



## Original Article

## Half-space albedo problem for İnönü, linear and quadratic anisotropic scattering

R.G. Tureci

Kırıkkale University, Kırıkkale Vocational School, Center Campus, Kırıkkale, 71450, Turkey



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## ABSTRACT

This study is concerned with the investigation of the half-space albedo problem for “İnönü-linear-quadratic anisotropic scattering” by the usage of Modified FN method. The method is based on Case's method. Therefore, Case's eigenfunctions and its orthogonality properties are derived for anisotropic scattering of interest. Albedo values are calculated for various linear, quadratic and İnönü anisotropic scattering coefficients and tabulated in Tables.

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## 1. Introduction

In this study one-speed neutron transport equation is used with İnönü-Linear-Quadratic anisotropic scattering. The one-speed neutron transport equation is the approximate equation of Boltzmann equation or Neutron transport equation in seven variables. Three of them are space variables, other three are velocity and the last one is time variable. This transport equation in seven variables does not have an analytical solution. Therefore, some reasonable approximations have to be made as follows in order to obtain analytical solution:

- Since the neutrons in a reactor are produced via certain nuclear reactions, it is assumed that they all have the same energy or velocity. This approximation is known as the one-speed approximation.
- It's assumed that the space is a planar geometry according to plane geometry approximation.
- If the medium structure is homogeneous, then all neutron interactions have the same cross sections. Therefore, the secondary neutron number,  $c$ , becomes a constant number.
- The fourth approximation is the “stationary state approximation”. The time variable drops from the Boltzmann equation by this approximation.

As a result of the above treatments, the number of variables is reduced to two, namely, a position variable and a direction cosine [1–3].

Case has reported a solution for one-speed, time-independent, homogenous medium neutron transport equation, which is known as Case's method [4,5]. The completeness of Case's eigenfunctions allows us to write a solution as a superposition of Case's eigenfunctions which correspond discrete eigenvalues and continuous eigenvalues.

Some semi-analytical methods such as Facile (FN) method [6,7], Complementary (CN) method [8], Hybrid (HN) (with the other name Modified FN) method [9] were developed in the literature. The Placzek lemma [10] is particularly important in these methods in terms of using the full-range orthogonality of Case's method. FN method uses Case's orthogonality relations to find the arbitrary expansion coefficients according to the boundary conditions. These arbitrary expansion coefficients are used in neutron flux. The neutron flux, which contains these coefficients, is written with  $N$  mesh points in order to get an equation system according to the angular variable. CN method uses the Green's function and the boundary neutron flux definitions in order to solve the transport equation. CN method is one of the most difficult method but it is an elegant method. Modified FN method, originally known as HN method, uses Case's orthogonality relations in order to find the arbitrary expansion coefficients. Finally, the moments of the neutron flux, which contains these arbitrary expansion coefficients and Case's eigenfunctions, is calculated to get an equation system.

E-mail address: [tureci@kku.edu.tr](mailto:tureci@kku.edu.tr).

According to the literature CN method is more convergent than FN method, and Modified FN method (HN method) provides the convergent results like CN method.

### 2. Linear-quadratic anisotropic scattering

The one-speed, plane geometry, homogenous medium and time independent neutron transport equation is given as

$$\mu \frac{\partial \Psi(x, \mu)}{\partial x} + \Psi(x, \mu) = \frac{c}{2} \int_{-1}^1 f(\mu, \mu') \Psi(x, \mu') d\mu' \quad (1)$$

where  $\mu$  is the direction cosine of the incident neutrons,  $\mu'$  is the direction cosine of the scattered neutrons,  $f(\mu, \mu')$  is the scattering function which defines the scattering probability of neutrons,  $\Psi(x, \mu)$  is the number of the neutrons at  $x$  position and  $\mu$  direction.  $c$  is the secondary neutron number which is a ratio between the cross sections. The scattering function in Eq. (1) is expanded with the Legendre polynomials, which is entitled as Mika's scattering [11] as

$$f(\mu, \mu') = \sum_{n=0}^N (2n+1) f_n P_n(\mu) P_n(\mu') \quad (2)$$

where  $P_n(\mu)$  and  $P_n(\mu')$  are Legendre polynomials and  $f_n$  is the scattering coefficient. Since the scattering function defines the probability of each scattering, the sum of these scatterings must equal to unity. Therefore, the definition interval of  $f_n$  coefficients must be determined for every scattering situation by physical meaning. The first and second terms in Eq. (2) are defined as isotropic and linear anisotropic scattering respectively. The probability of scattering is proportional to the multiplication of  $\mu$  and  $\mu'$ , which are  $P_1(\mu)$  and  $P_1(\mu')$ . Linear anisotropic scattering has been studied extensively with the different methods in the literature [12–14]. The third term is defined as quadratic anisotropic scattering and the probability of scattering is proportional to the multiplication  $P_2(\mu)$  and  $P_2(\mu')$ . This scattering has been studied in the literature [15–17].

