J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2020.27.1.1 ISSN(Online) 2287-6081
Volume 27, Number 1 (February 2020), Pages 1–11

FURTHER RESULTS INVOLVING THE HILBERT SPACE $L^2_{a,b}[0,T]$

Jae Gil Choi [a,*] and David Skoug [b]

Abstract. In this paper we determine conditions which a function $a(t)$ must satisfy to insure that the function $a'(t)$ is an element of the separable Hilbert space $L^2_{a,b}[0,T]$. We then proceed to illustrate our results with several pertinent examples and counter-examples.

1. Introduction

In the early 1920's, Nobert Wiener introduced the concept of "integration in function space" and hence it is fitting that the space of real-valued continuous functions $C_0[0,T]$ equipped with an appropriate Gaussian measure is called the one-parameter Wiener space. In [9], Yeh introduced a function space $C_{a,b}[0,T]$ related to a generalized Brownian motion. This theory was developed further by Chang and Chung in [3] for appropriate functions $a(t)$ and $b(t)$ on $[0,T]$. Also see $[2,4,5,7,8]$ for additional related results. The function $a(t)$ is often interpreted as the "drift" of the associated stochastic process, and will be our primary object of interest in this paper. In particular, we determine conditions under which translation by this function will result in an equivalent Gaussian measure on the corresponding generalized Wiener space.

Let $(\Omega,\mathcal{F},\mathrm{P})$ be a probability space. A real-valued stochastic process X on $(\Omega,\mathcal{F},\mathrm{P})$ and a time interval $[0,T]$ is called a generalized Brownian motion provided $X(0,\omega)=0$ a.e. ω and for $0=t_0<t_1<\cdots<t_n\le T$, the random vector $(X(t_1,\omega),\dots,X(t_n,\omega))$ has a normal distribution with density function

$$\bigg((2\pi)^n\prod_{i=1}^n[b(t_i)-b(t_{i-1})]\bigg)^{-1/2}\exp\bigg(-\frac{1}{2}\sum_{i=1}^n\frac{([u_i-a(t_i)]-[u_{i-1}-a(t_{i-1})])^2}{b(t_i)-b(t_{i-1})}\bigg),$$

Received by the editors October 28, 2018. Accepted January 26, 2020.
2010 *Mathematics Subject Classification.* 28C20, 60J65.
Key words and phrases. generalized Brownian motion process, separable Hilbert space, total variation function, Wiener space.
*Corresponding author.

with $u_0 = 0$, and where $a(t)$ and $b(t)$ are suitable continuous real-valued functions. In particular, we will assume $a(t)$ to be absolutely continuous with $a(0) = 0$ and $a' \in L^2[0,T]$ and will also assume $b(t)$ to be strictly increasing and continuously differentiable with $b(0) = 0$ and $b'(t) > 0$ for all $t \in [0,T]$. We note that the generalized Brownian motion process X determined by the functions a and b is Gaussian with mean $a(t)$ and covariance $r(s,t) = \min\{b(s), b(t)\}$. Let $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T], \mu)$ denote the complete generalized Wiener space. Note also that the generalized Wiener space reduces to the classical Wiener space precisely when $a(t) = 0$ and $b(t) = t$ for all $t \in [0,T]$.

2. Cameron–Martin Spaces

By their nature, Gaussian measures on function spaces do not enjoy the property of translation invariance. However, as first shown by Cameron and Martin in [1], it is possible to compute the Radon–Nikodym derivatives of measures resulting from certain translations. Specially, there is a particular class of functions, which along with translation results, yields an equivalent Gaussian measure. In the case of ordinary Wiener space, this collection of allowable translates coincides with the Sobolev space $H_0^1(0,T)$ of functions vanishing at 0 and having square-integrable weak derivatives on $(0,T)$. For a generalized Wiener space a similar, but more complicated, result holds.

