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SOLVABILITY OF SOME NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER VIA MEASURE OF NONCOMPACTNESS

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Abstract. In this article, we investigate the solvability of nonlinear fractional integro-differential equations of the Hammerstein type. The results are obtained using the technique of measure of noncompactness and the Darbo theorem in the real Banach space of continuous and bounded functions in the interval $[0, a]$. At the end, an example is presented to illustrate the effectiveness of the obtained results.

1. Introduction

The memory effecet and non-local properties of the non-integer order derivative with the appropriate analytical supporting, lead to the fractional differential equation convert to excellent instrument for modelling the physical and real phenomena. In fact, in many applications in engineering systems, fractional derivative-based ideas describe better adaptive models than the ideas based on conventional derivatives. Most recently, fractional calculations has been considered in the fields of applied sciences such as control theory, dynamics, viscoelasticity, electromagnetic theory and so on (for example, see [11, 14, 16] and the references therein).
Analytical investigations and numerical solutions of fractional differential equations have always been of interest to researchers [1–6, 9, 10, 12, 15, 18, 19, 21, 23–26]. In recent years, many researchers have investigated the existence of a unique solution of fractional integral and differential equations using the concept of a measure of noncompactness on bounded and unbounded intervals. For example, in [8], Banas and Zajac investigated solvability of a fractional integral equation in the class of functions with limits at infinity. By utilizing the theory of fixed point and a technique of

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the measure of noncompactness, Gou and Li [13] investigated the existence of local and global mild solution for an impulsive fractional functional integro-differential equation with noncompact semi-group in the Banach space.
In this work, we study the existence and uniqueness of solution of the following nonlinear fractional Volterra integro-differential equations of the Hammerstein type in the Banach space $BC([0, a])$:
(1.1)

$$^{C}D^{\beta}\left(\nu(\tau)+e(\tau,\nu(\tau))\right)=g(\tau,\nu(\tau))+f\left(\tau,\nu(\tau),\int_0^{\tau}k(\tau,s)\ H(\nu(s))ds\right),\ \tau\in[0,a],$$

with the initial conditions

$$\nu^{(j)}(0)=\nu_j,\qquad j=0,1,...,m-1, \tag{1.2}$$

where $0 < a < \infty$, $m \in \mathbb{N}$ and $^{c}D^{\beta}$ is the Caputos fractional derivative. The functions $e, g : [0, a] \times \mathbb{R} \to \mathbb{R}$, $f : [0, a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k : [0, a] \times [0, a] \to \mathbb{R}$ are appropriate continuous functions satisfying given conditions to be defined later.
To achieve the main purpose of this work, using the technique of measure of noncompactness and the Darbo theorem, we show that the problem (1.1) - (1.2) has at least one solution in the real Banach space $BC([0, a])$ which are continuous and bounded functions.

2. PRELIMINARIES

In this section, we give some basic preliminaries which are used further in this paper. First, we present some introductory concepts for fractional calculus (for more details see [17, 20, 22]).

Definition 2.1. The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $\nu(\tau)$, is defined as

$$(I^{\beta}\nu)(\tau)=\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\mu)^{\beta-1}\,\nu(\mu)d\mu,\ \tau>0, \tag{2.1}$$

where Γ is the Gamma function.

In this paper, we consider the definition of the Caputos derivative which is more useful in real-life usages since it can be better able to model phenomena and be consistent with the initial conditions of the problems.

Definition 2.2. The Caputo derivative of fractional order $\beta \geq 0$ for a function $\nu(\tau)$ is defined by

$$\left({}^{C}D^{\beta}\nu\right)(\tau) = \frac{1}{\Gamma(m-\beta)} \int_{0}^{\tau} (\tau-\mu)^{m-\beta-1}\, \nu^{(m)}(\mu) d\mu,$$

where $m = [\beta] + 1$ and $[\beta]$ denotes integer part of the real number β.