The scattering function in this study is written as a linear combination of İnönü's scattering [18,19]. Thus the scattering function, which is called as linear-quadratic anisotropic scatterings, is

$$f(\mu, \mu') = a[1 + 3f_1\mu\mu' + 5f_2P_2(\mu)P_2(\mu')] + b\delta(\mu - \mu') + d\delta(\mu + \mu') = af_{lq}(\mu, \mu') + b\delta(\mu - \mu') + d\delta(\mu + \mu') \quad (3)$$

where  $b$  is the forward scattering coefficient and  $d$  is the backward scattering coefficient. If  $b$  has a nonzero value then  $d$  is zero, and if  $d$  has a nonzero value then  $b$  is zero. The coefficient  $a$  obeys the rule of the sum of probabilities, i.e.  $a + b + d = 1$ . If  $b$  and  $d$  are equal to zero together in Eq. (3) and  $f_{1,2} \neq 0$ , then the scattering is linear-quadratic anisotropic scattering.

İnönü's scattering function is known as the synthetic kernel in the literature. This scattering function defines the forward scattering and backward scattering of neutrons, and moreover, İnönü showed that the solution of the one-speed neutron transport equation with this scattering function can be written by Case's eigenfunctions by using of the transformations which were described by İnönü. The İnönü's scattering function is used for lots of studies [20–26]. The combination of İnönü's scattering function and linear anisotropic scattering was defined as strong anisotropic scattering in the literature [27–29].

Case's method is a powerful method to solve the one-speed

neutron transport equation. Mika's anisotropic scattering function can be applied to Case's method. Similarly, İnönü's scattering function also can be applied to Case's method. Furthermore, both of them can be applied to Case's method simultaneously. The aim of this study is to investigate the effect of these scattering functions over Albedo values, in the Case's of simultaneous application of the both.

### 3. The solution for the anisotropic scattering

If Eq. (3) is used in Eq. (1), then the one-speed neutron transport equation could be written as

$$\begin{aligned} \mu \frac{\partial \Psi_{llq}(x, \mu)}{\partial x} + \Psi_{llq}(x, \mu) &= \frac{ac}{2} \int_{-1}^1 f_{lq}(\mu, \mu') \Psi_{llq}(x, \mu') d\mu' + cb\Psi_{llq}(x, \mu) + cd\Psi_{llq}(x, -\mu) \end{aligned} \quad (4)$$

where the subscript  $llq$  indicates İnönü-linear-quadratic anisotropic scattering. If the following variables, which were redefined by İnönü, are used

$$c' = \frac{ac}{1 - bc - dc}, \quad q = [(1 - bc)^2 - b^2c^2]^{1/2} \quad (5-a)$$

$$x' = qx, \quad v' = qv \quad (5-b)$$

$$A = \frac{1}{2} \left[ 1 + \left( \frac{1 - bc - dc}{1 - bc + dc} \right)^{1/2} \right], \quad B = \frac{1}{2} \left[ 1 - \left( \frac{1 - bc - dc}{1 - bc + dc} \right)^{1/2} \right] \quad (5-c)$$

where  $q$ ,  $A$  and  $B$  are the factors which come from the transformation, and  $v$  corresponds to the eigenvalue. Then the solution of Eq. (4) is written as

$$\Psi_{llq}(x, \mu) = A\Psi_{lq}(x', \mu) + B\Psi_{lq}(x', -\mu) \quad (6-a)$$

$$\Psi_{llq}(x, -\mu) = A\Psi_{lq}(x', -\mu) + B\Psi_{lq}(x', \mu) \quad (6-b)$$

where the  $lq$  subscript indicates the linear-quadratic anisotropic scattering. Thus the solution, which includes İnönü-linear-quadratic anisotropic scattering is written in Eq. (6). The neutron flux for the linear-quadratic anisotropic scattering,  $\Psi_{lq}(x', \pm\mu)$ , is the solution of the one-speed neutron transport equation for  $a = 1$ ,  $b = 0$  and  $d = 0$ ; but,  $f_1 \neq 0$  and  $f_2 \neq 0$ .

The linear-quadratic anisotropic scattering was investigated in Refs. [30–33]. The solution of the neutron transport equation with the linear-quadratic anisotropic scattering,

$$\begin{aligned} \Psi_{lq}(x, \mu) &= a_{0+}\phi_{lq}(v_0, \mu)e^{-x/v_0} + a_{0-}\phi_{lq}(-v_0, \mu)e^{x/v_0} \\ &+ \int_0^1 A(v)\phi_{lq}(v, \mu)e^{-x/v} dv + \int_0^1 A(-v)\phi_{lq}(-v, \mu)e^{x/v} dv \end{aligned} \quad (7)$$

where Case's eigenfunctions and the orthogonality relations are given as

$$\phi(\pm v_0, \mu) = \frac{cv_0}{2} \frac{1 - \eta_0 \pm wv_0\mu + 3\eta_0\mu^2}{v_0 \mp \mu} \quad (8-a)$$

$$\phi(v, \mu) = \frac{cv}{2} P \frac{1 - \eta + wv\mu + 3\eta\mu^2}{v - \mu} + \lambda(v)\delta(v - \mu) \quad (8-b)$$

where  $P$  symbol correspond Cauchy Principal Value. The constants  $w$ ,  $\eta_0$  and  $\eta$  are defined as

$$w = 3f_1(1 - c) \quad (8-c)$$

$$\eta_0 \equiv \eta(v_0) = \frac{5f_2}{4} [3v_0^2(1 - c)(1 - cf_1) - 1] \quad (8-d)$$

$$\eta \equiv \eta(v) = \frac{5f_2}{4} [3v^2(1 - c)(1 - cf_1) - 1] \quad (8-e)$$

The normalization integral of the continuous Case's eigenfunctions gives

$$\lambda(v) = (1 + wcv^2 + 3\eta cv^2) - cv(1 - \eta + wv^2 + 3\eta v^2) \arctan h(v). \quad (8-f)$$