Let $L^2_{a,b}[0,T]$ be the separable Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by the functions a and b; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(t)db(t) < \infty \text{ and } \int_0^T v^2(t)d|a|(t) < \infty \right\},$$

where $|a|(t)$ denotes the total variation of the function of a on $[0,t]$. Under our working assumptions on the functions a and b, from [3, 7] we have the following generalization of Cameron and Martin's celebrated Translation Theorem.

Theorem 2.1. *Let $z \in L^2_{a,b}[0,T]$ and let $x_0(t) = \int_0^t z(s)db(s)$. If F is an integrable function on $C_{a,b}[0,T]$, then*

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} F(x + x_0)\mu(dx) \\
&= \exp\left(-\frac{1}{2}\int_0^T z^2(s)db(s) - \int_0^T z(s)da(s)\right) \int_{C_{a,b}[0,T]} F(x)\exp(\langle z, x\rangle)\mu(dx),
\end{aligned}$$

where $\langle z, x\rangle$ denotes the Paley–Wiener–Zygmund stochastic integral of z (cf. [8]).

Notice that the Radon–Nikodym derivative involves three contributions. Of these, the stochastic integral $\langle z, x\rangle$ is exactly as observed in the Cameron–Martin Theorem and the term $-\frac{1}{2}\int_0^T z^2(s)db(s) = -\frac{\|z\|_b^2}{2}$ is the direct analog of the corresponding term $-\frac{\|z\|_2^2}{2}$ in the original theorem. The third contributor, $\int_0^T z(s)da(s)$, resulting from the interaction of the drift function $a(t)$ with the translation, is new. This additional interaction is what introduces additional difficulty in working in these more general spaces; for example see [6]. Therefore, it is necessary to develop a better understanding of its behavior.

Recall that $b(t)$ is assumed to be a strictly-increasing, continuously differentiable, real valued function on $[0,T]$ with $b(0)=0$ and with $b'(t)$ strictly positive. Let

$$m = \min_{[0,T]} b'(t) \tag{2.1}$$

and

$$M = \max_{[0,T]} b'(t). \tag{2.2}$$

It follows from the continuity of b' that

$$0 < m \le b'(t) \le M < +\infty \tag{2.3}$$

and that

$$0 < \frac{1}{M} \le \frac{1}{b'(t)} \le \frac{1}{m} < +\infty \tag{2.4}$$

for each $t \in [0,T]$.

Remark 2.2. In this paper as well as in [2,4,5,7,8], $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$ on $[0,T]$. It is easy to see that $|a|'(t) = |a'(t)|$ for m_L-a.e. $t \in [0,T]$ where m_L denotes Lebesgue measure on $[0,T]$. Thus it follows that

$$\mathrm{Var}(a) = \int_0^T d|a|(t) = \int_0^T |a'(t)|dt,$$

$$\int_0^T |a'(t)|d|a|(t) = \int_0^T |a'(t)|^2 dt$$

and

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T |a'(t)|^3 dt. \tag{2.5}$$

In our next theorem we show that the functions $b(t)$ and $b'(t)$ are elements of the Hilbert space $L^2_{a,b}[0,T]$.

Theorem 2.3. *The functions $b(t)$ and $b'(t)$ are both elements of the Hilbert space $L^2_{a,b}[0,T]$.*

Proof. We first note that the function $b(t)$ is an element of $L^2_{a,b}[0,T]$ because

$$\int_0^T b^2(t)db(t) = \frac{b^3(T)}{3} < +\infty$$

and

$$\int_0^T b^2(t)d|a|(t) \le b^2(T)\int_0^T d|a|(t) \le b^2(T)\sqrt{T}\|a'\|_2 < +\infty$$

by the Cauchy–Schwartz inequality.

