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SOME STABILITY RESULTS FOR COINCIDENCE POINT ITERATIVE ALGORITHMS WITH THREE MAPPINGS

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Abstract. In this paper, we introduce a new concept of stability of coincidence iterative algorithm for three mappings and derive a new three-step Jungck-type iterative algorithm. And, we prove a stability result and a strong convergence result for the Jungck-type algorithm using the M_J-contractive condition. Our results extend and unify the corresponding ones in [3, 6, 7, 13].

1. Introduction and Preliminaries

A concept of the stability of fixed point iterative algorithms was initiated by Harder and Hicks [1] in 1988. As their results show, the study of stability of iterative algorithms has been both theoretical and numerical interests. In fact, the study of stability of fixed point iterative algorithms for various mappings in normed spaces or metric spaces has been rapidly developed into many directions [6, 7, 8, 9, 11, 12, 14].

In 2004, Singh et al. [13] introduced a concept of the stability of coincidence point iterative algorithms for two mappings and proved some stability results of Jungck and Jungck-Mann iterative algorithms. In 2008, Olatinwo [7] introduced Jungck-Ishikawa iterative algorithm, and obtained some stability and strong convergence results for Jungck-Ishikawa iterative algorithm. Recently, Olatinwo [6] proved some stability and strong convergence results for Picard, Mann, Ishikawa and Jungck type iterative algorithms by M_J-contractive conditions.

Inspired by the above results, in this paper, we introduce a new concept of stability of coincidence iterative algorithm for three mappings and derive a new three-step Jungck-type iterative algorithm. And, we prove a stability result and a strong convergence result for our iterative algorithm using the M_J-contractive conditions in [6]. Our results extend and unify the corresponding ones in [6, 13, 3, 7].

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Let K be an arbitrary subset of a normed space E and $S, T, R \in M(K)(= \{T : T \text{ is a mapping from } K \text{ to } E\})$ with $T(K) \cup R(K) \subset S(K)$. We define a coincidence point iterative algorithm by

$$Sx_{n+1} = f(T, R, x_n) \text{ for } n \geq 0, \tag{1.1}$$

where $x_0 \in K$ is the initial approximation and f is a function $M(K) \times M(K) \times K$ to E.

In actual computations, since it's difficult to get the exact value of x_1 due to various errors (rounding errors, numerical approximation of functions, derivatives or integrals, etc.), where $Sx_1 = f(T, R, x_0)$, the following method is used to get an approximation of $\{Sx_n\}$. Take y_1 closely enough to x_1, so that $Sy_1 \approx Sx_1$. Take y_2 closely enough to x_2 so that $Sy_2 \approx Sx_2 = f(T, R, y_1)$. Continuing this process, we obtain a sequence $\{Sy_{n+1}\}$ approximating closely to $\{Sx_{n+1}\}$ with $Sx_{n+1} = f(T, R, y_n)$ for $n \geq 0$. Now, we introduce a new concept of stability for the coincidence point iterative algorithm (1.1) as follows;

Definition 1.1. Let $q \in C(S, T, R)(:= \{q \in K : Sq = Tq = Rq\})$. For any $x_0 \in K$, let the sequence $\{Sx_n\}$ generated by (1.1) converge to Sq, say p. Let $\{Sy_n\} \subset E$ be an arbitrary sequence and set $\varepsilon_n = \|Sy_{n+1} - f(T, R, y_n)\|$. Then, the iterative algorithm (1.1) is said to be (S, T, R)-stable if $\lim_{n\to\infty} Sy_n = p$ for $\lim_{n\to\infty} \varepsilon_n = 0$.

Definition 1.1 reduces to that of the stability of iterative algorithm due to Singh et al. [13] when $K = E$ and $f(T, R, x_n) = f(T, x_n)$.