The orthogonality relations are

$$\int_{-1}^1 \mu \phi_{lq}(\pm v_0, \mu) \phi_{lq}(\pm v_0, \mu) d\mu = N_{lq}(\pm v_0) \text{ and } N_{lq}(-v_0) = -N_{lq}(v_0) \quad (8-g)$$

$$\int_{-1}^1 \mu \phi_{lq}(\pm v_0, \mu) \left\{ \begin{array}{l} \phi_{lq}(v, \mu) d\mu \\ \phi_{lq}(\mp v_0, \mu) d\mu \end{array} \right\} = \left\{ \begin{array}{l} 0 \\ 0 \end{array} \right. \quad (8-h)$$

$$\int_{-1}^1 \mu \phi_{lq}(v, \mu) \phi_{lq}(v', \mu) d\mu = N_{lq}(v) \delta(v - v') \text{ and } N_{lq}(-v) = -N_{lq}(v) \quad (8-i)$$

where

$$\begin{aligned} N_{lq}(v_0) = & \left(\frac{cv_0}{2}\right)^2 \left\{ \frac{2v_0}{v_0^2 - 1} [(1 - \eta_0)^2 + 2(1 - \eta_0)w + (6\eta_0(1 - \eta_0) + w^2 v_0^2) \right. \\ & + 6w\eta_0 + 9\eta_0^2] - [(1 - \eta_0)^2 + 4(1 - \eta_0)wv_0^2 + 3v_0^2(6\eta_0(1 - \eta_0) + w^2 v_0^2) \\ & + 24w\eta_0 v_0^4 + 45\eta^2 v_0^4] \frac{2}{cv_0} \frac{1 + wcv_0^2 + 3\eta cv_0^2}{1 - \eta_0 + wv_0^2 + 3\eta_0 v_0^2} + 8(1 - \eta_0)wv_0 + 6v_0(6\eta_0(1 - \eta_0) + w^2 v_0^2) + 48w\eta_0 v_0^3 \\ & \left. + 16w\eta_0 v_0 + 90\eta_0^2 v_0^3 + 30\eta_0^2 v_0 \right\} \end{aligned} \quad (8-j)$$

$$N_{lq}(v) = v \left[ (1 + wcv^2 + 3\eta cv^2) - cv(1 - \eta + wv^2 + 3\eta v^2) \arctan h(v) \right]^2 + \frac{c^2 \pi^2 v^3}{4} (1 - \eta + wv^2 + 3\eta v^2)^2 \quad (8-k)$$

#### 4. Albedo problem

It is well known that albedo is the ratio from a surface between the net outgoing and the net incoming neutron (or any neutral

particle). The surface is located at  $x = 0$ . The right side of this surface is the medium which consists of moderator material, and the left side of this surface is vacuum. The neutron flux which is given in Eq. (6) must be convergent to zero when  $x$  goes to infinity. Therefore, the arbitrary expansion coefficients, which are the coefficients of the positive exponential terms, must be selected equal to zero,  $a'_{0-} = 0$  and  $A(-v') = 0$ .

$$\begin{aligned} \Psi_{llq}(x, -\mu) = & A \left[ a'_{0+} \phi_{lq}(-v'_0, \mu) \exp\left(-\frac{x'}{v'_0}\right) + \int_0^1 A(v') \phi_{lq}(-v', \mu) \exp\left(-\frac{x'}{v'}\right) dv' \right] \\ & + B \left[ a'_{0+} \phi_{lq}(v'_0, \mu) \exp\left(-\frac{x'}{v'_0}\right) + \int_0^1 A(v') \phi_{lq}(v', \mu) \exp\left(-\frac{x'}{v'}\right) dv' \right], \quad \mu \in [-1, 1] \end{aligned} \quad (9)$$

The suggested neutron flux definitions over the surface are given by power series as

$$\Psi_{llq}(0, \mu) = \mu^\gamma, \quad \gamma = 0, 1, 2, \dots \quad (10-a)$$

$$\Psi_{llq}(0, -\mu) = \sum_{\ell=0}^N a_\ell \mu^\ell \quad (10-b)$$

Thus, the albedo relation is found by using Eq. (10)

$$\beta = \frac{\int_0^1 \mu \Psi(0, -\mu) d\mu}{\int_0^1 \mu \Psi(0, \mu) d\mu} = (\gamma + 2) \sum_{\ell=0}^N \frac{a_\ell}{\ell + 2} \quad (11)$$

Therefore, if  $a_\ell$  coefficients are found as numerical, then the albedo values can be calculated by using Eq. (11).