Next, using (2.1) through (2.4) above, we note that $b'(t)$ is also an element of $L^2_{a,b}[0,T]$ since

$$\int_0^T |b'(t)|^2 db(t) = \int_0^T [b'(t)]^2 b'(t)dt = \int_0^T [b'(t)]^3 dt \le M^3 T < +\infty$$

and

$$\int_0^T |b'(t)|^2 d|a|(t) \le M^2 \int_0^T d|a|(t) \le M^2\sqrt{T}\|a'\|_2 < +\infty$$

by the Cauchy–Schwartz inequality. □

Theorem 2.3 above tells us that the functions $b(t)$ and $b'(t)$ will always be allowable translates. Our next theorem will help us determine in Section 3 below whether or not the mean function $a(t)$ enjoys this same property.

Theorem 2.4. *The following three statements are equivalent:*

(i) $\int_0^T |a'(t)|^2 d|a|(t) < +\infty$,

(ii) $a' \in L^2_{a,b}[0,T]$,

(iii) $\frac{a'}{b'} \in L^2_{a,b}[0,T]$.

Proof. Statement (ii) holds if and only if statement (iii) holds by (2.4) above. It is immediate that statement (ii) implies statement (i). What remains to be shown is that statement (i) implies that

$$\int_0^T \left[\frac{a'(t)}{b'(t)}\right]^2 d[b(t) + |a|(t)] < +\infty.$$

But, by the assumptions on the function $a(t)$ and using (2.4) above, it follows that

$$\int_0^T \left[\frac{a'(t)}{b'(t)}\right]^2 db(t) = \int_0^T \frac{[a'(t)]^2}{b'(t)} dt \le \frac{1}{m}\int_0^T [a'(t)]^2 dt < +\infty,$$

and also that

$$\int_0^T \left[\frac{a'(t)}{b'(t)}\right]^2 d|a|(t) \leq \frac{1}{m^2}\int_0^T [a'(t)]^2 d|a|(t) < +\infty.$$

Hence statement (i) implies statement (ii) and so the proof of Theorem 2.4 is complete. □

3. When is $a' \in L^2_{a,b}[0,T]$?

In this section we are interested in finding conditions on $a(t)$ which will guarantee that the function $a'(t)$ is an element of $L^2_{a,b}[0,T]$. That is, must it necessarily be true that

$$\int_0^T |a'(t)|^2 db(t) < +\infty$$

and that

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty \ ?$$

As the following example shows, the answer to this question is "no" in general.

Example 3.1. Let $a(t) = t^{2/3}$ and let $b(t) = t$ on $[0,T]$. Then

$$\int_0^T |a'(t)|^2 db(t) = \frac{4}{9}\int_0^T t^{-2/3} dt = \frac{4T^{1/3}}{3} < +\infty$$

and

$$\int_0^T |a'(t)|^2 d|a|(t) = \frac{8}{27}\int_0^T \frac{1}{t} dt = +\infty.$$

Thus in this case $a' \notin L^2_{a,b}[0,T]$.

Theorem 3.2. *Let $b(t)$ be as in Section 2 above and let $a(t) = t^\alpha$ for $\alpha > 0$. Then $a' \in L^2_{a,b}[0,T]$ if and only if $\alpha > \frac{2}{3}$.*

Proof. (i) Let $\alpha > \frac{2}{3}$ be given. In view of Theorem 2.4 above it will suffice to show that $\int_0^T |a'(t)|^2 d|a|(t) < +\infty$. But because $\alpha > \frac{2}{3}$ it follows that $3\alpha - 2 > 0$ and so

$$\begin{aligned}\int_0^T |a'(t)|^2 d|a|(t) &= \int_0^T |a'(t)|^3 dt = \int_0^T |\alpha t^{\alpha-1}|^3 dt = \alpha^3 \int_0^T t^{3\alpha-3} dt \\ &= \frac{\alpha^3 T^{3\alpha-2}}{3\alpha - 2} < +\infty\end{aligned}$$

as desired.

