2F_1\left(1,\frac{\lambda}{2\beta},j,1-e^{-\frac{2\beta}{u}x}\right)\right.$$

$$\left.+\frac{\lambda}{2\beta j}e^{-\frac{2\beta}{u}x}\,{}_2F_1\left(2,\frac{\lambda}{2\beta}+1,j+1,1-e^{-\frac{2\beta}{u}x}\right)\right\} \tag{30}$$

2.2. Turbulent parameterizations

The turbulent parameterizations plays an important role for contaminant dispersion modeling in the atmospheric diffusion problems. It is an approximation for the natural phenomenon in a view point of Physics, where details are hidden in the parameters used, that have to be adjusted in order to reproduce experimental findings. For the performance of predicted model (25), the turbulent parameters can be formulated as follows:

$$k_z = ku_* \tag{31}$$

where u_* is the fraction velocity, $k = 0.4$, for the lateral dispersion parameter σ_y, from Ref. [13], it has the form:

$$\frac{\sigma_y}{h} = \left[\frac{0.26X}{1+0.91X}\right]^{\frac{1}{2}} \tag{32}$$

where X is a non-dimensional distance $(X = xw_*/uh)$,w_* is the convective velocity scale with wind speed u. For the small variations of the wind direction in the relation to the position of the receptors the expression (28) was utilized as the lateral dispersion, along with the following expression for the large variations [13]:

$$\frac{\sigma_y}{x} = \frac{\sigma_\theta}{(1+0.031x^{0.46})} \tag{33}$$

where $\sigma_\theta = 15$ for the unstable/neutral conditions [2].

3. Experimental data and model evaluation

Our present model is evaluated against the experimental ground level concentration data from the dispersion experiment carried out at the nuclear reactor site Angra dos Reis in the Rio de Janeiro state, Brazil. The details of the dispersion experiment is described elsewhere [3]. The experiment consisted in the controlled releases of the radioactive tritiated water vapour from the meteorological tower at 100 m height during five days (28 November to 4 December 1984). During the whole experiment, four meteorological towers collected the relevant meteorological data. The wind speed and the direction were measured at three levels (10 m, 60 m and 100 m) together with the temperature gradients between 10 m and 100 m. Some additional data of a relative humidity were available in some of the sampling sites, and were used to calculate the concentration of the radioactive tritiated water in the air (after measuring the radioactivity of the collected samples). All the relevant details, as well as the synoptic meteorological conditions during the dispersion campaign are described in Ref. [3]. The data collected from the 5 experiments are shown in Table 1, w_* is the convective velocity. The data are used for evaluating our present model.

3.1. Comparison between our proposed approach and the experimental one

D. M. Moreira et al. [10] used a dispersion model that employs a new analytical solution of the advection-diffusion equation for non-stationary and non-homogeneous conditions and radioactive contaminant by applying the Laplace transform, considering the Planetary boundary layer subdivided in N multilayers where the meteorological parameters can be considered constant in a complex terrain. D. M. Moreira et al. [11] developed the model for the growth of the turbulence in convective boundary layer by applying dimensional analysis to parameterize the unknown inertial transport and convective source term in the dynamic equation for the three dimensional spectrum and solved the problem using general integral Laplace transform using micro-meteorological parameters and wind profile generated by large eddy simulation. Buske et al. [6] solved advection diffusion equation for the atmospheric boundary layer where the eddy diffusivity coefficients and the wind profile are assumed to be space dependent and solved it using integral transform and spectral theory. Convergence of the solution is discussed in terms of a convergence criterion using a new interpretation of the Cardinal Theorem of Interpolation theory and Parseval's theorem. G.J. Weymar et al. [16] presented the dispersion model of radioactive pollutants that undergo chemical reactions and solved the two dimensional advection-diffusion-reaction equation to represent the dispersion of the pollutant in the atmospheric boundary layer for a source term by the combination of the methods of separating variables with the generalized integral Laplace transform technique. The other models were solved considering the wind velocity along z-direction w is negligible to u for the atmospheric boundary layer for a point source.

For the simulations we used the micro-meteorological parameters of the experiments 2 and 3 and in a period 3 of the Angra dos Reis experiment (see Table 1).

In the present study we introduce $\beta = 0.001\left(\frac{w_*}{u}\right)^2$, $\lambda = 1.55 \times 10^{-4} d^{-1}$, $v_d = 0.005 m/s$, the vertical and lateral dispersion parameters (equations (33–35)) in the predicted model (equation (32))to calculate the ground level concentration of emissions released from an point source in an unstable/neutral atmospheric boundary layer in centerline. In this manner the results of the predicted model are evaluated using mathmod (mathematica and maple programmes).