The first stage is to find a relationship for the arbitrary expansion coefficients,  $a'_{0+}$  and  $A(v')$ , by using the neutron flux definitions over the surface which are given in Eq. (10) and Case's orthogonality relations. If Eq. (9) is written by taking  $x = 0$  and it is multiplied by  $\mu \phi_{lq}(-v'_0, \mu)$  and  $\mu \phi_{lq}(-v', \mu)$  respectively and integrating over  $\mu \in [-1, 1]$ , then arbitrary expansion coefficients are

found as

$$a'_{0+} = \frac{c\nu'_0}{2} \frac{1}{A} \frac{1}{N_{lq}(\nu'_0)} \left[ B_\gamma(\nu'_0) - \sum_{\ell}^N a_\ell A_\ell(\nu'_0) \right] \quad (12)$$

$$A(\nu') = \frac{c\nu'}{2} \frac{1}{A} \frac{1}{N_{lq}(\nu')} \left[ B_\gamma(\nu') - \sum_{\ell}^N a_\ell A_\ell(\nu') \right] \quad (13)$$

where  $A_n(\xi)$  and  $B_\gamma(\xi)$  are defined as

$$A_n(\xi) = \frac{2}{c\xi} \int_0^1 \mu^{n+1} \phi_{lq}(\xi, -\mu) d\mu \quad (14)$$

and

$$B_n(\xi) = \frac{2}{c\xi} \int_0^1 \mu^{n+1} \phi_{lq}(\xi, \mu) d\mu \quad (15)$$

then the relation defines the outgoing neutron flux from the surface.

$$\Psi_{llq}(0, -\mu) = A \left[ a'_{0+} \phi_{lq}(-\nu'_0, \mu) + \int_0^1 A(\nu') \phi_{lq}(-\nu', \mu) d\nu' \right] + B \left[ a'_{0+} \phi_{lq}(\nu'_0, \mu) + \int_0^1 A(\nu') \phi_{lq}(\nu', \mu) d\nu' \right], \quad \mu \in [0, 1] \quad (9a)$$

In FN method to get a system of linear equations, Eq. (9a) is written with  $N+1$  mesh points for the angular variable. The  $N$  represents the number of meshes. In Modified FN method to get a system of linear equations, Eq. (9a) is multiplied with  $\mu^{m+1}$  and integrated over  $\mu \in [0, 1]$ . Thus, Eq. (20) is written by replacing the expansion coefficients Eq.12 and 13 and by using Eqs. 14 and 15 in this integrated equation.

$$\sum_{\ell=0}^N a_\ell \left[ \frac{1}{m+\ell+2} + \left(\frac{c\nu'_0}{2}\right)^2 \frac{A_\ell(\nu'_0)A_m(\nu'_0)}{N_{lq}(\nu'_0)} + \frac{B}{A} \left(\frac{c\nu'_0}{2}\right)^2 \frac{A_\ell(\nu'_0)B_m(\nu'_0)}{N_{lq}(\nu'_0)} \right. \\ \left. + \int_0^1 \left(\frac{c\nu'}{2}\right)^2 \frac{A_\ell(\nu')A_m(\nu')}{N_{lq}(\nu')} d\nu' + \frac{B}{A} \int_0^1 \left(\frac{c\nu'}{2}\right)^2 \frac{A_\ell(\nu')B_m(\nu')}{N_{lq}(\nu')} d\nu' \right] = \left(\frac{c\nu'_0}{2}\right)^2 \frac{B_\gamma(\nu'_0)A_m(\nu'_0)}{N_{lq}(\nu'_0)} + \frac{B}{A} \left(\frac{c\nu'_0}{2}\right)^2 \frac{B_\gamma(\nu'_0)B_m(\nu'_0)}{N_{lq}(\nu'_0)} + \int_0^1 \left(\frac{c\nu'}{2}\right)^2 \frac{B_\gamma(\nu')A_m(\nu')}{N_{lq}(\nu')} d\nu' \\ + \frac{B}{A} \int_0^1 \left(\frac{c\nu'}{2}\right)^2 \frac{B_\gamma(\nu')B_m(\nu')}{N_{lq}(\nu')} d\nu' \quad (20)$$

and  $n$  (or  $\gamma$ ) corresponds to the integer numbers according to  $N$  number in Eq. (10) which is the approximation number.  $\xi$  correspond to  $\nu'_0$  and  $\nu'$  for discrete and the continuous eigenvalues respectively. These integrals have separately recurrence relations:

$$A_n(\xi) = \frac{1-\eta_\xi}{n+1} - \frac{w\xi}{n+2} + \frac{3\eta_\xi}{n+3} - \xi A_{n-1}(\xi) \quad (16)$$

$$A_0(\xi) = 1 - \eta_\xi - \frac{w\xi(1-2\xi)}{2} + \frac{\eta_\xi}{2} (2 - 3\xi + 6\xi^2) - \xi \left( 1 - \eta_\xi + w\xi^2 + 3\eta_\xi \xi^2 \right) \ln \left( 1 + \frac{1}{\xi} \right) \quad (17)$$