Example 1.1. Let $S, T, R : [0, 1] \to [0, 2]$ be mappings defined by

$$Sx = \begin{cases} x + 1, & x \neq \frac{2}{3} \\ 0, & x = \frac{2}{3}, \end{cases} \qquad Tx = \begin{cases} 1, & x \in [0, \frac{1}{2}] \\ 2, & x = \frac{2}{3} \\ \frac{3}{2}, & x \in (\frac{1}{2}, 1] \setminus \{\frac{2}{3}\} \end{cases} \qquad \text{and } Rx = \begin{cases} 1, & x \in [0, \frac{1}{2}] \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then, $S0 = T0 = R0 = 1$. Let $x_0 \in [0, 1]$ and

$$Sx_{n+1} = f(T, R, x_n) = (1 - a_n - b_n)Sx_n + a_nTx_n + b_nRx_n \text{ for } n \geq 0. \tag{1.2}$$

Take $a_n = \frac{1}{2}$ and $b_n = \frac{1}{4}$ for $n \geq 0$. If $x_0 \in [0, \frac{1}{2}]$, then

$$Sx_1 = \frac{1}{4}(x_0 + 1) + \frac{1}{2} + \frac{1}{4} = \frac{1}{4}x_0 + 1, \;\; x_1 = \frac{1}{4}x_0;$$

$$
\begin{aligned}
Sx_2 &= \frac{1}{4}\Big(\frac{1}{4}x_0+1\Big)+\frac{1}{2}+\frac{1}{4}=\frac{1}{4^2}x_0+1,\ \ x_2=\frac{1}{4^2}x_0;\\
&\ \vdots\\
Sx_n &= \frac{1}{4^n}x_0+1,\ \ x_n=\frac{1}{4^n}x_0\ \text{ for }\ n\geq 0.
\end{aligned}
$$

If $x_0=\frac{2}{3}$, then

$$
\begin{aligned}
Sx_1 &= \frac{1}{4}\cdot 0+\frac{1}{2}\cdot 2=1,\ \ x_1=0;\\
Sx_2 &= \frac{1}{4}\cdot 1+\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 1=1,\ \ x_2=0;\\
&\ \vdots\\
Sx_n &= \frac{1}{4}\cdot 1+\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 1,\ \ x_n=0\ \text{ for }\ n\geq 0.
\end{aligned}
$$

If $x_0\in(\frac{1}{2},1]\setminus\{\frac{2}{3}\}$, then we have

$$
Sx_1 = \frac{1}{4}(x_0+1)+\frac{1}{2}\cdot\frac{3}{2}=\frac{1}{4}x_0+1,\ \ x_1=\frac{1}{4}x_0.
$$

Thus, $x_1\in[0,\frac{1}{2}]$, so $Sx_n=\frac{1}{4^n}x_0+1$ for $n\geq 2$. Hence, we obtain $\lim\limits_{n\to\infty}Sx_n=1$.

Now, we show that the iterative algorithm (1.2) is (S,T,R)-stable. Take a sequence $\{Sy_n\}=\{\frac{1}{n}+1\}$ for $n\geq 0$, then

$$
\begin{aligned}
\varepsilon_n &= |Sy_{n+1}-(1-a_n-b_n)Sy_n-a_nTy_n-b_nRy_n|\\
&= \Big|\frac{1}{n+1}+1-\frac{1}{4}\Big(\frac{1}{n}+1\Big)-\frac{1}{2}-\frac{1}{4}\Big|\\
&= \Big|\frac{1}{n+1}-\frac{1}{4n}\Big|\ \text{ for }\ n\geq 2.
\end{aligned}
$$

Thus, we have $\lim\limits_{n\to\infty}\varepsilon_n=0$ and $\lim\limits_{n\to\infty}Sy_n=\lim\limits_{n\to\infty}(\frac{1}{n}+1)=1$. Hence, the iterative algorithm (1.2) is (S,T,R)-stable.

