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KINK WAVE SOLUTIONS TO KDV-BURGERS EQUATION WITH FORCING TERM

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Abstract. In this paper, we used modified tanh-coth method, combined with Riccati equation and secant hyperbolic ansatz to construct abundantly many real and complex exact travelling wave solutions to KdV-Burgers (KdVB) equation with forcing term. The real part is the sum of the shock wave solution of a Burgers equation and the solitary wave solution of a KdV equation with forcing term, while the imaginary part is the product of a shock wave solution of Burgers with a solitary wave travelling solution of KdV equation. The method gives more solutions than the previous methods.

1. Introduction

Nonlinear waves in dynamical systems are of fundamental importance that is receiving much attention, most especially in the field of wave propagation in nonlinear systems. They are written in nonlinear partial differential equations (NPDE). Application of nonlinear waves cuts across many fields, which include mixture of gas bubble in liquid [22], waves in elastic tubes [2], systems incorporating damping and dispersion [21], KP lump in ferrimagnet [23], chemical physics and geochemistry [8]. There are many methods of solving nonlinear wave developed by researchers over the years, including inverse scattering method [34], sine-cosine method [20], homogeneous balance method [15], first integral method [18], variational iteration method [29], Adomian decomposition method [4], homotopy analysis method [10, 11], reduced transformed method [19], Jacobi elliptic expansion method [5], F-expansion method [26], exp-function method [12]. Tangent hyperbolic method to find the travelling wave solutions of evolution equations was initially introduced by Malfliet [24], by assuming the solution to be a series of tangent hyperbolic function, the method received considerable attention and underwent through many improvements. Fan [16] extended the method, follows by Wazwaz [27] and obtained some other exact solutions apart from the once derived before. The extended

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method was later modified by El-wakil [6] and then by Soliman [3] to obtain new solutions. It is important to note that the later improvements were given different names by different authors.

One-dimensional KdV equation was first derived in 1895 to model shallow water waves with small but finite amplitudes by Korteweg-de-Vries, which is completely integrable and has many soliton solutions [28]. Its has applications in mixture of bubbly liquid [31] and many more [25]. Many methods of solving the equation was presented in [28]. On the other hand, a combination of Burger's equation and KdV equation gives rise to what is called KdVB (Korteweg-de-Vries-Burgers) equation. Solution to the KdVB equation is a combination of solitary wave solution and shock wave solution, whose solutions were obtained in [7, 14, 28, 30] with applications in many area of science and engineering [30].

The aim of this paper is to establish an extension to the above methods that would enable us to find both real and complex travelling wave solutions to the Korteweg-de-Vries (KdV) equation with forcing term [33] and KdVB equation with forcing term [1, 32] using the modified tanh-coth method.

2. Description of the modified tanh-coth method

We illustrate the modified tanh-coth method to obtain many exact travelling wave solutions for nonlinear evolution equations. Let

$$u_t = P\left(u,\ u_x,\ u_{xx},\ u_{xxx},\ \ldots\right) \tag{1}$$

be a partial differential equation (PDE) in two independent variables x and t. We introduced a travelling wave solution of the form

$$u\left(x,t\right) = u\left(\Omega\right), \quad \Omega = k\left(x - \int c\left(t\right) dt\right), \tag{2}$$

where k and $\int c\left(t\right) dt$ are the wave number and velocity of the travelling wave to be determined later. We suppose that

$$u(\Omega) = w(\Omega) + \int f\left(t\right) dt. \tag{3}$$

Equation (1) using (2) and (3) is reduced to an ordinary differential equation (ODE) of the form

$$-c\left(t\right) k w^{'}\left(\Omega\right) = P\left(w\left(\Omega\right),\ k w^{'}\left(\Omega\right),\ k^2 w^{''}\left(\Omega\right),\ \ldots\right). \tag{4}$$