$$B_n(\xi) = \xi B_{n-1}(\xi) - \frac{1-\eta_\xi}{n+1} - \frac{w\xi}{n+2} - \frac{3\eta_\xi}{n+3} \quad (18)$$

$$B_0(\xi) = \frac{2}{c} \left( 1 + cw\xi^2 + 3c\eta_\xi \xi^2 \right) - (1 - \eta_\xi) - \frac{w\xi(1+2\xi)}{2} - \frac{\eta_\xi}{2} (2 + 3\xi + 6\xi^2) - \xi \left( 1 - \eta_\xi + w\xi^2 + 3\eta_\xi \xi^2 \right) \ln \left( 1 + \frac{1}{\xi} \right) \quad (19)$$

where  $\eta_\xi$  is related to the kind of eigenvalue.  $\eta_0$  is related to the discrete eigenvalues and  $\eta$  is related to the continuous eigenvalues. If  $a_{0+}'$  and  $A(\nu')$  coefficients, which are found in Eqs. 12 and 13, are put into Eq. (9) and this relation is written in the interval  $\mu \in [0, 1]$ ,

Eq. (20) can be rewritten as

$$\sum_{\ell=0}^N a_\ell T_{m\ell} = U_{m\gamma} \quad (21)$$

$m$  and  $\ell$  indices change together from zero to  $N$ , thus we get a square matrix.  $a_\ell$  coefficients are calculated by using ordinary matrix operations.

$$A = T^{-1} \cdot U \quad (22)$$

where the elements of  $A$  matrix are  $a_\ell$  coefficients.  $T^{-1}$  is the inverse matrix of  $T$ , and its elements are  $T_{m\ell}$ .  $U$  matrix is the column matrix and its elements are  $U_{m\gamma}$ . Finally  $a_\ell$  coefficients are used in Eq. (11) and so the albedo values are found. Here  $T_{m\ell}$  is

$$T_{m\ell} = \frac{1}{m+\ell+2} + \left(\frac{c\nu'_0}{2}\right)^2 \frac{A_\ell(\nu'_0)A_m(\nu'_0)}{N_{lq}(\nu'_0)} + \frac{B}{A} \left(\frac{c\nu'_0}{2}\right)^2 \frac{A_\ell(\nu'_0)B_m(\nu'_0)}{N_{lq}(\nu'_0)} + \int_0^1 \left(\frac{c\nu'}{2}\right)^2 \frac{A_\ell(\nu')A_m(\nu')}{N_{lq}(\nu')} d\nu' \\ + \frac{B}{A} \int_0^1 \left(\frac{c\nu'}{2}\right)^2 \frac{A_\ell(\nu')B_m(\nu')}{N_{lq}(\nu')} d\nu'$$

and  $U_{m\gamma}$  is

$$U_{m\gamma} = \left(\frac{c'v'_0}{2}\right)^2 \frac{B_\gamma(v'_0)A_m(v'_0)}{N_{lq}(v'_0)} + \frac{B}{A} \left(\frac{c'v'_0}{2}\right)^2 \frac{B_\gamma(v'_0)B_m(v'_0)}{N_{lq}(v'_0)} + \int_0^1 \left(\frac{c'v'}{2}\right)^2 \frac{B_\gamma(v')A_m(v')}{N_{lq}(v')} dv' + \frac{B}{A} \int_0^1 \left(\frac{c'v'}{2}\right)^2 \frac{B_\gamma(v')B_m(v')}{N_{lq}(v')} dv'$$

5. Numerical results

Case's eigenfunction is normalized to 1. The discrete eigenvalue corresponds to  $v \notin [-1, 1]$  and represented as  $v_0$ . By using the normalization condition of the discrete eigenfunction we get a distribution which is a transcendental equation.

$$\Lambda(v'_0) = \ln\left(\frac{1 + 1/v'_0}{1 - 1/v'_0}\right) - \frac{2}{cv'_0} \frac{1 + wcv'_0 + 3\eta_0cv'^2_0}{1 - \eta_0 + wv'^2_0 + 3\eta_0v'^2_0} = 0 \quad (23)$$

The numerical solution of this equation gives the discrete eigenvalues,  $\pm v_0$ .

The Mika's part of the scattering function is

$$f(\mu, \mu') = 1 + 3f_1\mu\mu' + 5f_2P_2(\mu)P_2(\mu') \quad (24)$$

where  $f_1 \in [-1/3, 1/3]$  for pure-linearly anisotropic scattering,  $f_2 = 0$ , and also  $f_2 \in [-0.2, 0.4]$  for pure-quadratic anisotropic scattering,  $f_1 = 0$ . The maximum and the minimum values of the direction cosines are +1 and -1 respectively. If  $f_2$  equal to zero, i.e. the scattering is pure-linear anisotropic scattering, then the scattering function is

$$f(\mu, \mu') = 1 + 3f_1\mu\mu'. \quad (25)$$

Eq. (25) is defined in interval [0, 1] according to the rule of the sum of probabilities. In other words, the summation equals one. The minimum value of  $\mu \cdot \mu'$  is -1, and the maximum value is +1. Thus, the definition interval is determined for  $f_1$  which is  $f_1 \in [-1/3, 1/3]$ . If  $f_1$  equals zero in Eq. (24) which corresponds to