(ii) If $\alpha \le 2/3$, then $3\alpha - 3 \le -1$, and so

$$\begin{aligned}\int_0^T |\alpha t^{\alpha-1}|^3 dt &= |\alpha|^3 \int_0^T t^{3\alpha-3} dt \\ &= |\alpha|^3 \int_0^1 t^{3\alpha-3} dt + |\alpha|^3 \int_1^T t^{3\alpha-3} dt \\ &\ge |\alpha|^3 \int_0^1 t^{-1} dt + |\alpha|^3 \int_1^T t^{3\alpha-3} dt \\ &= +\infty + |\alpha|^3 \int_1^T t^{3\alpha-3} dt \\ &= +\infty.\end{aligned}$$

□

Corollary 3.3. *If $a(t)$ is a polynomial, then $a' \in L^2_{a,b}[0,T]$.*

Example 3.4. Let $a(t) = \begin{cases} 0 & , \ 0 \le t \le T/2 \\ 2(t - T/2) & , \ T/2 \le t \le T \end{cases}$. Then

$$a'(t) = \begin{cases} 0 & , \ 0 \le t < T/2 \\ 2 & , \ T/2 < t \le T \end{cases},$$

$$\int_0^T |a'(t)|^2 dt = 2T,$$

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_{T/2}^T |a'(t)|^3 dt = 4T,$$

and

$$\int_0^T |a'(t)|^2 db(t) = 4[b(T) - b(T/2)].$$

Example 3.5. Let $b(t)$ be as in Section 2 above with $T = \pi/2$ and let $a(t) = \sin(t)$ on $[0, \pi/2]$. Then $a'(t) = \cos(t) \in L^2_{a,b}[0,T]$ and

$$\int_0^{\pi/2} |a'(t)|^2 db(t) = \int_0^{\pi/2} \cos^2(t) b'(t) dt.$$

Furthermore

$$\int_0^{\pi/2} |a'(t)|^2 d|a|(t) = \int_0^{\pi/2} \cos^3(t) dt.$$

Remark 3.6. The following well-known integration formula will be useful in Example 3.7 below. For $x > 0$ and $k > 0$,

$$\int \ln^3(kx)dx = x\ln^3(kx) - 3\int \ln^2(kx)dx. \tag{3.1}$$

Example 3.7. In this example we will assume that $T \geq 1$ so that $\ln(t+T) \geq 0$ for all $t \in [0,T]$. Let $b(t)$ be as in Section 2 above and let

$$a(t) = (T+t)\ln(t+T) - (t + T\ln(T))$$

on $[0,T]$. Then $a(0) = 0$,

$$a(T) = 2T\ln(2T) - T(1+\ln(T)) = T[2\ln(2T) - 1 - \ln(T)],$$

and

$$a'(t) = \ln(t+T)$$

on $[0,T]$. Furthermore

$$\int_0^T |a'(t)|^2 db(t) = \int_0^T [\ln(t+T)]^2 db(t) = \int_0^T [\ln(t+T)]^2 b'(t)dt.$$

In addition, using (3.1) above, it follows that

$$\begin{aligned}
&\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T |a'(t)|^3 dt \\
&= \int_0^T \ln^3(t+T)dt \\
&= (t+T)\ln^3(t+T)\Big|_0^T - 3\int_0^T \ln^2(t+T)dt \\
&= 2T\ln^3(2T) - T\ln^3(T) - 3(t+T)[\ln^2(t+T) - 2\ln(t+T) + 2]\Big|_0^T \\
&= 2T\ln^3(2T) - T\ln^3(T) - 6T[\ln^2(2T) - 2\ln(2T) + 2] + 3T[\ln^2(T) - 2\ln(T) + 2] \\
&= 2T\ln^3(2T) - T\ln^3(T) - 6T\ln^2(2T) + 12T\ln(2T) - 12T \\
&\quad + 3T\ln^2(T) - 6T\ln(T) + 6T \\
&= 2T[\ln^3(2T) - 3\ln^2(2T) + 6\ln(2T)] - T[\ln^3(T) - 3\ln^2(T) + 6\ln(T) + 6].
\end{aligned}$$