Remark 1.1. (i) We derive a new three-step iterative scheme from (1.1) as follows;

$$
(1.3)\qquad \begin{cases} Sx_{n+1}=f(T,R,x_n)=(1-a_n-b_n)Sx_n+a_nTr_n+b_nRr_n,\\ Sr_n=(1-a'_n-b'_n)Sx_n+a'_nTs_n+b'_nRs_n,\\ Ss_n=(1-a''_n-b''_n)Sx_n+a''_nTx_n+b''_nRx_n\ \text{ for }\ n\geq 0,\end{cases}
$$

where S is injective and $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$ are sequences in $[0,1]$.
(ii) If $K=E$, $S=I$ and $b_n=b'_n=b''_n=0$ $(n\geq 0)$ in (1.3), then we obtain the

following Noor iterative algorithm [5]

$$(1.4)\qquad \begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T r_n, \\ r_n = (1 - a'_n)x_n + a'_n T s_n, \\ s_n = (1 - a''_n)x_n + a''_n T x_n \ \text{ for } \ n \geq 0, \end{cases}$$

where $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$ are sequences in $[0, 1]$.
(iii) If $a''_n = 0$ $(n \geq 0)$ in (1.4), then we obtain the following Ishikawa iterative algorithm [2]

$$(1.5)\qquad \begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T r_n, \\ r_n = (1 - a'_n)x_n + a'_n T x_n \ \text{ for } \ n \geq 0, \end{cases}$$

where $\{a_n\}$ and $\{a'_n\}$ are sequences in $[0, 1]$.
(iv) If $a''_n = 0$ $(n \geq 0)$ in (1.5), then we obtain the following Mann iterative algorithm [4]

$$\Big\{ x_{n+1} = (1 - a_n)x_n + a_n T x_n \ \text{ for } \ n \geq 0,$$

where $\{a_n\}$ is a sequence in $[0, 1]$.

Definition 1.2 ([6])**.** Let $S, T : K \to E$ be mappings with $T(K) \subset S(K)$, where $S(K)$ is a complete subspace of E and let $\alpha : \mathbb{R}^3_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous mapping satisfying the following condition $(*)$;
$(*)$: an inequality $a \leq \alpha(b, b, a)$ guarantees the existence of $k \in [0, 1)$ with $a \leq kb$.
A pair (S, T) is said to be a M_J-contraction with respect to a mapping α with condition $(*)$ if it satisfies the following inequality;

$$\begin{aligned} \|Tx - Ty\| \ \leq \ & \alpha(\|Sx - Sy\|, max\{\|Sx - Tx\|, \|Sy - Tx\|\}, \\ & max\{\|Sy - Ty\|^m \cdot \|Sx - Tx\|^l, \|Sx - Ty\|\}) \end{aligned} \qquad (1.6)$$

for $x, y \in K$ and $m, l \in \mathbb{R}_{\geq 0}$.

Lemma 1.3 ([10])**.** *If $d \in [0, 1)$ and $\{v_n\}$ is a sequence of nonnegative real numbers such that $\lim_{n\to\infty} v_n = 0$, then for any sequence of nonnegative real numbers $\{u_n\}$ satisfying*

$$u_{n+1} \leq d u_n + v_n \ \textit{ for } \ n \geq 0,$$

we have $\lim_{n\to\infty} u_n = 0$.

2. Stability Result

In this section, we establish a stability result of iterative algorithm (1.3).

Theorem 2.1. *Let S, T, R be mappings from $K(\subset E)$ to a normed space E with $T(K) \cup R(K) \subset S(K)$, S be an injective mapping and $C(S, T, R) \neq \emptyset$. Assume that (S, T) and (S, R) are M_J-contractions with respect to α and α' with condition (*), respectively. For $x_0 \in K$, let $\{Sx_n\} \subset E$ be an iterative algorithm defined by (1.3) converging to $p(= Sq = Tq = Rq)$, where $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$ are sequences in $[0, 1]$ such that $0 < w = \inf_{n \geq 0} a_n$. Then, $\{Sx_n\}$ is (S, T, R)-stable.*