Table 1 The quadratic anisotropic scattering coefficients and the corresponding discrete eigenvalues according to  $f_1$  for fixed secondary neutron number.  $c = 0.8$

b or d	c'	$f_1 = 0.1$		$f_1 = 0.2$		$f_1 = 0.3$	
		$f_2 \in [-0.14, 0.06]$		$f_2 \in [-0.08, 0.12]$		$f_2 \in [-0.02, 0.18]$	
		$f_2$	$\pm v'_0$	$f_2$	$\pm v'_0$	$f_2$	$\pm v'_0$
0.2	0.7619047619048	-0.14	1.3503367489803	-0.08	1.41334586878385	-0.01	1.4867844987356
0.4	0.7058823529412		1.2419111919982		1.29509510276002		1.3567170391812
0.6	0.6153846153846		1.1291274965985		1.16975744761954		1.2166527676182
0.8	0.4444444444444		1.0262359306891		1.04702282933634		1.0723781218970
0.2	0.7619047619048	-0.12	1.3521486330776	-0.01	1.42066290480843	0.05	1.4938635927823
0.4	0.7058823529412		1.2438736414053		1.30288836413616		1.3641417225914
0.6	0.6153846153846		1.1312089296472		1.17784635913769		1.2242044885072
0.8	0.4444444444444		1.0278887618233		1.05374548049964		1.0786818616781
0.2	0.7619047619048	-0.1	1.3540147671943	0.1	1.43395481263645	0.1	1.5003196844828
0.4	0.7058823529412		1.2458903282655		1.31687706972736		1.3708625968616
0.6	0.6153846153846		1.1333393250400		1.19206935333486		1.2309538595848
0.8	0.4444444444444		1.0295808620024		1.06525719311210		1.0842024211896
0.2	0.7619047619048	0.06	1.3712190807226	0.12	1.43664749310913	0.18	1.5118821198339
0.4	0.7058823529412		1.2642676247910		1.31968466887168		1.3827801177440
0.6	0.6153846153846		1.1523638009773		1.19487937971909		1.2427251544163
0.8	0.4444444444444		1.0445661970991		1.06748354209275		1.0935813926334

Table 2 The albedo values for  $c = 0.8, f_1 = 0.1, \gamma = 0$  varying  $f_2$ , forward and backward.

$f_2 = -0.14$								
N	d = 0				b = 0			
	b = 0.2	b = 0.4	b = 0.6	b = 0.8	d = 0.2	d = 0.4	d = 0.6	d = 0.8
1	0.2875967	0.2434477	0.1873575	0.1118563	0.3597839	0.3920832	0.4227940	0.4553374
2	0.2868021	0.2428643	0.1870181	0.1117550	0.3588239	0.3912132	0.4221210	0.4549991
3	0.2867814	0.2428470	0.1870063	0.1117507	0.3588000	0.3911897	0.4221006	0.4549872
4	0.2867802	0.2428459	0.1870055	0.1117503	0.3587987	0.3911883	0.4220993	0.4549863
5	0.2867801	0.2428458	0.1870054	0.1117503	0.3587985	0.3911881	0.4220991	0.4549862
6	0.2867800	0.2428458	0.1870054	0.1117502	0.3587985	0.3911881	0.4220991	0.4549862
7	0.2867800	0.2428458	0.1870054	0.1117502	0.3587985	0.3911880	0.4220991	0.4549862
$f_2 = 0.06$								
N	d = 0				b = 0			
	b = 0.2	b = 0.4	b = 0.6	b = 0.8	d = 0.2	d = 0.4	d = 0.6	d = 0.8
1	0.2896088	0.2456510	0.1897130	0.1140646	0.3616931	0.3940646	0.4247977	0.4571087
2	0.2888531	0.2451073	0.1894057	0.1139769	0.3608007	0.3932910	0.4242337	0.4568503
3	0.2888233	0.2450826	0.1893890	0.1139709	0.3607663	0.3932572	0.4242047	0.4568335
4	0.2888215	0.2450810	0.1893877	0.1139703	0.3607643	0.3932552	0.4242027	0.4568322
5	0.2888213	0.2450808	0.1893876	0.1139703	0.3607641	0.3932549	0.4242025	0.4568321
6	0.2888213	0.2450807	0.1893876	0.1139702	0.3607641	0.3932549	0.4242025	0.4568320
7	0.2888213	0.2450807	0.1893875	0.1139702	0.3607641	0.3932549	0.4242024	0.4568320

