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ON THE STABILITY OF THE FUNCTIONAL EQUATION $g(x+y+xy) = g(x) + f(y) + xf(y) + yg(x)$

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Abstract. In this note, we investigate the Hyers-Ulam stability and the hyperstability of the Pexider type functional equation

$$g(x+y+xy) = g(x) + f(y) + xf(y) + yg(x).$$

1. Introduction

The problem of stability of functional equations was originally raised by S.M. Ulam [10] in 1940: given a group V, a metric group W with metric $d(\cdot,\cdot)$, and a $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : V \to W$ satisfies $d(f(xy), f(x)f(y)) \le \delta$ for all $x,\ y \in V$, then a homomorphism $g : V \to W$ exists with $d(f(x), g(x)) \le \epsilon$ for all $x \in V$?

For Banach spaces the Ulam problem was first solved by D.H. Hyers [3] in 1941, which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for all $x,\ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \delta$$

for all $x,\ y \in X$.

Due to this fact, the additive functional equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability property on (X,Y). This terminology is also applied to other functional equations and the generalization of this property has been studied by many authors (see, for example, [1], [2], [4], [7]).

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During the 34th International Symposium on Functional Equations, Gy. Maksa [5] posed the problem concerning the Hyers-Ulam stability of the functional equation

$$f(xy) = xf(y) + yf(x) \tag{1.1}$$

on the interval $(0, 1]$, which is usually called a *derivation* (or *left derivation*).

J. Tabor [9] gave an answer to the question of Maksa by proving the Hyers-Ulam stability of the functional equation (1.1) on the interval $(0, 1]$. Also Zs. Páles [6] remarked that the functional equation (1.1) for real-valued functions has a hyperstability on the interval $[1, \infty)$.

We here introduce the Pexider type functional equation motivated by the functional equation (1.1):

$$g(x + y + xy) = g(x) + f(y) + xf(y) + yg(x). \tag{1.2}$$

In this note, we will solve the Pexider type functional equation (1.2) and then by referring the ideas of J. Tabor [9] and Zs. Páles [6], the Hyers-Ulam stability and the hyperstability of the equation (1.2) will be investigated, respectively.

2. Solutions of the Functional Equation (1.2)

In this section, we set the interval $I = (-1, \infty)$. It is easy to see that the pair of the real-valued functions $f(x) = (x + 1)\ln(x + 1)$ and $g(x) = f(x) + a(x + 1)$ for some nonzero $a \in \mathbb{R}$ is a solution of the functional equation (1.2) on I. In the following theorem, we will find out the general solution of the functional equation (1.2) on I.

Theorem 2.1. *Let X be a real (or complex) vector space. The pair of mappings $f, g : I \to X$ satisfies the functional equation (1.2) for all $x, y \in I$ if and only if there exists a solution $D : (0, \infty) \to X$ of the functional equation (1.1) such that*

$$f(x) = D(x + 1) \quad \text{and} \quad g(x) = D(x + 1) + (x + 1)a$$

hold for some $a \in X$ and for all $x \in I$

Proof. (*Necessity*) Let us define a mapping $D : (0, \infty) \to X$ by $D(x) = f(x - 1)$ for all $x \in (0, \infty)$. Let $x = 0$ in the equation (1.2). Then we have

$$g(y) = f(y) + (y + 1)g(0),$$

that is, $g(x) = f(x) + (x + 1)a$ for all $x \in I$, where $a = g(0)$. We claim that D is a solution of the functional equation (1.1).

Indeed, for all $x,\ y \in (0,\infty)$, we have

$$\begin{aligned} D(xy) &= f(xy-1) = g(xy-1) - (xy)a \\ &= g((x-1)+(y-1)+(x-1)(y-1)) - (xy)a \\ &= g(x-1)+f(y-1)+(x-1)f(y-1)+(y-1)g(x-1)-(xy)a \\ &= xf(y-1)+yg(x-1)-(xy)a \\ &= xf(y-1)+y\{g(x-1)-xa\} = xD(y)+yD(x). \end{aligned}$$

Therefore, D is a solution of the functional equation (1.1), as claimed, and

$$f(x) = D(x+1) \quad \text{and} \quad g(x) = D(x+1)+(x+1)a$$

are true for all $x \in I$.