Proof. Take $\{Sy_n\}$ in E with $\varepsilon_n = \|Sy_{n+1} - (1 - a_n - b_n)Sy_n - a_n Tc_n - b_n Rc_n\|$, $Sc_n = (1 - a'_n - b'_n)Sy_n + a'_n Td_n + b'_n Rd_n$ and $Sd_n = (1 - a''_n - b''_n)Sy_n + a''_n Ty_n + b''_n Ry_n$ $(n \geq 0)$. Assume that $\lim_{n\to\infty} \varepsilon_n = 0$. From (1.6), we have

$$\begin{aligned} \|Tq - Tc_n\| &\leq \alpha(\|Sq - Sc_n\|, \max\{\|Sq - Tq\|, \|Sc_n - Tq\|\}, \\ &\qquad \max\{\|Sc_n - Tc_n\|^m \cdot \|Sq - Tq\|^l, \|Sq - Tc_n\|\}) \\ &= \alpha(\|Sq - Sc_n\|, \|Sc_n - Sq\|, \|Tq - Tc_n\|) \end{aligned}$$

and

$$\begin{aligned} \|Rq - Rc_n\| &\leq \alpha'(\|Sq - Sc_n\|, \max\{\|Sq - Rq\|, \|Sc_n - Rq\|\}, \\ &\qquad \max\{\|Sc_n - Rc_n\|^m \cdot \|Sq - Rq\|^l, \|Sq - Rc_n\|\}) \\ &= \alpha'(\|Sq - Sc_n\|, \|Sc_n - Sq\|, \|Rq - Rc_n\|). \end{aligned}$$

From the above inequalities and condition (*), we obtain

$$\|Tq - Tc_n\| \leq k_1 \|Sq - Sc_n\| \tag{2.1}$$

and

$$\|Rq - Rc_n\| \leq k_2 \|Sq - Sc_n\| \tag{2.2}$$

for some $k_1, k_2 \in [0, 1)$. By the same method, from (1.6) and condition (*), we get

$$\|Tq - Td_n\| \leq k_3 \|Sq - Sd_n\|, \tag{2.3}$$

$$\|Rq - Rd_n\| \leq k_4 \|Sq - Sd_n\|, \tag{2.4}$$

$$\|Tq - Ty_n\| \leq k_5 \|Sq - Sy_n\|, \tag{2.5}$$

and

$$\|Rq - Ry_n\| \leq k_6 \|Sq - Sy_n\| \tag{2.6}$$

for some $k_3, k_4, k_5, k_6 \in [0,1)$. From (2.1)-(2.6), we get

$$\begin{aligned}\|Sy_{n+1}-p\| &= \|Sy_{n+1}+(1-a_n-b_n)Sy_n-(1-a_n-b_n)Sy_n+a_nTc_n-a_nTc_n\\ &\quad +b_nRc_n-b_nRc_n-(1-a_n-b_n+a_n+b_n)p\|\\ &\le (1-a_n-b_n)\|Sy_n-p\|+a_n\|Tc_n-p\|+b_n\|Rc_n-p\|+\varepsilon_n\\ &= (1-a_n-b_n)\|Sy_n-p\|+a_n\|Tq-Tc_n\|+b_n\|Rq-Rc_n\|+\varepsilon_n\\ &\le (1-a_n-b_n)\|Sy_n-p\|+a_nk_1\|Sq-Sc_n\|+b_nk_2\|Sq-Sc_n\|+\varepsilon_n, \qquad (2.7)\end{aligned}$$

$$\begin{aligned}\|Sc_n-Sq\| &= \|(1-a'_n-b'_n)Sy_n+a'_nTd_n+b'_nRd_n-Sq\|\\ &\le (1-a'_n-b'_n)\|Sy_n-Sq\|+a'_n\|Td_n-Sq\|+b'_n\|Rd_n-Sq\|\\ &= (1-a'_n-b'_n)\|Sy_n-Sq\|+a'_n\|Td_n-Tq\|+b'_n\|Rd_n-Rq\|\\ &\le (1-a'_n-b'_n)\|Sy_n-p\|+a'_nk_3\|Sd_n-Sq\|+b'_nk_4\|Sd_n-Sq\| \qquad (2.8)\end{aligned}$$