**Table 3**  
The albedo values for  $c = 0.8, f_1 = 0.2, \gamma = 0$  varying  $f_2$ , forward and backward.

$f_2 = -0.08$								
N	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2691058	0.2262191	0.1724083	0.1013803	0.3422094	0.3765424	0.4100238	0.4468974
2	0.2682567	0.2256016	0.1720538	0.1012768	0.3411726	0.3756014	0.4092971	0.4465348
3	0.2682329	0.2255817	0.1720403	0.1012718	0.3411448	0.3755736	0.4092727	0.4465202
4	0.2682315	0.2255804	0.1720393	0.1012713	0.3411432	0.3755720	0.4092711	0.4465191
5	0.2682314	0.2255803	0.1720392	0.1012713	0.3411431	0.3755718	0.4092709	0.4465190
6	0.2682313	0.2255803	0.1720392	0.1012713	0.3411430	0.3755718	0.4092709	0.4465190
7	0.2682313	0.2255802	0.1720391	0.1012712	0.3411430	0.3755718	0.4092709	0.4465190
$f_2 = 0.12$								
N	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2713509	0.2286496	0.1749641	0.1037134	0.3443460	0.3787399	0.4122137	0.4487824
2	0.2705373	0.2280691	0.1746399	0.1036226	0.3433772	0.3779003	0.4116034	0.4485049
3	0.2705032	0.2280409	0.1746208	0.1036156	0.3433375	0.3778608	0.4115689	0.4484846
4	0.2705012	0.2280391	0.1746194	0.1036151	0.3433352	0.3778585	0.4115667	0.4484830
5	0.2705010	0.2280388	0.1746192	0.1036150	0.3433350	0.3778582	0.4115665	0.4484829
6	0.2705010	0.2280388	0.1746192	0.1036150	0.3433350	0.3778582	0.4115664	0.4484828
7	0.2705010	0.2280388	0.1746192	0.1036150	0.3433349	0.3778582	0.4115664	0.4484828

the pure-quadratic anisotropic scattering, then the scattering function is

$$f(\mu, \mu') = 1 + 5f_2P_2(\mu)P_2(\mu') \tag{26}$$

The definition interval  $f_2$  is determined as  $f_2 \in [-0.2, 0.4]$  by using the same rule. But, there is a linear dependent relation between  $f_1$  and  $f_2$  in the presence of both of the scatterings. This linear dependent relation is determined as

$$\frac{3f_1 - 1}{5} \leq f_2 \leq \frac{3f_1}{5} \tag{27}$$

by using the same rule and taking the set of intersections between  $f_1$  and  $f_2$ . This interval was written for  $f_1$  because the linear anisotropic scattering is more dominant scattering than the quadratic anisotropic scattering. Mathematica software v.12.0 [34] was used

in all calculations. Table 1 represents the discrete eigenvalues for varying  $f_1, f_2, b$  and  $d$ .

Tables 2–4 represent the albedo values which are calculated by Eq. (11) for  $c = 0.8, f_1 = 0.1, \gamma = 0$  and varying selected  $f_2$  values. Table 5 represents the albedo values for varying  $\gamma$  values.

### 6. Conclusion

In this study; one-speed, homogeneous medium, plane geometry and time-independent neutron transport equation for half-space albedo has been investigated for the anisotropic scattering which contains İnönü's scattering, linear and quadratic anisotropic scatterings in Mika's anisotropic scattering function. To write the solution, Case's eigenfunctions and the orthogonality properties should be written for the scattering function of interest. There is also a linear dependent relation in order to determine the

**Table 4**  
The albedo values for  $c = 0.8, f_1 = 0.3, \gamma = 0$  varying  $f_2$ , forward and backward.

$f_2 = -0.02$								
N	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2486893	0.2073941	0.1563186	0.0903775	0.3227452	0.3594641	0.3961745	0.4379663
2	0.2477801	0.2067398	0.1559487	0.0902720	0.3216204	0.3584413	0.3953859	0.4375763
3	0.2477530	0.2067172	0.1559333	0.0902664	0.3215883	0.3584089	0.3953569	0.4375585
4	0.2477514	0.2067158	0.1559322	0.0902659	0.3215866	0.3584070	0.3953551	0.4375573
5	0.2477513	0.2067156	0.1559321	0.0902658	0.3215864	0.3584068	0.3953549	0.4375571
6	0.2477512	0.2067156	0.1559321	0.0902658	0.3215864	0.3584068	0.3953549	0.4375571
7	0.2477512	0.2067156	0.1559321	0.0902658	0.3215864	0.3584068	0.3953549	0.4375571
$f_2 = 0.18$								
N	$d = 0$				$b = 0$			
	$b = 0.2$	$b = 0.4$	$b = 0.6$	$b = 0.8$	$d = 0.2$	$d = 0.4$	$d = 0.6$	$d = 0.8$
1	0.2512097	0.2100882	0.1591012	0.0928463	0.3251515	0.3619145	0.3985774	0.4399763
2	0.2503296	0.2094663	0.1587584	0.0927522	0.3240933	0.3609976	0.3979131	0.4396769
3	0.2502911	0.2094345	0.1587370	0.0927445	0.3240477	0.3609517	0.3978725	0.4396525
4	0.2502889	0.2094325	0.1587355	0.0927439	0.3240452	0.3609491	0.3978700	0.4396508
5	0.2502886	0.2094322	0.1587353	0.0927438	0.3240450	0.3609489	0.3978698	0.4396506
6	0.2502886	0.2094322	0.1587353	0.0927438	0.3240449	0.3609488	0.3978697	0.4396506
7	0.2502886	0.2094322	0.1587353	0.0927438	0.3240449	0.3609488	0.3978697	0.4396506