Theorem 3.8. *Let $b(t)$ be as in Section 2 above. Assume that $a(t)$ satisfies a Lipschitz condition on $[0,T]$. Then $a' \in L^2_{a,b}[0,T]$.*

Proof. It is a well known fact that an absolutely continuous function a satisfies a Lipschitz condition on $[0,T]$ if and only if $|a'|$ is bounded. For each $t \in [0,T]$, let $|a'(t)| \le K$ for some $K > 0$. Then

$$\int_0^T |a'(t)|^2 db(t) \le K^2 \int_0^T db(t) = K^2 b(T) < +\infty$$

and

$$\int_0^T |a'(t)|^2 d|a|(t) \le K \int_0^T |a'(t)|^2 dt < +\infty.$$

□

Theorem 3.9. *Let $b(t)$ be as in Section 2 above. Let n be a positive integer and let*

$$a(t) = t^n e^t$$

on $[0,T]$. Then $a' \in L^2_{a,b}[0,T]$ and satisfies the equation

$$\int_0^T |a'(t)|^2 d|a(t)| = \int_0^T |a'(t)|^3 dt = \int_0^T t^{3(n-1)} e^{3t} (t+n)^3 dt.$$

Proof. We first note that for all $t \in [0,T]$,

$$a'(t) = t^{n-1} e^t (t+n).$$

Hence

$$\int_0^T |a'(t)|^2 db(t) = \int_0^T t^{2(n-1)} e^{2t} (t+n)^2 db(t) < +\infty$$

and that

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T |a'(t)|^3 dt = \int_0^T t^{3(n-1)} e^{3t} (t+n)^3 dt < +\infty$$

for each positive integer n. □

Example 3.10. Let $n = 2$ in Theorem 3.9 above. Then

$$a(t) = t^2 e^t,$$
$$a'(t) = t(t+2)e^t,$$

and

$$\begin{aligned}\int_0^T |a'(t)|^2 d|a|(t) &= \int_0^T |a'(t)|^3 dt \\ &= \int_0^T t^3 (t^3 + 6t^2 + 12t + 8) e^{3t} dt \\ &= \int_0^T [t^6 + 6t^5 + 12t^4 + 8t^3] e^{3t} dt.\end{aligned}$$

Example 3.11. Let $a(t) = t^3e^t$. Then proceeding as in Example 3.10 above, it follows that

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T [t^9 + 9t^8 + 27t^7 + 27t^6]e^{3t}dt.$$

Example 3.12. More generally it follows quite easily that:

(i) If $a(t) = t^4e^t$ then

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T [t^{12} + 12t^{11} + 48t^{10} + 64t^9]e^{3t}dt.$$

(ii) If $a(t) = t^5e^t$ then

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T [t^{15} + 15t^{14} + 75t^{13} + 125t^{12}]e^{3t}dt.$$

(iii) If $a(t) = t^6e^t$ then

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T [t^{18} + 18t^{17} + 108t^{16} + 216t^{15}]e^{3t}dt.$$

(iv) If $a(t) = t^7e^t$ then

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T [t^{21} + 21t^{20} + 147t^{19} + 343t^{18}]e^{3t}dt.$$

Example 3.13. In Example 3.5 above, we discussed the functions $b(t)$ and $a(t)$ on the interval $[0, T]$ with $T = \pi/2$ so that the function $a(t)$ is increasing on $[0, \pi/2]$. But, the restriction $T = \pi/2$ is not needed. For example, let $b(t)$ be as in Section 2 above with $T = \pi$ and let $a(t) = \sin t$ on $[0, \pi]$. Then

$$|a|(t) = \begin{cases} \sin(t) & , \ t \in [0, \pi/2] \\ -\sin(t) + 2 & , \ t \in [\pi/2, \pi] \end{cases},$$