If $\beta = m \in \mathbb{N}_0$ and the usual derivative $\nu^{(m)}(\tau)$ of order m exists, then $({}^{C}D^{m}\nu)(\tau)$ coincides with $\nu^{(m)}(\tau)$. Also, this definition implies that ${}^{C}D^{\beta}\nu^{(n)}(\tau) = {}^{C}D^{\beta+n}\nu(\tau)$ and ${}^{C}D^{\beta}c = 0$ (c is a constant).

Propositon 2.3. *Let $\beta > 0$ and $m = [\beta] + 1$. If $\nu(\tau) \in C^{m}[0, a]$, then*

(i) $\left(I^{\beta}\, {}^{C}D^{\beta}\nu\right)(\tau) = \nu(\tau) - \sum_{j=0}^{m-1} \dfrac{\nu^{(j)}(0)}{j!}\tau^{j}$,

(ii) $\left({}^{C}D^{\beta}\, I^{\beta}\nu\right)(\tau) = \nu(\tau)$.

Throughout this paper, we consider the Banach space $BC([0, a])$ with the norm $\|\,.\,\|$ which is defined as follows:

$$\|\nu\| = \sup\left\{|\nu(\tau)| : \ \tau \in [0, a]\right\}.$$

Next we recall some basic facts concerning measures of noncompactness.
Suppose that S is an infinite dimensional Banach space with a norm $\|\,.\,\|$. If M is a subset of S then $\overline{M}$ and $ConvM$ stand for the closure and convex closure of M, respectively. The family of all nonempty and bounded subsets of S will be shown by $\mathfrak{M}_S$ and its subfamily consisting of all compact sets is denoted by $\mathfrak{N}_S$.

Now we give the concept of a measure of noncompactness [7]:

Definition 2.4. A mapping $\eta : \mathfrak{M}_S \longrightarrow \mathbb{R}_+$ is called the *measure of noncompactness* in S if it has the following conditions:

1) The family $ker\eta = \{M \in \mathfrak{M}_S : \ \eta(M) = 0\}$ is nonempty and $ker\eta \subset \mathfrak{N}_S$, the kernel of the measure of noncompactnees η is called by $ker\eta$,
2) $M \subset N \Rightarrow \eta(M) \leq \eta(N)$,
3) $\eta(\overline{M}) = \eta(M)$,
4) $\eta(ConvM) = \eta(M)$,
5) $\eta(\lambda M + (1-\lambda)N) \leq \lambda\eta(M) + (1-\lambda)\eta(N)$ for $\lambda \in [0, 1]$,
6) If (M_n) is a sequence of closed sets from $\mathfrak{M}_S$ such that $M_{n+1} \subset M_n$ $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \eta(M_n) = 0$, then the intersection $M_\infty = \bigcap_{n=1}^{\infty} M_n$ is nonempty.

Theorem 2.5 ([7]). *Suppose Ψ be a nonempty, bounded, closed and convex subset of the space S and let $\Phi : \Psi \to \Psi$ be a continuous mapping. Suppose that there exists a constant $\lambda \in [0,1)$ such that $\eta(\Phi M) \le \lambda\eta(M)$ for any nonempty subset M of Ψ. Then Φ has a fixed point in the set Ψ.*

Now we introduce the definition of a special measure of noncompactness in $BC([0,a])$ which will be used in this paper. Suppose $M \in \mathfrak{M}_{BC([0,a])}$, $\epsilon > 0$ and $\nu \in M$. So the modulus of continuity of the function ν is defined by

$$\varpi(\nu,\epsilon) = \sup\left\{|\nu(\tau_1) - \nu(\tau_2)| : \ \tau_1, \tau_2 \in [0,a], \ |\tau_1 - \tau_2| \le \epsilon\right\}.$$

Also, let us put

$$\varpi(M,\epsilon) = \sup\left\{\varpi(\nu,\epsilon) : \ \nu \in M\right\},$$
$$\varpi_0(M) = \lim_{\epsilon\to 0} \varpi(M,\epsilon).$$

It may be shown [7] that

$$\eta(M) = \frac{1}{2}\varpi_0(M),$$

which $\eta(M)$ is a regular measure of noncompactness in the space $BC([0,a])$.