The validation of the predicted model against the experimental data from Angra site [3] and the other model [6] is shown in Table 2. Fig. 2 shows the observed, the other model [6] and our predicted results of the ground-level concentrations with a downwind distance.

Table 1
The micro-meteorological parameters and the emission rate for the Angra dos Reis experiments 2 and 3 with the period 3.

Exp.	Period	$u(m/s)$	$u_*(m/s)$	$w_*(m/s)$	$h(m)$	Q (MBq/s)
2	3	2.18	0.38	0.54	1133.98	25.34
3	3	2.61	0.46	0.66	1367.21	20.46

4. The statistical evaluation

The statistical analysis of the predictions and the observations is central to the model performance evaluation. The recommendations of two workshops sponsored by the American Meteorological Society (AMS) to review the statistical approach to the air quantity model evaluation and the model uncertainty are summarized by Fox [9]. The predicted and the corresponding observed concentrations are treated as pairs in this evaluation.The statistical index FB indicates whether the predicted quantities underestimate or overestimate the observed ones. The statistical index NMSE represents the quadratic error of the predicted quantities in the relation to the observed ones. The best results are indicated by the values nearest zero in NMSE, FB, and nearest 1 in R and FA2. The statistical measures chosen to compare the performances of the models described here:

(i) fractional bias FB is defined as:

$$FB = \frac{\overline{C_0} - \overline{C_p}}{0.5(\overline{C_0} + \overline{C_p})}$$

where the subscripts o and p refer to the observed and the predicted values, respectively, and the overline indicate the mean values. A good model should have FB a value close to zero.

Table 2
The comparison between the observed, the other model and the calculated concentrations (Bq/m^3).

Exp.	Period	Distance(m)	Observed (Bq/m^3)	*ourPredicted*(Bq /m^3)	other model
2	3	600	0.50	0.49	0.4
2	3	610	0.58	0.56	0.4
2	3	750	0.39	0.39	0.46
2	3	815	0.61	0.60	0.47
2	3	935	0.40	0.41	0.48
2	3	1070	0.86	0.63	0.48
3	3	700	24.09	23.06	31.02
3	3	705	38.89	38.24	31.13
3	3	960	19.61	20.24	32.97
3	3	970	36.22	36.26	32.95
3	3	1070	33.50	33.63	32.44

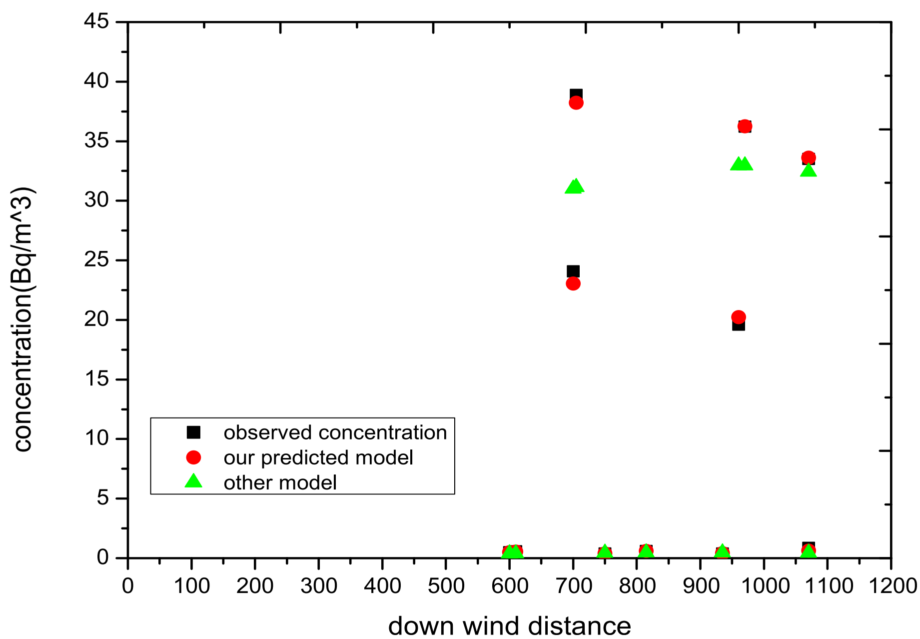

Fig. 2. Comparison between observed, our calculated and other model concentrations (Bq/m^3) with down wind distance.

(ii) The normalized mean square error (NMSE) is defined as:

$$\text{NMSE} = \frac{\overline{(C_0 - C_p)^2}}{\overline{C_0}\,\overline{C_p}}$$

This provides the information on the overall deviations between the predicted and the observed concentrations. It is a dimensionless statistic and its value should be as small as possible for a good model.