The ODE (4) is then solved using the modified tanh-coth method [26], by assuming finite series of functions of the form

$$w\left(\Omega\right) = S\left(Y\right) = \sum_{i=0}^{m} a_i Y^i + \sum_{i=1}^{m} b_i Y^{-i}, \tag{5}$$

together with the Riccati equation

$$\frac{dY}{d\Omega} = A + BY + CY^2 . \tag{6}$$

Change of variables leads to

$$\begin{aligned} \frac{d}{d\Omega} &= \left(A + BY + CY^2\right)\frac{d}{dY} , \\ \frac{d^2}{d\Omega^2} &= \left(A + BY + CY^2\right)\left((B + 2CY)\frac{d}{dY} + \left(A + BY + CY^2\right)\frac{d^2}{dY^2}\right), \end{aligned} \tag{7}$$

where $a_i\,(i = 0, 1\ , 2, \ldots, m)$, $b_i\,(i = 1\ , 2, \ldots, m)$, A, B and C are constants to be obtained later. The parameter m is a positive real number obtained by balancing the highest order linear term with the nonlinear term in equation (4) [21]. Inserting equation (5) into (4) using (2) produces a system of equations in powers of Y^i. As the Y^i vanishes, we can obtain $a_i\,(i = 0, 1, 2, \ldots, m)$, $b_i\,(i = 1, 2, \ldots, m)$, k and $c(t)$. We will stick to the following Riccati equation solutions

If $A = 1$, $B = 1$ and $C = 0$, then $Y = e^{\Omega} - 1$,

If $A = 1/2$, $B = 0$ and $C = -1/2$, then $Y = \coth\Omega\ \pm \operatorname{csch}\Omega$.

2.1. The real solutions to KdVB equation with forcing term

The KdVB equation with forcing term is [1, 32]

$$u_t + puu_x - qu_{xx} + ru_{xxx} = f(t), \tag{8}$$

where $f(t)$ is the forcing term which is time dependent and p, q and r are real constants. Using the transformation given by (3), equation (8) is reduced to a homogeneous equation of the form

$$w_t + p\left(w + \int f(t)\,dt\right)w_x - qw_{xx} + rw_{xxx} = 0. \tag{9}$$

Introducing the wave transformation given by (2), we make

$$w(x,t) = w(\Omega), \quad \Omega = k\left(x - \int c(t)\,dt\right), \tag{10}$$

to obtain the travelling wave solutions to equation (9). Equation (9) using (10) is reduced to an ODE of the form

$$\begin{aligned} &- c(t)\,kw^{'}(\Omega) + \frac{p}{2}kw^{2'}(\Omega) + pk\,w^{'}(\Omega)\int f(t)\,dt \\ &- qk^2w^{''}(\Omega) + k^3rw^{'''}(\Omega) = 0, \end{aligned} \tag{11}$$

where $' = d/d\Omega$. Integrating (11) with respect to Ω and letting the integration constant to be zero, we have

$$- c(t)\,w + \frac{p}{2}w^2 + pw\int f(t)\,dt - qkw^{'} + k^2rw^{''} = 0. \tag{12}$$

Balancing the nonlinear term w^2 with the higher order linear term w'' in equation (12), i.e., $2m = m + 2$ implies that $m = 2$. The solution of (12) takes the form

$$w(\Omega) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{13}$$

Substituting equation (13) into equation (12) using equation (7), and equating the corresponding coefficients of all powers of Y to zero, yields system of equations in terms of $a_0,\ a_1,\ a_2,\ b_1,\ b_2,\ k$ and $c(t)$. Solving the equations we get Case I: When $A = 1$, $B = 1$ and $C = 0$,

$$\begin{aligned} &a_0 = a_1 = a_2 = 0,\ b_1 = -\frac{24q^2}{25pr},\ b_2 = -\frac{12q^2}{25pr},\ k = \frac{q}{5r}, \\ &c(t) = \frac{6q^2 - 25pr\int f(t)\,dt}{25r}, \end{aligned} \tag{14}$$

$$\begin{aligned} &a_0 = -\frac{12q^2}{25pr},\ a_1 = a_2 = 0,\ b_1 = -\frac{24q^2}{25pr},\ b_2 = -\frac{12q^2}{25pr},\ k = \frac{q}{5r}, \\ &c(t) = \frac{-6q^2 + 25pr\int f(t)\,dt}{25r}. \end{aligned} \tag{15}$$

Using (14) into (3), we have

$$\begin{aligned} u_1 &= -\frac{24q^2}{25pr}\left(\frac{1}{e^{\Omega}-1}\right) - \frac{12q^2}{25pr}\left(\frac{1}{e^{\Omega}-1}\right)^2 + \int f(t)\,dt, \\ \Omega &= \frac{q}{5r}\left(x - \int \frac{6q^2 - 25pr\int f(t)\,dt}{25r}dt\right). \end{aligned} \tag{16}$$