3. Main Result

In this section, we investigate the solvability of the fractional integro-differential equation (1.1). Since g and f are continuous, we can exert the operator I^β to both sides of Eq. (1.1). Thus using Proposition 2.3, we obtain

$$\nu(\tau) = \sum_{j=0}^{m-1} \frac{\nu_j + e^{(j)}(0,\nu_0)}{j!}\tau^j - e(\tau,\nu(\tau)) + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{g(s,\nu(s))}{(\tau-s)^{1-\beta}}ds + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{f(s,\nu(s),(K\nu)(s))}{(\tau-s)^{1-\beta}}ds, \tag{3.1}$$

where $(K\nu)(\tau) := \int_0^\tau k(\tau,s)H(\nu(s))ds$. The Eq. (1.1) is equivalent to the above fractional integral equation. That is, every solution of (3.1) is also a solution of (1.1) and vice versa. In what follows, we consider Eq. (3.1) under the following conditions:

(H_1) $e, g : [0,a]\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous and there exist the continuous functions $a_1, a_2 : [0,a] \longrightarrow [0,a]$ such that

$$|e(\tau,\nu) - e(\tau,\upsilon)| \le a_1(\tau)|\nu - \upsilon|, \qquad B = \sup\left\{|e(\tau,0)| : \ \tau \in [0,a]\right\},$$
$$|g(\tau,\nu) - g(\tau,\upsilon)| \le a_2(\tau)|\nu - \upsilon|, \qquad C = \sup\left\{|g(\tau,0)| : \ \tau \in [0,a]\right\}.$$

(H_2) $f:[0,a]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and there exist the continuous function $a_3:[0,a]\longrightarrow[0,a]$ such that

$$|f(\tau,\nu_1,\upsilon_1)-f(\tau,\nu_2,\upsilon_2)|\leq a_3(\tau)\left(|\nu_1-\nu_2|+|\upsilon_1-\upsilon_2|\right),$$
$$D=\sup\left\{|f(\tau,0,0)|:\ \tau\in[0,a]\right\},$$

for all $\nu_j,\upsilon_j\in\mathbb{R},\ j=1,2,$ let $A=\max_i\left\{|a_i(\tau)|:\ \tau\in[0,a]\right\}$, in which $i=1,2,3$ and $0\leq A<1$. Also, let there exists a nonnegetive constant W such that $B,C,D\leq W$.

(H_3) The function $H:BC([0,a])\to BC([0,a])$ satisfies the Lipschitz condition, i.e. there exists a constant $N>0$, such that for any $\tau\in[0,a]$ and for all $\nu,\upsilon\in BC([0,a])$ the following relationship holds

$$|(H\nu)(\tau)-(H\upsilon)(\tau)|\leq N|\nu(\tau)-\upsilon(\tau)|. \tag{3.2}$$

(H_4) The operator H transforms the space $BC([0,a])$ continuously into itself, and there exists a nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\|H\nu\|\leq\psi(\|\nu\|)$ for any $\nu\in BC([0,a])$. Moreover, for every function $\nu\in BC([0,a])$ which is nonnegative on $[0,a]$, the function $H\nu$ is nonnegative on $[0,a]$.

(H_5) If

$$\left|\sum_{j=0}^{m-1}\frac{\nu_j+e^{(j)}(0,\nu_0)}{j!}a^j\right|\leq L,$$

then there exists a positive solution $r_0>0$ of the inequality

$$L+W+Ar+\frac{a^\beta}{\Gamma(\beta+1)}\left[2Ar+2W+Aa\|k\|\psi(r)\right]\leq r. \tag{3.3}$$

Now let's consider the operators E,F,G defined on the Banach space $BC([0,a])$ as follows:

$(E\nu)(\tau)=e\left(\tau,\nu(\tau)\right)$,

$(F\nu)(\tau)=\frac{1}{\Gamma(\beta)}\int_0^\tau\dfrac{f(s,\nu(s),(K\nu)(s))}{(\tau-s)^{1-\beta}}ds$,

$(G\nu)(\tau)=\frac{1}{\Gamma(\beta)}\int_0^\tau\dfrac{g(s,\nu(s))}{(\tau-s)^{1-\beta}}ds$.