**Table 5**  
The albedo values for  $c = 0.8, f_1 = 0.1, f_2 = -0.14$ , forward and backward for varying  $\gamma$

$c = 0.8, f_1 = 0.1, f_2 = -0.14$										
$b = 0.2$ and $d = 0$ (forward)										
N	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$	$\gamma = 7$	$\gamma = 8$	$\gamma = 9$
1	0.2875967	0.2692079	0.2590674	0.2526693	0.2482730	0.2450691	0.2426318	0.2407158	0.2391704	0.2378977
2	0.2868021	0.2684446	0.2583050	0.2519013	0.2474983	0.2442881	0.2418450	0.2399240	0.2383743	0.2370977
3	0.2867814	0.2684337	0.2582982	0.2518968	0.2474951	0.2442858	0.2418435	0.2399231	0.2383737	0.2370976
4	0.2867802	0.2684335	0.2582984	0.2518971	0.2474956	0.2442863	0.2418440	0.2399236	0.2383743	0.2370982
5	0.2867801	0.2684335	0.2582985	0.2518972	0.2474957	0.2442864	0.2418441	0.2399237	0.2383744	0.2370983
6	0.2867800	0.2684335	0.2582985	0.2518972	0.2474957	0.2442865	0.2418442	0.2399238	0.2383745	0.2370983
7	0.2867800	0.2684336	0.2582985	0.2518972	0.2474957	0.2442865	0.2418442	0.2399238	0.2383745	0.2370983
$b = 0$ and $d = 0.2$ (backward)										
N	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$	$\gamma = 7$	$\gamma = 8$	$\gamma = 9$
1	0.3597839	0.3413852	0.3312436	0.3248465	0.3204515	0.3172490	0.3148129	0.3128981	0.3113536	0.3100818
2	0.3588239	0.3410050	0.3311891	0.3250012	0.3207524	0.3176579	0.3153048	0.3134558	0.3119649	0.3107375
3	0.3588000	0.3409927	0.3311512	0.3249313	0.3206515	0.3175291	0.3151516	0.3132811	0.3117714	0.3105274
4	0.3587987	0.3409926	0.3311515	0.3249341	0.3206586	0.3175411	0.3151687	0.3133034	0.3117987	0.3105595
5	0.3587985	0.3409926	0.3311515	0.3249342	0.3206584	0.3175404	0.3151672	0.3133010	0.3117952	0.3105548
6	0.3587985	0.3409927	0.3311516	0.3249343	0.3206584	0.3175405	0.3151674	0.3133013	0.3117957	0.3105555
7	0.3587985	0.3409927	0.3311516	0.3249343	0.3206584	0.3175405	0.3151674	0.3133012	0.3117956	0.3105554

scattering coefficients,  $f_1$  and  $f_2$ . This linear dependent relation is found by taking the set of intersections for  $f_1$  and  $f_2$ . The solution is written according to the İnönü's results.

Case's method is a powerful method in order to write the solution for the one-speed neutron transport equation. Moreover, Case's method is well defined for Mika's scattering function, İnönü's scattering function and the linear combination of these scattering functions. This is an important result.

Albedo definitions are written for the selected surface neutron distributions. Calculations have been performed with HN (Modified FN) method and the convergent results are obtained like CN method results in the earlier different problems. The method also gives convergent numerical results for this study.

According to the tabulated values:

- 1 Since the net incoming neutron flux decreases by the increasing  $\gamma$ , the albedo values decrease by the increasing  $\gamma$  number. This expectation seems true according to Table 5.
- 2 Albedo values decrease by increasing linear anisotropic scattering coefficient. This decline is fast for the increasing  $b$ , and slowly for increasing  $d$ .
- 3 Albedo values decrease by increasing forward scattering in the case of certain  $f_1$  and  $f_2$ . This is an expected result because the increasing forward scattering gives attribute to incoming flux. While albedo values increase by increasing backward scattering in the case of certain  $f_1$  and  $f_2$ . This is an expected result because the increasing backward scattering gives attribute to outgoing flux.
- 4 The albedo values for the backward scattering is bigger than forward anisotropic scattering in the case of certain  $f_1$  and  $f_2$  because the effect of backward scattering gives the attribute to the outgoing neutron flux.

According to the results, İnönü's scattering or forward-backward scattering becomes the dominant scattering. The variation of the albedo values for the certain forward or backward scattering is changed more quickly than the variation of linear-quadratic anisotropic scattering. The linear anisotropic scattering is more dominant scattering than the quadratic anisotropic scattering for the certain forward or backward scattering. This is an expected result because the linear anisotropic scattering is proportional to the multiplication of  $P_1(\mu)$  and  $P_1(\mu')$ , and the

quadratic anisotropic scattering is proportional to the multiplication of  $P_2(\mu)$  and  $P_2(\mu')$ .

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