(*Sufficiency*) For all $x, y \in I$, we get

$$\begin{aligned} g(x+y+xy) &= D(x+y+xy+1)+(x+y+xy+1)a \\ &= D((x+1)(y+1))+(x+1)(y+1)a \\ &= (x+1)D(y+1)+(y+1)D(x+1)+(x+1)(y+1)a \\ &= xD(y+1)+D(y+1)+(y+1)\{D(x+1)+(x+1)a\} \\ &= g(x)+f(y)+xf(y)+yg(x). \end{aligned}$$

□

3. Hyers-Ulam Stability of the Functional Equation (1.2)

In this section, we will denote $I = (-1, 0]$. We first need a theorem of F. Skof [8] concerning the stability of the additive functional equation $f(x+y) = f(x)+f(y)$ on a restricted domain:

Theorem 3.1. *Let X be a real (or complex) Banach space. Given $c > 0$, let a mapping $f : [0,c) \to X$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for some $\delta \ge 0$ and for all $x,\ y \in [0,c)$ with $x+y \in [0,c)$. Then there exists an additive mapping $A : \mathbb{R} \to X$ such that

$$\|f(x) - A(x)\| \le 3\delta$$

for all $x \in [0,c)$.

Theorem 3.2. *Let X be a real (or complex) Banach space, and let $g, f : I \to X$ be mappings satisfying the inequality*

$$\|g(x+y+xy) - g(x) - f(y) - xf(y) - yg(x)\| \le \delta \tag{3.1}$$

for some $\delta > 0$ and for all $x,\ y \in I$. Then there exist mappings $h, p : I \to X$ satisfying the functional equation (1.2) such that

$$\|g(x) - h(x)\| \le (12e+1)\delta \tag{3.2}$$

and

$$\|f(x) - p(x)\| \le (12e+2)\delta \tag{3.3}$$

for all $x \in I$.

Proof. Let us define the mappings $G, H : I \to X$ by

$$G(x) = \frac{g(x)}{x+1} \quad \text{and} \quad H(x) = \frac{f(x)}{x+1}$$

for all $x \in I$. Then, from (3.1), we see that G and H satisfy the inequality

$$\|G(x+y+xy) - G(x) - H(y)\| \le \frac{\delta}{(x+1)(y+1)}$$

for all $x, y \in I$. Define the mappings $U, V : [0, \infty) \to X$ by

$$U(u) = G(e^{-u} - 1) \quad \text{and} \quad V(u) = H(e^{-u} - 1)$$

for all $u \in [0, \infty)$. Then we obtain

$$\|U(u+v) - U(u) - V(v)\| \le \delta e^{u+v} \tag{3.4}$$

for all $u, v \in [0, \infty)$. Putting $v = 0$ in (3.4), we have

$$\|V(0)\| \le \delta e^{u} \tag{3.5}$$

for all $u \in [0, \infty)$. Analogously, if we set $u = 0$ in (3.4), then we get

$$\|U(v) - U(0) - V(v)\| \le \delta e^{v} \tag{3.6}$$

for all $v \in [0, \infty)$. We now define a mapping $F : [0, \infty) \to X$ by

$$F(u) = U(u) - U(0) - V(0) \tag{3.7}$$

for all $u \in [0, \infty)$. We claim that the inequality

$$\|F(u+v) - F(u) - F(v)\| \le 3\delta e^{u+v} \tag{3.8}$$

holds for all $u, v \in [0, \infty)$. In fact, it follows from (3.4), (3.5), (3.6) and (3.7) that

$$\begin{aligned}
&\|F(u+v) - F(u) - F(v)\| \\
&= \|U(u+v) - U(u) - U(v) + U(0) + V(0)\| \\
&\le \|U(u+v) - U(u) - V(v)\| + \|V(0)\| + \|V(v) - U(v) + U(0)\| \\
&\le \delta e^{u+v} + \delta e^{u} + \delta e^{v} \\
&\le 3\delta e^{u+v}.
\end{aligned}$$

for all $u,\ v \in [0, \infty)$. The inequality (3.8) means that the inequality

$$\|F(u+v) - F(u) - F(v)\| \le 3\delta e^{c}$$

is valid for all $u,\ v \in [0, c)$ with $u+v < c$, where $c > 1$ is an arbitrary given constant.