$|a|'(t) = |a'(t)| = a'(t) = |\cos(t)|$ in $L^2_{a,b}[0, T]$,

$$\int_0^\pi |a'(t)|^2 db(t) = \int_0^\pi \cos^2(t)b'(t)dt,$$

and

$$\int_0^\pi |a'(t)|^2 d|a|(t) = \int_0^{\pi/2} \cos^3(t)dt - \int_{\pi/2}^\pi \cos^3(t)dt = 2\int_0^{\pi/2} \cos^3(t)dt.$$

Remark 3.14. One can see that the derivatives $a'(t)$ of the suggested functions $a(t)$ in Examples 3.4 through 3.13 are bounded on $[0, T]$. Thus it follows that the $a'(t)$'s are elements of $L^2_{a,b}[0, T]$ in view of Theorem 3.8.

We finish this paper with the following theorem which gives a different condition on the function $a(t)$.

Theorem 3.15. *Let $b(t)$ be as in Section 2 above. Given a function $a(t)$ on $[0,T]$, assume that a' is an element of $L^3[0,T]$. Then $a' \in L^2_{a,b}[0,T]$.*

Proof. Using (2.2), it first follows that

$$\int_0^T |a'(t)|^2 db(t) \leq M \int_0^T |a'(t)|^2 dt < +\infty.$$

Next using (2.5), it also follows that

$$\int_0^T |a'(t)|^2 d|a|(t) = \int_0^T |a'(t)|^3 dt < +\infty.$$

□

Remark 3.16. If a is an indefinite integral of a function in $L^3[0,T]$, that is,

$$a(t) = \int_0^t u(s)ds \quad \text{for some } u \in L^3[0,T],$$

then, in view of Theorem 3.15, the function a' is in $L^2_{a,b}[0,T]$. The class $L^3[0,T]$ contains many unbounded functions on $[0,T]$.

REFERENCES

1. R.H. Cameron & W.T. Martin: Transformations of Wiener integrals under translations. Ann. of Math. **45** (1944), no. 2, 386-396.
2. S.J. Chang, J.G. Choi & D. Skoug: A generalized Fourier-Feynman transform on the product function space $C_{a,b}[0,T]$ and related topics. arXiv:1309.7176 (2013).
3. S.J. Chang & D.M. Chung: Conditional function space integrals with applications. Rocky Mountain J. Math. **26** (1996), 37-62.
4. S.J. Chang, H.S. Chung & D. Skoug: Integral transforms of functionals in $L_2(C_{a,b}[0,T])$. J. Fourier Anal. Appl. **15** (2009), 441-462.
5. S.J. Chang, H.S. Chung & D. Skoug: Some basic relationships among transforms, convolution products, first variation, and inverse transforms. Cent. Eur. J. Math. **11** (2013), 538-551.
6. S.J. Chang & D. Skoug: The effect of drift on conditional Fourier–Feynman transforms and conditional convolution products. Int. J. Appl. Math. **2** (2000), 505-527.
7. S.J. Chang & D. Skoug: Generalized Fourier-Feynman transforms and a first variation on function space. Integral Transforms Spec. Funct. **14** (2003), 375-393.

8. I. Pierce & D. Skoug: Integration formulas for functionals on the function space $C_{a,b}[0,T]$ involving Paley-Wiener-Zygmund stochastic integrals. Panamerican Math. J. **18** (2008), 101-112.
9. J. Yeh: Singularity of Gaussian measures on function spaces induced by Brownian motion processes with non-stationary increments. Illinois J. Math. **15** (1971), 37-46.
10. J. Yeh: Stochastic Processes and the Wiener Integral. Marcel Dekker, Inc., New York, 1973.

[a]School of General Education, Dankook University, Cheonan 330-714, Republic of Korea
Email address: jgchoi@dankook.ac.kr

[b]Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA
Email address: dskoug@math.unl.edu