Then we have the following Theorem.

Theorem 3.1. *Suppose that assumptions (H_1) - (H_5) hold. Then, problem (1.1) - (1.2) has at least one solution in the Banach space $BC([0,a])$.*

Proof. Let's consider the operator Υ on the space $BC([0,a])$ using (3.1) as follows:

$$(\Upsilon\nu)(\tau)=\sum_{j=0}^{m-1}\frac{\nu_j+e^{(j)}(0,\nu_0)}{j!}\tau^j-(E\nu)(\tau)+(G\nu)(\tau)+(F\nu)(\tau).$$

Observe that by virtue of our assumptions, for any function $\nu\in BC([0,a])$ the function $(E\nu)$ is continuous on $[0,a]$. We demonstrate that the functions $F\nu$ and $G\nu$ are continuous on $[0,a]$. For this purpose, consider an arbitrary function $\nu\in BC([0,a])$ and $\epsilon>0$ and suppose that $\tau_1,\tau_2\in[0,a]$ such that $|\tau_2-\tau_1|\le\epsilon$. Without diminishing the whole issue, we can take $\tau_1<\tau_2$. Then, taking into account our assumptions we get

$$\begin{aligned}
&|(G\nu)(\tau_2)-(G\nu)(\tau_1)|\\
&=\frac{1}{\Gamma(\beta)}\left|\int_0^{\tau_1}\frac{g(s,\nu(s))}{(\tau_2-s)^{1-\beta}}ds+\int_{\tau_1}^{\tau_2}\frac{g(s,\nu(s))}{(\tau_2-s)^{1-\beta}}ds-\int_0^{\tau_1}\frac{g(s,\nu(s))}{(\tau_1-s)^{1-\beta}}ds\right|\\
&\le\frac{1}{\Gamma(\beta)}\int_0^{\tau_1}|g(s,\nu(s))|\left|\frac{1}{(\tau_2-s)^{1-\beta}}-\frac{1}{(\tau_1-s)^{1-\beta}}\right|ds\\
&\quad+\frac{1}{\Gamma(\beta)}\int_{\tau_1}^{\tau_2}\frac{|g(s,\nu(s))|}{(\tau_2-s)^{1-\beta}}ds\le\frac{1}{\Gamma(\beta)}\int_0^{\tau_1}[|g(s,\nu(s))-g(s,0)|\\
&\quad+|g(s,0)|]\left[\frac{1}{(\tau_1-s)^{1-\beta}}-\frac{1}{(\tau_2-s)^{1-\beta}}\right]ds\\
&\quad+\frac{1}{\Gamma(\beta)}\int_{\tau_1}^{\tau_2}[|g(s,\nu(s))-g(s,0)|+|g(s,0)|]\frac{ds}{(\tau_2-s)^{1-\beta}}\\
&\le\frac{A\|\nu\|+C}{\Gamma(\beta)}.\ \frac{\tau_1^\beta-\tau_2^\beta+(\tau_2-\tau_1)^\beta}{\beta}+\frac{A\|\nu\|+C}{\Gamma(\beta)}.\ \frac{(\tau_2-\tau_1)^\beta}{\beta}\\
&\le\frac{2\epsilon^\beta}{\Gamma(\beta+1)}\left(A\|\nu\|+W\right).
\end{aligned}\tag{3.4}$$

According to the above, we conclude that function $G\nu$ is continuous on interval $[0,a]$. The same way, we have

$$\begin{aligned}
|(F\nu)(\tau_2)-(F\nu)(\tau_1)|\le&\frac{1}{\Gamma(\beta)}\int_0^{\tau_1}[|f(s,\nu(s),(K\nu)(s))-f(s,0,0)|\\
&+|f(s,0,0)|]\left|\frac{1}{(\tau_2-s)^{1-\beta}}-\frac{1}{(\tau_1-s)^{1-\beta}}\right|ds\\
&+\frac{1}{\Gamma(\beta)}\int_{\tau_1}^{\tau_2}[|f(s,\nu(s),(K\nu)(s))-f(s,0,0)|
\end{aligned}$$