According to Theorem 3.1, there exists an additive mapping $A : \mathbb{R} \to X$ such that

$$\|F(u) - A(u)\| \le 9\delta e^{c}$$

for all $u \in [0, c)$. If we let $c \to 1$ in the last inequality, we then get

$$||F(u) - A(u)|| \le (9e)\delta \tag{3.9}$$

for all $u \in [0, 1]$. Moreover, it follows from (3.8) that

$$\begin{aligned}
\|F(u+1) - F(u) - F(1)\| &\le 3\delta e^{u+1} \\
\|F(u+2) - F(u+1) - F(1)\| &\le 3\delta e^{u+2} \\
&\vdots \\
\|F(u+k) - F(u+k-1) - F(1)\| &\le 3\delta e^{u+k}
\end{aligned}$$

for all $u \in [0, 1]$ and $k \in \mathbb{N}$. Summing up these inequalities, we obtain

$$||F(u+k) - F(u) - kF(1)|| \le (3e)\delta \cdot e^{u+k} \tag{3.10}$$

for all $u \in [0, 1]$ and $k \in \mathbb{N}$. We assert that

$$||F(v) - A(v)|| \le (12e)\delta \cdot e^{v} \tag{3.11}$$

for all $v \in [0, \infty)$. For, let $v \ge 0$ and let $k \in \{0\} \cup \mathbb{N}$ be given with $v - k \in [0, 1]$. Then, by (3.9) and (3.10), we have

$$\begin{aligned}\|F(v)-A(v)\| &\le ||F(v)-F(v-k)-kF(1)\| \\ &\quad+\|F(v-k)-A(v-k)\|+\|A(k)-kF(1)\| \\ &\le (3e)\delta\cdot e^{v}+(9e)\delta+\|A(k)-kF(1)\| \\ &\le (3e)\delta\cdot e^{v}+(9e)\delta+k\|A(1)-F(1)\| \\ &\le (3e)\delta\cdot e^{v}+(9e)\delta+(9e)\delta v \\ &\le (3e)\delta(e^{v}+3(1+v)) \\ &\le (12e)\delta\cdot e^{v}.\end{aligned}$$

Now, from (3.11) and the definitions of F, G, H and V, it follows that

$$\|G(x)-G(0)-H(0)-A(-\ln(x+1)\| \le (12e)\delta\cdot e^{-\ln(x+1)} = \frac{(12e)\delta}{x+1}$$

for all $x \in I$. Using (3.5), we have

$$\|H(0) = \|V(0)\| \le \delta e^{-\ln(x+1)} = \frac{\delta}{x+1}$$

for all $x \in I$. Therefore, we obtain

$$\begin{aligned}&\|G(x)-A(-\ln(x+1))-G(0)\| \\ &\le \|G(x)-G(0)-H(0)-A(-\ln(x+1)\|+\|H(0)\| \\ &\le \frac{(12e+1)\delta}{x+1},\end{aligned}$$

i.e.,

$$\left\|\frac{g(x)}{x+1}-g(0)-A(-\ln(x+1))\right\| \le \frac{(12e+1)\delta}{x+1} \tag{3.12}$$

for all $x \in I$.

Let $D:(0,1]\to X$ be a mapping defined by $D(x)=xA(-\ln x)$ for all $x\in(0,1]$. Then D satisfies the functional equation (1.1). If we set $p(x)=D(x+1)$ and $h(x)=p(x)+(x+1)g(0)$ for all $x\in I$, then h and p satisfy the functional equation (1.2), i.e.,

$$h(x+y+xy)=h(x)+p(y)+xp(y)+yh(x)$$

and we obtain the inequality (3.2) by (3.12). Using (3.6), (3.7) and (3.11), we have

$$\begin{aligned}\|V(v)-A(v)-V(0)\| &= \|V(v)-A(v)+F(v)-U(v)+U(0)\| \\ &\le \|F(v)-A(v)\|+\|U(v)-U(0)-V(v)\| \\ &\le (12e)\delta e^{v}+\delta e^{v} = (12e+1)\delta e^{v}\end{aligned}$$