Similarly, using (15) into (3), we get

$$\begin{aligned} u_2 &= -\frac{12q^2}{25pr} - \frac{24q^2}{25pr}\left(\frac{1}{e^{\Omega}-1}\right) - \frac{12q^2}{25pr}\left(\frac{1}{e^{\Omega}-1}\right)^2 + \int f(t)\,dt, \\ \Omega &= \frac{q}{5r}\left(x - \int \frac{-6q^2 + 25pr\int f(t)\,dt}{25r}dt\right). \end{aligned} \tag{17}$$

Case II: When $A = 1/2$, $B = 0$ and $C = -1/2$, we have the following solutions

$$\begin{aligned} &a_0 = a_2 = -\frac{3q^2}{25pr},\ a_1 = -\frac{6q^2}{25pr},\ b_1 = b_2 = 0,\ k = \frac{q}{5r}, \\ &c(t) = \frac{-6q^2 + 25pr\int f(t)\,dt}{25r}, \end{aligned} \tag{18}$$

$$\begin{aligned} &a_0 = \frac{9q^2}{25pr},\ a_1 = -\frac{6q^2}{25pr},\ a_2 = -\frac{3q^2}{25pr},\ b_1 = b_2 = 0,\ k = \frac{q}{5r}, \\ &c(t) = \frac{6q^2 + 25pr\int f(t)\,dt}{25r}, \end{aligned} \tag{19}$$

$$a_0 = \frac{9q^2}{25pr},\ a_1 = a_2 = 0,\ b_1 = \frac{-6q^2}{25pr},\ b_2 = -\frac{3q^2}{25pr},\ k = \frac{q}{5r},$$
$$c(t) = \frac{6q^2 + 25pr \int f(t)\, dt}{25r}, \tag{20}$$

$$a_1 = a_2 = 0, \quad a_0 = b_2 = \frac{-3q^2}{25pr}, \quad b_1 = -\frac{6q^2}{25pr}, \quad k = \frac{q}{5r},$$
$$c(t) = \frac{-6q^2 + 25pr \int f(t)\, dt}{25r}, \tag{21}$$

$$a_0 = \frac{3q^2}{100pr},\ a_1 = b_1 = \frac{-3q^2}{25pr},\ a_2 = b_2 = -\frac{3q^2}{100pr},\ k = \frac{q}{10r},$$
$$c(t) = \frac{6q^2 + 25pr \int f(t)\, dt}{25r}, \tag{22}$$

$$a_0 = \frac{-9q^2}{50pr},\ a_1 = b_1 = \frac{-3q^2}{25pr},\ a_2 = b_2 = -\frac{3q^2}{100pr},\ k = \frac{q}{10r},$$
$$c(t) = \frac{-6q^2 + 25pr \int f(t)\, dt}{25r}. \tag{23}$$

Using (18) into (3) we have

$$u_3 = -\frac{3q^2}{25pr} - \frac{6q^2}{25pr}(\coth\Omega \pm \operatorname{csch}\Omega) - \frac{3q^2}{25pr}(\coth\Omega \pm \operatorname{csch}\Omega)^2$$
$$+ \int f(t)\, dt, \quad \Omega = \frac{q}{5r}\left(x - \int \frac{-6q^2 + 25pr \int f(t)\, dt}{25r} dt\right). \tag{24}$$

Using (19) into (3)

$$u_4 = \frac{9q^2}{25pr} - \frac{6q^2}{25pr}(\coth\Omega \pm \operatorname{csch}\Omega) - \frac{3q^2}{25pr}(\coth\Omega \pm \operatorname{csch}\Omega)^2$$
$$+ \int f(t)\, dt, \quad \Omega = \frac{q}{5r}\left(x - \int \frac{6q^2 + 25pr \int f(t)\, dt}{25r} dt\right). \tag{25}$$

Using (20) into (3)

$$u_5 = \frac{9q^2}{25pr} - \frac{6q^2}{25pr}\frac{1}{(\coth\Omega \pm \operatorname{csch}\Omega)} - \frac{3q^2}{25pr}\frac{1}{(\coth\Omega \pm \operatorname{csch}\Omega)^2}$$
$$+ \int f(t)\, dt, \quad \Omega = \frac{q}{5r}\left(x - \int \frac{6q^2 + 25pr \int f(t)\, dt}{25r} dt\right). \tag{26}$$