$$
\begin{aligned}
&+|f(s,0,0)|]\,\frac{1}{(\tau_2-s)^{1-\beta}}ds \le \frac{1}{\Gamma(\beta)}\int_0^{\tau_1}[A(|\nu(s)| \\
&+|(K\nu)(s)|)+D]\left[\frac{1}{(\tau_1-s)^{1-\beta}}-\frac{1}{(\tau_2-s)^{1-\beta}}\right]ds \\
&+\frac{1}{\Gamma(\beta)}\int_{\tau_1}^{\tau_2}[A(|\nu(s)|+|(K\nu)(s)|)+D]\,\frac{1}{(\tau_2-s)^{1-\beta}}ds \\
\le\ & \frac{A\|\nu\|+Aa\ \|k\|\ \psi(\|\nu\|)+D}{\Gamma(\beta)}\ .\ \left[\frac{\tau_1^\beta-\tau_2^\beta+(\tau_2-\tau_1)^\beta}{\beta}\right. \\
&\left.+\frac{(\tau_2-\tau_1)^\beta}{\beta}\right] \le\ (A\|\nu\|+Aa\ \|k\|\ \psi(\|\nu\|)+W)\ .\ \frac{2\epsilon^\beta}{\Gamma(\beta+1)}. \qquad (3.5)
\end{aligned}
$$

Therefore, the function $F\nu$ is continuous on the interval $[0,a]$. Finally, we deduce $T\nu\in C[0,a]$.

We now consider the arbitrary function $\nu\in BC([0,a])$ and using our assumptions, for a constant $\tau\in[0,a]$ we have

$$
\begin{aligned}
|(\Upsilon\nu)(\tau)| \le & \left|\sum_{j=0}^{m-1}\frac{\nu_j+e^{(j)}(0,\nu_0)}{j!}\tau^j\right| + |e(\tau,\nu(\tau))-e(\tau,0)|+|e(\tau,0)| \\
&+\frac{1}{\Gamma(\beta)}\int_0^\tau\frac{|g(s,\nu(s))-g(s,0)|+|g(s,0)|}{(\tau-s)^{1-\beta}}ds \\
&+\frac{1}{\Gamma(\beta)}\int_0^\tau\frac{|f(s,\nu(s),(k\nu)(s))-f(s,0,0)|+|f(s,0,0)|}{(\tau-s)^{1-\beta}}ds \\
\le & \left|\sum_{j=0}^{m-1}\frac{\nu_j+e^{(j)}(0,\nu_0)}{j!}a^j\right| + A\|\nu\|+B+\frac{A\|\nu\|+C}{\Gamma(\beta+1)}a^\beta \\
&+\frac{A\|\nu\|+Aa\|k\|\psi(\|\nu\|)+D}{\Gamma(\beta+1)}a^\beta \\
\le & L+W+A\|\nu\|+\frac{a^\beta}{\Gamma(\beta+1)}\left[2A\|\nu\|+2W+Aa\|k\|\psi(\|\nu\|)\right]. \qquad (3.6)
\end{aligned}
$$

Therefore $\Upsilon\nu\in B([0,a])$. From the continuity of $\Upsilon\nu$ on $[0,a]$ along with the Boundedness $\Upsilon\nu$ in the same interval, we conclude that $\Upsilon\nu\in BC([0,a])$. Moreover, from the estimates (3.6) and assumption H_5, we conclude that there exists $r_0>0$ such that the operator Υ transforms the ball B_{r_0} into itself where $B_{r_0}=\{\nu\in BC([0,a]):\ \|\nu\|\le r_0\}$ is a closed bounded and convex subset of $BC([0,a])$. Let $U\subset B_{r_0}$ and $\nu,\upsilon\in U$ such that $\|\nu-\upsilon\|\le\epsilon$ for every $\tau\in[0,a]$. Using the assumptions (H_1) - (H_3) , we obtain