for all $v \in [0, \infty)$ and so employing the inequality (3.5), we get

$$\begin{aligned} \|V(v) - A(v)\| &\le \|V(v) - A(v) - V(0)\| + \|V(0)\| \\ &\le (12e + 1)\delta e^v + \delta e^v = (12e + 2)\delta e^v \end{aligned} \tag{3.13}$$

for all $v \in [0, \infty)$. From (3.13) and the definitions of V and H, it follows that

$$\left\| \frac{f(x)}{x+1} - A(-\ln(x+1)) \right\| \le (12e + 2)\delta e^{-\ln(x+1)} = \frac{(12e+2)\delta}{x+1}$$

for all $x \in I$. Hence we arrive at the inequality (3.3). This completes the proof. □

4. Hyperstability of the Functional Equation (1.2)

In this section, we will investigate the hyperstability of the functional equation (1.2) on the interval I, where $I = [0, \infty)$.

Theorem 4.1. *Let X be a real (or complex) Banach space. If there exist mappings $g, f : I \to X$ satisfying the inequality*

$$\|g(x + y + xy) - g(x) - f(y) - xf(y) - yg(x)\| \le \delta \tag{4.1}$$

for some $\delta > 0$ and for all $x, y \in I$ under the condition $g(0) = f(0)$, then $g = f$ on I and $g(= f)$ satisfies the functional equation (1.2) for all $x, y \in I$.

Proof. Defining the mappings $G, H : I \to X$ by

$$G(x) = \frac{g(x)}{x+1} \quad \text{and} \quad H(x) = \frac{f(x)}{x+1}$$

for all $x \in I$ and defining the mappings $U, V : I \to X$ by

$$U(u) = G(e^u - 1) \quad \text{and} \quad V(u) = H(e^u - 1)$$

for all $u \in I$, it follows from (4.1) that

$$\|U(u + v) - U(u) - V(v)\| \le \delta e^{-(u+v)} \tag{4.2}$$

for all $u, v \in I$. We will show that $U(u + v) = U(u) + V(v)$ for all $u, v \in I$.

Let $\varepsilon > 0$ be given. Then the inequality (4.2) implies that there exists a $c > 0$ satisfying

$$\|U(u + v) - U(u) - V(v)\| < \frac{\varepsilon}{36} \tag{4.3}$$

for all $u, v \in I$ with $u + v > c$.

Suppose that positive real numbers u and v are given. Let m and n be integers great than 1. Then it is easy to see that

$$\begin{aligned}(4.4)\qquad V(nu) - V((n-1)u) - V(u) &= U(mv+u) - U(mv) - V(u)\\ &+ U((mv+u)+(n-1)u) - U(mv+u) - V((n-1)u))\\ &- (U(mv+nu) - U(mv) - V(nu))\end{aligned}$$

If m is so large that $m > (c+u)/v$, the last equality implies

$$\|V(nu) - V((n-1)u) - V(u)\| < \frac{\varepsilon}{12} \tag{4.5}$$

for all $u \in I$ and all integers $n > 1$. The relation

$$V(nu) - nV(u) = \sum_{k=2}^{n} (V(ku) - V((k-1)u) - V(u))$$

together with (4.5), produces the inequality

$$\|V(nu) - nV(u)\| < \frac{n-1}{12}\varepsilon \tag{4.6}$$

for all $u \in I$ and all integers $n > 1$. Let $u = 0$ in (4.3). Then we obtain the inequality

$$\|U(v) - U(0) - V(v)\| < \frac{\varepsilon}{36}$$

for all $v \in I$ with $v > c$ which gives the inequality

$$\|U(v) - V(v)\| \le \|U(v) - V(v) - U(0)\| + \|U(0)\| < \frac{\varepsilon}{36} + \|U(0)\| \tag{4.7}$$

for all $v \in I$ with $v > c$.