and

$$\begin{aligned}\|Sd_n-Sq\| &\le (1-a''_n-b''_n)\|Sy_n-Sq\|+a''_n\|Ty_n-Sq\|+b''_n\|Ry_n-Sq\|\\ &= (1-a''_n-b''_n)\|Sy_n-p\|+a''_n\|Ty_n-Tq\|+b''_n\|Ry_n-Rq\|\\ &\le (1-a''_n-b''_n)\|Sy_n-p\|+a''_nk_5\|Sy_n-p\|+b''_nk_6\|Sy_n-p\|. \qquad (2.9)\end{aligned}$$

Applying (2.8) and (2.9) to (2.7) and putting $k=\max_{1\le i\le 6} k_i$, we obtain

$$\begin{aligned}\|Sy_{n+1}-p\| &\le (1-a_n-b_n)\|Sy_n-p\|+(a_n+b_n)k\|Sq-Sc_n\|+\varepsilon_n\\ &\le \{(1-a_n-b_n)+(a_n+b_n)k(1-a'_n-b'_n)\}\|Sy_n-p\|\\ &\quad +(a_n+b_n)(a'_n+b'_n)k^2\|Sd_n-Sq\|+\varepsilon_n\\ &\le \{(1-a_n-b_n)+(a_n+b_n)k(1-a'_n-b'_n)\}\|Sy_n-p\|\\ &\quad +(a_n+b_n)(a'_n+b'_n)k^2\{(1-a''_n-b''_n)\|Sy_n-p\|\\ &\quad +(a''_n+b''_n)k\|Sy_n-p\|\}+\varepsilon_n\\ &\le \{(1-a_n-b_n)+a_nk+b_n-(a_n+b_n)k(a'_n+b'_n)\\ &\quad +(a_n+b_n)k(a'_n+b'_n)-(a_n+b_n)(a'_n+b'_n)k^2(a''_n+b''_n)\\ &\quad +(a_n+b_n)(a'_n+b'_n)k^2(a''_n+b''_n)\}\|Sy_n-p\|+\varepsilon_n\\ &= \{1-(1-k)a_n\}\|Sy_n-p\|+\varepsilon_n\\ &\le \{1-(1-k)w\}\|Sy_n-p\|+\varepsilon_n.\end{aligned}$$

From Lemma 1.3, we have $\lim_{n\to\infty}\|Sy_n-p\|=0$, i.e., $\lim_{n\to\infty}Sy_n=p$. □

Remark 2.1. By putting $\alpha(t_1, t_2, t_3) = kt_1$ for $t_1, t_2, t_3 \in \mathbb{R}_{\geq 0}$, $k \in [0, 1)$ and $a_n = 1$, $b_n = a'_n = b'_n = a''_n = b''_n = 0$ $(n \geq 0)$ in Theorem 2.1, we obtain Theorem 3.1 in [13] for the sequence $\{Sx_n\}$ defined as

$$Sx_{n+1} = Tx_n,$$

which is the Jungck iterative algorithm considered in [3].

By putting $b_n = b'_n = a''_n = b''_n = 0$ $(n \geq 0)$ in Theorem 2.1, we obtain the following theorem in [6] for the sequence $\{Sx_n\}$ defined as

$$\begin{cases} Sx_{n+1} = (1 - a_n)Sx_n + a_nTr_n, \\ Sr_n = (1 - a'_n)Sx_n + a'_nTx_n \text{ for } n \geq 0, \end{cases} \tag{2.10}$$

which is the Jungck-Ishikawa iterative algorithm considered in [7].