$$\begin{aligned}|(\Upsilon\nu)(\tau)-(\Upsilon\upsilon)(\tau)| \leq& |e(\tau,\upsilon(\tau))-e(\tau,\nu(\tau))| + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{|g(s,\nu(s))-g(s,\upsilon(s))|}{(\tau-s)^{1-\beta}}ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{|f(s,\nu(s),(K\nu)(s))-f(s,\upsilon(s),(K\upsilon)(s))|}{(\tau-s)^{1-\beta}}ds \\ \leq& A\|\nu-\upsilon\| + \frac{Aa^\beta\|\nu-\upsilon\|}{\Gamma(\beta+1)} + \frac{Aa^\beta\ \|\nu-\upsilon\|\ (1+aN\ \|k\|)}{\Gamma(\beta+1)} \\ <& \left(1+\frac{a^\beta}{\Gamma(\beta+1)}+\frac{a^\beta(1+aN\ \|k\|)}{\Gamma(\beta+1)}\right)\ .\ \|\nu-\upsilon\|.\end{aligned}$$

We infer from the above that the operator Υ is continuous on B_{r_0}. Now, we show that the operator Υ satisfies the Darbo condition with respect to the measure ϖ_0 in the ball B_{r_0}. Consider a nonempty subset U of B_{r_0} such that $\nu \in U$. suppose that $\tau_1, \tau_2 \in [0,a]$ such that $|\tau_2-\tau_1| \leq \epsilon$, $\epsilon > 0$. Without diminishing the whole issue, we can take $\tau_1 < \tau_2$. Then, using the our assumptions and the estimates (3.4) and (3.5), we have

$$\begin{aligned}|(\Upsilon\nu)(\tau_2)-(\Upsilon\nu)(\tau_1)| \leq& \Big|\sum_{j=0}^{m-1}\frac{\nu_j+e^{(j)}(0,\nu_0)}{j!}(\tau_2^j-\tau_1^j)\Big| + |(E\nu)(\tau_1)-(E\nu)(\tau_2)| \\ &+ |(G\nu)(\tau_2)-(G\nu)(\tau_1)| + |(F\nu)(\tau_2)-(F\nu)(\tau_1)| \\ \leq& |e(\tau_1,\nu(\tau_1))-e(\tau_1,\nu(\tau_2))| + |e(\tau_1,\nu(\tau_2))-e(\tau_2,\nu(\tau_2))| \\ &+ |(G\nu)(\tau_2)-(G\nu)(\tau_1)| + |(F\nu)(\tau_2)-(F\nu)(\tau_1)| \\ \leq& A\varpi(\nu,\epsilon)+\varpi(e,\epsilon)+\frac{2\epsilon^\beta}{\Gamma(\beta+1)}(A\|\nu\|+W) \\ &+ (A\|\nu\|+Aa\ \|k\|\ \psi(\|\nu\|)+W)\ .\ \frac{2\epsilon^\beta}{\Gamma(\beta+1)},\end{aligned}$$

where

$$\begin{aligned}\varpi(e,\epsilon) &= \sup\{\ |e(\tau_1,\nu)-e(\tau_2,\nu)|\ :\ \tau_1,\tau_2\in[0,a]\ ,\ |\tau_1-\tau_2|\leq\epsilon\ ,\ \nu\in[-r_0,r_0]\}\ , \\ \varpi(\nu,\epsilon) &= \sup\{\ |\nu(\tau_1)-\nu(\tau_2)|\ :\ \tau_1,\tau_2\in[0,a]\ ,\ |\tau_1-\tau_2|\leq\epsilon\}\ .\end{aligned}$$

Given that the function e is uniformly continuous on the bounded subset $[0,a] \times [-r_0, r_0]$, we conclude that $\varpi(e,\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$. Consequently, from the above estimate we get

$$\varpi_0\ (\Upsilon U) \leq A\varpi_0\ (U).$$

We conclude that the operator Υ satisfies the Darbo condition with respect to the measure of noncompactness ϖ_0 with the constant $A < 1$. This means that the operator Υ is a contraction on the ball B_{r_0}, with respect to ϖ_0. Consequently, according to Theorem 2.5, we conclude that the operator Υ has a fixed point in the ball B_{r_0} and this completes the proof. □

4. An Example

In this section, we give an example to demonstrate the application of the obtained results.