On the other hand, let $u \in [0, \infty)$ with $u > c$. Then the inequality (4.3) with such an u and $v = 0$ yields

$$\|V(0)\| < \frac{\varepsilon}{36}. \tag{4.8}$$

Note that we have $\|U(0)\| = \|V(0)\|$ since $g(0) = f(0)$. Applying (4.3), (4.7) and (4.8) with this equality, we see that the inequality

$$\begin{aligned}\|U(u+v) - U(u) - U(v)\| &\le \|U(u+v) - U(u) - V(v)\| + \|V(v) - U(v)\|\\ &< \frac{\varepsilon}{36} + \frac{\varepsilon}{36} + \|U(0)\| < \frac{\varepsilon}{12}\end{aligned}$$

holds for all $u, v \in I$ with $v > c$. Replacing V by U in (4.4) and then following the same process to obtain the inequality (4.6), we have

$$\|U(nu) - nU(u)\| < \frac{n-1}{4}\varepsilon \tag{4.9}$$

for all $u \in I$ and all integers $n > 1$. Obviously, it follows from (4.6) and (4.9) that

$$\begin{aligned}
&\|(U(nu+nv) - U(nu) - V(nv)) - n(U(u+v) - U(u) - V(v))\| \\
&\quad \leq \|U(nu+nv) - nU(u+v)\| + \|U(nu) - nU(u)\| + \|V(nv) - nV(v)\| \\
&\quad < \frac{n-1}{4}\varepsilon + \frac{n-1}{4}\varepsilon + \frac{n-1}{12}\varepsilon = \frac{7}{12}(n-1)\varepsilon
\end{aligned}$$

for all $u, v \in [0, \infty)$ with $(u, v) \neq (0,\ 0)$. Dividing by n both sides of the last inequality and then letting $n \to \infty$ and considering the fact that $(1/n)(U(nu + nv) - U(nu) - V(nv)) \to 0$ as $n \to \infty$ on account of (4.3), we get

$$\|U(u+v) - U(u) - V(v)\| < \frac{7}{12}\varepsilon$$

for all $u, v \in I$ with $(u, v) \neq (0, 0)$. Also, we see that

$$\|U(0+0) - U(0) - V(0)\| = \|V(0)\| < \frac{\varepsilon}{36}$$

by using (4.8). Therefore, the inequality (4.3) holds for all $u, v \in I$. Since $\varepsilon > 0$ was arbitrary, we conclude that the relation $U(u+v) = U(u) + V(v)$ is established for all $u, v \in I$.

Now, according to the definitions of U and G, we get

$$\frac{g(x)}{x+1} = G(x) = U(\ln(x+1))$$

for all $x \in I$, i.e.,

$$g(x) = (x+1)U(\ln(x+1))$$

for all $x \in I$. From the definitions of V and H, we have

$$\frac{f(x)}{x+1} = H(x) = V(\ln(x+1))$$

for all $x \in I$, i.e.,

$$f(x) = (x+1)V(\ln(x+1))$$

for all $x \in I$. Since $U(u+v) = U(u) + V(v)$ for all $u, v \in I$, we see that g and f satisfy the functional equation (1.2) for all $x, y \in I$. In fact, $g = f$ on I.

Letting $x = 0$ in (1.2), we obtain

$$g(y) = f(y) + (y+1)g(0) \tag{4.10}$$

for all $y \in I$. Since $g(0) = f(0)$, it follows from (4.10) that $g(0) = 0$. Hence the equality (4.10) gives $g(x) = f(x)$ for all $x \in I$. The proof of the theorem is completed. □

REFERENCES

1. T. Aoki: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan **2** (1950), 64-66.
2. P. Găvruţă: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184** (1994), 431-436.
3. D.H. Hyers: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. **27** (1941), 222-224.
4. Y.-S. Jung: On the stability of higher ring left derivations. Indian J. Pure Appl. Math. **47** (2016), no. 3, 523-533.
5. Gy. Maksa: Problems 18, In 'Report on the 34th ISFE'. Aequationes Math. **53** (1997), 194.
6. Zs. Páles: Remark 27, In 'Report on the 34th ISFE'. Aequationes Math. **53** (1997), 200-201
7. Th.M. Rassias: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc. **72** (1978), 297-300.
8. F. Skof: Sull'approssimazione delle appliazioni localmente δ-additive. Atti Accad. Sc. Torino. **117** (1983), 377-389
9. J. Tabor: Remarks 20, In 'Report on the 34th ISFE'. Aequationes Math. **53** (1997), 194-196
10. S.M. Ulam: A Collection of Mathematical Problems. Interscience Publ., New York, 1960.

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