Theorem 2.2. *Let S and T be mappings from $K(\subset E)$ to a normed space E with $T(K) \subset S(K)$, S be an injective mapping and $C(S, T) \neq \emptyset$. Assume that (S, T) is a M_J-contraction with respect to α with condition (∗). For $x_0 \in K$, let $\{Sx_n\} \subset E$ be an iterative algorithm defined by (2.10) converging to $p(= Sq = Tq)$. Then, $\{Sx_n\}$ is (S, T)-stable.*

Remark 2.2. In [6], Theorem 2.2 is proved under the assumption that $0 < w \leq a_n$ and $0 < w' \leq a'_n$ for some $w, w' \in [0, 1]$. However, the assumption "$0 < w' \leq a'_n$ for some $w' \in [0, 1]$" is superfluous.

3. Strong Convergence Result

In this section, we prove a stong convergence of iterative algorithm (1.3).

Theorem 3.1. *Let S, T, R be mappings from $K(\subset E)$ to a normed space E with $T(K) \cup R(K) \subset S(K)$, S be injective and $C(S, T, R) \neq \emptyset$. Assume that (S, T) and (S, R) are M_J-contractions with respect to α and α' with condition (∗), respectively. For $x_0 \in K$, let $\{Sx_n\}$ be an iterative algorithm defined by (1.3), where $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$ are sequences in $[0, 1]$ such that $0 < w \leq a_n$ for some $w \in [0, 1]$. Then, $\{Sx_n\}$ converges strongly to a coincidence point of S, T and R.*

Proof. For $q \in C(S, T, R)$, since M_J-contractions (S, T) and (S, R) satisfy (1.6), we have

$$\begin{aligned}\|Tq - Tr_n\| &\leq \alpha(\|Sq - Sr_n\|, \max\{\|Sq - Tq\|, \|Sr_n - Tq\|\}, \\ &\quad \max\{\|Sr_n - Tr_n\|^m \cdot \|Sq - Tq\|^l, \|Sq - Tr_n\|\}) \\ &= \alpha(\|Sq - Sr_n\|, \|Sr_n - Sq\|, \|Tq - Tr_n\|)\end{aligned}$$

and

$$\begin{aligned}\|Rq - Rr_n\| &\leq \alpha'(\|Sq - Sr_n\|, \max\{\|Sq - Rq\|, \|Sr_n - Rq\|\}, \\ &\quad \max\{\|Sr_n - Rr_n\|^m \cdot \|Sq - Rq\|^l, \|Sq - Rr_n\|\}) \\ &= \alpha'(\|Sq - Sr_n\|, \|Sr_n - Sq\|, \|Rq - Rr_n\|).\end{aligned}$$

From the above inequalities and condition $(*)$, we obtain

$$\|Tq - Tr_n\| \leq k_1\|Sq - Sr_n\| \tag{3.1}$$

and

$$\|Rq - Rr_n\| \leq k_2\|Sq - Sr_n\| \tag{3.2}$$

for some $k_1, k_2 \in [0, 1)$. Above and the same way, from (1.6) and condition $(*)$, we get

$$\|Tq - Ts_n\| \leq k_3\|Sq - Ss_n\|, \tag{3.3}$$

$$\|Rq - Rs_n\| \leq k_4\|Sq - Ss_n\|, \tag{3.4}$$

$$\|Tq - Tx_n\| \leq k_5\|Sq - Sx_n\|, \tag{3.5}$$

and

$$\|Rq - Rx_n\| \leq k_6\|Sq - Sx_n\| \tag{3.6}$$

for some $k_3, k_4, k_5, k_6 \in [0, 1)$. If we put $p = Sq$ and apply (3.1)-(3.6) to (1.3), then we get

$$\begin{aligned}\|Sx_{n+1} - p\| &= \|(1 - a_n - b_n)Sx_n + a_nTr_n + b_nRr_n - (1 - a_n - b_n + a_n + b_n)p\| \\ &\leq (1 - a_n - b_n)\|Sx_n - p\| + a_n\|Tr_n - p\| + b_n\|Rr_n - p\| \\ &= (1 - a_n - b_n)\|Sx_n - p\| + a_n\|Tq - Tr_n\| + b_n\|Rq - Rr_n\| \\ &\leq (1 - a_n - b_n)\|Sx_n - p\| + a_nk_1\|Sq - Sr_n\| + b_nk_2\|Sq - Sr_n\| \\ &= (1 - a_n - b_n)\|Sx_n - p\| + (a_nk_1 + b_nk_2)\|Sq - Sr_n\|, \qquad (3.7)\end{aligned}$$