Consider the following nonlinear integro-differential equation of fractional order

$$
\begin{aligned}
{}^{C}D^{1.25}(\nu(\tau)+\frac{1}{10}\sin\nu(\tau)) &= \frac{e^{\tau}}{5+|\nu(\tau)|}+\frac{1}{(\tau+3)^2}.\frac{|\nu(\tau)|}{1+|\nu(\tau)|}\\
&\quad+\frac{1}{9}\int_0^{\tau}\frac{s\tau^2}{1+\tau}\left(\frac{1}{4}+\int_0^{s}\mu\nu(\mu)d\mu\right)ds,
\end{aligned}
$$

$$\nu^{(j)}(0)=\nu_j,\ j=0,1,\ \tau\in[0,1]. \tag{4.1}$$

This equation is a special case of Eq. (1.1), where

$$
\begin{aligned}
&\beta=1.25,\ m=2,\ a=1,\\
&e(\tau,\nu(\tau))=\frac{1}{10}\sin\nu(\tau),\ g(\tau,\nu(\tau))=\frac{e^{\tau}}{5+|\nu(\tau)|},\\
&(H\nu)(\tau)=\frac{1}{4}+\int_0^{\tau}\mu\nu(\mu)d\mu\\
&f(\tau,\nu,K\nu)=\frac{1}{(\tau+3)^2}.\frac{|\nu(\tau)|}{1+|\nu(\tau)|}+\frac{1}{9}(K\nu)(\tau),\\
&(K\nu)(\tau)=\int_0^{\tau}\frac{s\tau^2}{1+\tau}\left(\frac{1}{4}+\int_0^{s}\mu\nu(\mu)d\mu\right)ds.
\end{aligned}
$$

We have

$$
\begin{aligned}
|e(\tau,\nu(\tau))-e(\tau,\upsilon(\tau))| &\le \frac{1}{10}\|\nu-\upsilon\|,\\
|g(\tau,\nu(\tau))-g(\tau,\upsilon(\tau))| &\le e^{\tau}\|\frac{1}{5+\nu}-\frac{1}{5+\upsilon}\|\le\frac{e}{25}\|\nu-\upsilon\|,\\
\|f(\tau,\nu,K\nu)-f(\tau,\upsilon,K\upsilon)\| &\le \frac{1}{(\tau+3)^2}|\nu-\upsilon|+\frac{1}{9}\|K\nu-K\upsilon\|\\
&\le\frac{1}{9}\left(|\nu-\upsilon|+\|K\nu-K\upsilon\|\right).
\end{aligned}
$$

From the above relationships, we conclude that conditions (H_1) and (H_2) are satisfied with $A=\frac{1}{9}$ and $W=\frac{e}{25}$. Also, easily check that the function H satisfies conditions (H_3) and (H_4) with $N=1$ and $\psi(\|\nu\|)=\frac{1}{4}+\|\nu\|$. By assumption

$\nu_0 = 0,\ \nu_1 = 1$, we have $L = 1.1$. Finally, in order to verify assumption (H_5), the corresponding inequality has the form

$$1.1 + \frac{e}{25} + \frac{1}{9}r + \frac{1}{\Gamma(2.25)}\left[\frac{2}{9}r + \frac{2e}{25} + \frac{2}{45}(\frac{1}{4} + r)\right] \leq r.$$

We can show that $r_0 = 1.29$ is a solution of the above inequality and finally using the Theorem 3.1, we conclude that Eq. (4.1) has at least one solution.

5. Conclusion

In this article, using the technique of measure of noncompactness and the Darbo theorem under the appropriate assumptions, we studied the existence of solution of nonlinear fractional integro-differential equations of the Hammerstein type in the Banach space. In future work, we can investigate the existence and stability of solution for fractional integro-differential equations of the Hammerstein type using the techniques of noncompactness measures in an infinite interval.

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