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QUADRATIC (ρ_1, ρ_2)-FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

Siriluk Paokanta [a] and Dong Yun Shin [b,*]

Abstract. In this paper, we consider the following quadratic (ρ_1, ρ_2)-functional equation

(0.1)

$$N(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho_1(f(x+y) + f(x-y) \\ - 2f(x) - 2f(y)) - \rho_2(4f\left(\frac{x+y}{2}\right) + f(x-y) - f(x) - f(y)), t) \geq \frac{t}{t + \varphi(x,y)},$$

where ρ_1, ρ_2 are fixed nonzero real numbers with $\rho_2 \neq 1$ and $2\rho_1 + 2\rho_2 \neq 1$, in fuzzy normed spaces.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional equation (0.1) in fuzzy Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 13]).

We recall a fundamental result in fixed point theory.

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*Corresponding author.

Theorem 1.1 ([4, 9]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 22]).

Katsaras [15] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [10, 17, 25]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [16]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 19, 20] to investigate the Hyers-Ulam stability of functional equations in fuzzy Banach spaces.

Definition 1.2 ([2, 19, 20, 21]). Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$.

(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [19, 20, 21].

Definition 1.3 ([2, 19, 20, 21])**.** Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.4 ([2, 19, 20, 21])**.** Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

In this paper, we prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional equation (0.1) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that ρ_1, ρ_2 are fixed nonzero real numbers with $\rho_2 \neq 1$ and $2\rho_1 + 2\rho_2 \neq 1$.

2. Quadratic (ρ_1, ρ_2)-functional Equation (0.1)

In this section, we investigate the additive (ρ_1, ρ_2)-functional equation (0.1) in fuzzy Banach spaces.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) &= \rho_1(f(x+y) + f(x-y) \\ -2f(x) - 2f(y)) &+ \rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - f(x) - f(y)\right) \end{aligned} \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Letting $y = x$ in (2.1), we get $-\rho_1(f(2x) - 4f(x)) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned} &\frac{1}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \\ &\quad = (\rho_1 + \rho_2)(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so $f(x+y)+f(x-y)-2f(x)-2f(y)=0$ for all $x,y \in X$, since $2\rho_1+2\rho_2 \neq 1$. Thus $f: X \to Y$ is quadratic. □

We prove the Hyers-Ulam stability of the quadratic (ρ_1,ρ_2)-functional equation (0.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y)$$

for all $x,y \in X$. Let $f: X \to Y$ be a mapping satisfying $f(0)=0$ and

(2.2)

$$N(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)-\rho_1(f(x+y)+f(x-y)$$
$$-2f(x)-2f(y))-\rho_2\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-f(x)-f(y)\right),t) \ge \frac{t}{t+\varphi(x,y)}$$

for all $x,y \in X$ and all $t>0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q: X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \ge \frac{(1-\rho_2)(1-L)t}{(1-\rho_2)(1-L)t+\varphi(x,0)} \tag{2.3}$$

for all $x \in X$ and all $t>0$.

Proof. Letting $y=0$ in (2.2), we get

$$N((1-\rho_2)\left(4f\left(\frac{x}{2}\right)-f(x)\right),t) \ge \frac{t}{t+\varphi(x,0)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y, \quad g(0)=0\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \ge \frac{t}{t+\varphi(x,0)},\ \forall x \in X, \forall t>0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [18, Lemma 2.1]).

Now we consider the linear mapping $J: S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(Jg(x) - Jh(x), L\varepsilon t) = N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right)$$

$$\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.4) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{t}{1 - \rho_2}\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence $d(f, Jf) \leq \frac{1}{1-\rho_2}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.5}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{(1 - \rho_2)(1 - L)}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$N(4^n \left(2f\left(\frac{x+y}{2^n}\right) + 2f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)$$
$$- 4^n\rho_1 \left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)$$
$$- 4^n\rho_2 \left(4f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), 4^n t) \geq \frac{t}{t + \varphi(\frac{x}{2^n}, \frac{y}{2^n})}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N(4^n \left(2f\left(\frac{x+y}{2^n}\right) + 2f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)$$
$$- 4^n\rho_1 \left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)$$
$$- 4^n\rho_2 \left(4f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), t) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}$$

Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) = \rho_1(Q(x+y) + Q(x-y)$$
$$-2Q(x) - 2Q(y)) + \rho_2\left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - Q(x) - Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho_1(f(x+y) + f(x-y) - 2f(x) \tag{2.6}$$
$$-2f(y)) - \rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - f(x) - f(y)\right), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(1-\rho_2)(2^p - 4)t}{(1-\rho_2)(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2) *and $f(0) = 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(1-\rho_2)(1-L)t}{(1-\rho_2)(1-L)t + L\varphi(x, 0)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4(1-\rho_2)}t\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{4}f(2x), t\right) \geq \frac{4(1-\rho_2)t}{4(1-\rho_2)t + 4L\varphi(x, 0)} = \frac{(1-\rho_2)t}{(1-\rho_2)t + L\varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$. Then $d(f, Jf) \leq \frac{L}{1-\rho_2}$. Hence

$$d(f, Q) \leq \frac{L}{(1-\rho_2)(1-L)},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying* (2.6) *and $f(0) = 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(1-\rho_2)(4-2^p)t}{(1-\rho_2)(4-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

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COMPETING INTERESTS

The authors declare that they have no competing interests.

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[a]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
Email address: siriluk22@hanyang.ac.kr

[b]Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea
Email address: dyshin@uos.ac.kr