$$
\begin{aligned}
\|Sq - Sr_n\| &= \|(1-a'_n-b'_n+a'_n+b'_n)Sq-(1-a'_n-b'_n)Sx_n-a'_nTs_n-b'_nRs_n\| \\
&\leq (1-a'_n-b'_n)\|Sq-Sx_n\|+a'_n\|Sq-Ts_n\|+b'_n\|Sq-Rs_n\| \\
&= (1-a'_n-b'_n)\|Sx_n-p\|+a'_n\|Tq-Ts_n\|+b'_n\|Rq-Rs_n\| \\
&\leq (1-a'_n-b'_n)\|Sx_n-p\|+a'_nk_3\|Sq-Ss_n\|+b'_nk_4\|Sq-Ss_n\| \\
&= (1-a'_n-b'_n)\|Sx_n-p\|+(a'_nk_3+b'_nk_4)\|Sq-Ss_n\| \quad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\|Sq - Ss_n\| &= \|(1-a''_n-b''_n+a''_n+b''_n)Sq-(1-a''_n-b''_n)Sx_n-a''_nTx_n-b''_nRx_n\| \\
&\leq (1-a''_n-b''_n)\|Sq-Sx_n\|+a''_n\|Sq-Tx_n\|+b''_n\|Sq-Rx_n\| \\
&= (1-a''_n-b''_n)\|Sx_n-p\|+a''_n\|Tq-Tx_n\|+b''_n\|Rq-Rx_n\| \\
&\leq (1-a''_n-b''_n)\|Sx_n-p\|+a''_nk_5\|Sq-Sx_n\|+b''_nk_6\|Sq-Sx_n\| \\
&= (1-a''_n-b''_n)\|Sx_n-p\|+(a''_nk_5+b''_nk_6)\|Sq-Sx_n\|. \quad (3.9)
\end{aligned}
$$

Combining (3.7) with (3.8) and (3.9), we have

$$
\begin{aligned}
\|Sx_{n+1}-p\| &\leq (1-a_n-b_n)\|Sx_n-p\|+(a_n+b_n)k\|Sq-Sr_n\| \\
&\leq (1-a_n-b_n)\|Sx_n-p\|+(a_n+b_n)k\{(1-a'_n-b'_n)\|Sx_n-p\| \\
&\quad +(a'_n+b'_n)k\|Sq-Ss_n\|\} \\
&\leq (1-a_n-b_n)\|Sx_n-p\|+(1-a'_n-b'_n)(a_n+b_n)k\|Sx_n-p\| \\
&\quad +(a'_n+b'_n)(a_n+b_n)k^2(1-a''_n-b''_n+a''_nk+b''_nk)\|Sx_n-p\| \\
&\leq (1-a_n-b_n)\|Sx_n-p\|+(a_n+b_n)k\|Sx_n-p\| \\
&\leq \{1-(1-k)a_n\}\|Sx_n-p\| \\
&\leq \{1-(1-k)w\}\|Sx_n-p\| = \theta\|Sx_n-p\| \\
&\leq \theta^2\|Sx_{n-1}-p\| \leq \cdots \\
&\leq \theta^{n+1}\|Sx_0-p\| \to 0 \text{ as } n\to\infty,
\end{aligned}
$$

where $k = \max\{k_1, k_2, k_3, k_4, k_5, k_6\}$ and $0 < \theta = 1-(1-k)w < 1$. Hence, $\{Sx_n\}$ converges strongly to p (as $n \to \infty$). □

Remark 3.1. By putting $b_n = b'_n = a''_n = b''_n = 0$ $(n \geq 0)$ in Theorem 3.1, we obtain Theorem 3.1 in [6] for the sequence $\{Sx_n\}$ defined as (2.10).

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