(iii) The correlation coefficient (R) is defined as:

$$R = \frac{\overline{(C_0 - \overline{C_0})(C_p - \overline{C_p})}}{\sigma_0 \sigma_p}$$

where σ_0 and σ_p are the standard deviations of the observed and the predicted concentrations, respectively. The square of R is called the Co-efficient of the determination and is useful measure of performance when evaluating two or more models with the same data set. Its value lies between 0 and 1 and for a good performance of a model it should be close to unity.(iv) (FA2) is the fraction of data (%) and satisfies the equation below for a good performance of the model

$$0.5 \le (C_p / C_0) \le 2$$

Table 3 presents the statistics for the normalized peak concentrations for the results of the experiments 2 and 3 in period 3 by our model and the other model [6].

From the statistical method, we find that the Laplace transform method agrees well with the observed data than Buske et al. [6] where *NMSE* and *FB* are nearest to zero, R is nearest 1 and the predicted model is *FA*2 with the observed data and is shown in Fig. 3.

Table 3
Statistics indices for the predicted model.

Case	FB	NMSE	R	FA2
The predicted model	0.01	0.01	0.91	1.03
The other model	0.13	0.38	0.83	0.88

5. Summary

In this work, the three-dimensional advection diffusion equation for an air radioactive pollutant (Tritium-3) released from an area source with constant emission rate and dry deposition for the

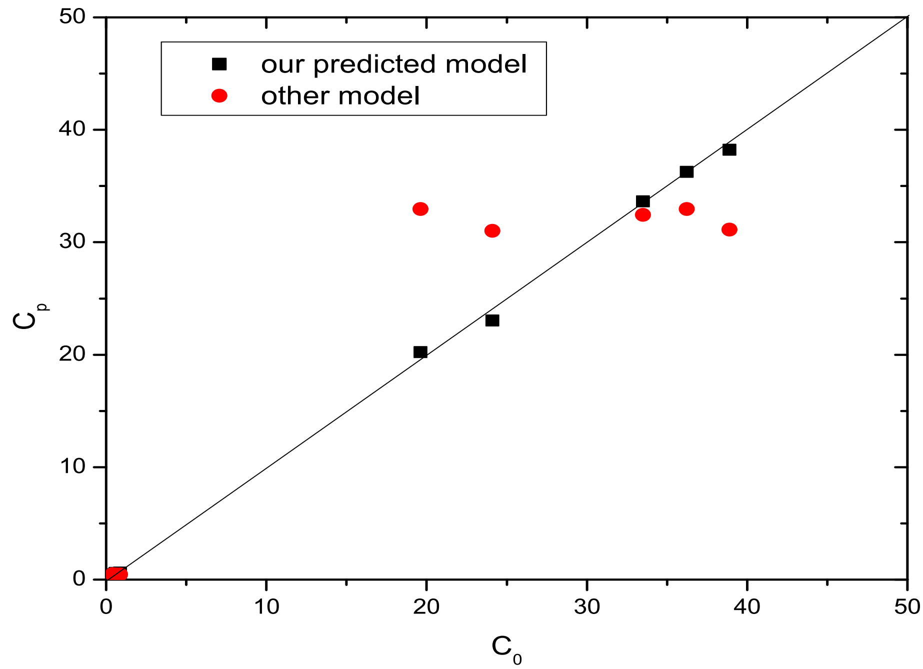

Fig. 3. Scatter plot of the predicted concentration is compared with the observed concentration and the other model for the experiments 2 and 3 in period 3.

flux is solved analytically using Laplace transform method. From the mathematical treatment, the inversion Laplace transform of hypergeometric function of first kind is formulated analytically using inversion Laplace transform of incomplete gamma function. The reliability of each model strongly depends on the way of the turbulent parameters (the wind velocity and eddy diffusivity profiles) which are taken and related to the current planetary boundary layer, in the present study, the wind velocity can be taken as function of the vertical direction z and eddy diffusivity profile along z-direction as constant functions. The obtained mathematical formulae of the present model can be validated for the comparison with the available data of the atmospheric dispersion experiments that were carried out in the nuclear power plant of Angra dos Reis (Brazil) for a point source. The concentration for a point source is obtained mathematically from which is formulated for an area source to perform the present model with the experimental data and the previous work. The analysis of the results shows a very good agreement between the computed values by the present model and the observed ones when comparing to the previous work of Busk et al. [6] which is the numerical solution of our problem.

The good agreement between the present analytical model and the experimental data gives us confidence to extend this work for a future work which can be applied in the advection-diffusion reaction equation involving eddy diffusivity profile as function of z, for more than a radioactive pollutant and extends the application to the experimental data for an area source.

Decleration of competing interests

The authors declare that they have no competing interests.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.net.2019.09.018.

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