Using (21) into (3)

$$u_6 = \frac{-3q^2}{25pr} - \frac{6q^2}{25pr}\frac{1}{(\coth\Omega \pm \operatorname{csch}\Omega)} - \frac{3q^2}{25pr}\frac{1}{(\coth\Omega \pm \operatorname{csch}\Omega)^2}$$
$$+ \int f(t)\, dt, \quad \Omega = \frac{q}{5r}\left(x - \int \frac{-6q^2 + 25pr \int f(t)\, dt}{25r} dt\right). \tag{27}$$

Using (22) into (3)

$$u_7 = \frac{3q^2}{100pr} - \frac{3q^2}{25pr}\left(\coth\Omega \pm \mathrm{csch}\Omega\right) - \frac{3q^2}{100pr}\left(\coth\Omega \pm \mathrm{csch}\Omega\right)^2$$
$$- \frac{3q^2}{25pr}\frac{1}{\left(\coth\Omega \pm \mathrm{csch}\Omega\right)} - \frac{3q^2}{100pr}\frac{1}{\left(\coth\Omega \pm \mathrm{csch}\Omega\right)^2} + \int f(t)\,dt, \tag{28}$$
$$\Omega = \frac{q}{5r}\left(x - \int \frac{6q^2 + 25pr\int f(t)\,dt}{25r}dt\right).$$

Using (23) into (3)

$$u_8 = \frac{-9q^2}{50pr} - \frac{3q^2}{25pr}\left(\coth\Omega \pm \mathrm{csch}\Omega\right) - \frac{3q^2}{100pr}\left(\coth\Omega \pm \mathrm{csch}\Omega\right)^2$$
$$- \frac{3q^2}{25pr}\frac{1}{\left(\coth\Omega \pm \mathrm{csch}\Omega\right)} - \frac{3q^2}{100pr}\frac{1}{\left(\coth\Omega \pm \mathrm{csch}\Omega\right)^2} + \int f(t)\,dt, \tag{29}$$
$$\Omega = \frac{q}{10r}\left(x - \int \frac{-6q^2 + 25pr\int f(t)\,dt}{25r}dt\right).$$

Some of the solutions obtained here using this method are new compared to the ones obtained in [1,9].

2.2. A complex solution to KdVB with forcing term

We introduced a secant hyperbolic ansatz in the form

$$w(\Omega) = \varphi(\Omega) + \Phi(\Omega)\,\mathrm{sech}\Omega\,. \tag{30}$$

Substituting (30) into (12), we have

$$\frac{1}{2}p\Phi^2\left(1 - \tanh^2\Omega\right) - c(t)\,\varphi + \frac{1}{2}p\varphi^2 + p\varphi\int f(t)\,\mathrm{d}t - qk\varphi'$$
$$+ rk^2\varphi'' + \left[2\,rk^2\Phi\tanh^2\Omega - 2\,rk^2\Phi'\tanh\Omega + qk\Phi\tanh\Omega - rk^2\Phi\right. \tag{31}$$
$$\left. + rk^2\Phi'' + p\varphi\Phi + p\Phi\int f(t)\,\mathrm{d}t - qk\Phi' - c(t)\,\Phi\right]\mathrm{sech}\Omega = 0,$$

where $\mathrm{sech}^2\Omega = 1 - \tanh^2\Omega$ is used. Setting the coefficients of $\mathrm{sech}\Omega$ equals to zero, we obtained the following set of ordinary differential equations

$$\left[2\,rk^2\Phi\tanh^2\Omega - 2\,rk^2\Phi'\tanh\Omega + qk\Phi\tanh\Omega - rk^2\Phi\right.$$
$$\left. + rk^2\Phi'' + p\varphi\Phi + p\Phi\int f(t)\,\mathrm{d}t - qk\Phi' - c(t)\,\Phi\right] = 0, \tag{32}$$
$$\frac{1}{2}p\Phi^2\left(1 - \tanh^2\Omega\right) - c(t)\,\varphi + \frac{1}{2}p\varphi^2 + p\varphi\int f(t)\,\mathrm{d}t$$
$$- qk\varphi' + rk^2\varphi'' = 0.$$

Assuming the degrees of φ and Φ are m and n, balancing their nonlinear terms and highest order derivatives, we have $m = 2$ and $n = 1$. Therefore, the tanh-coth equations for φ and Φ take the following forms

$$\text{(33)} \qquad \begin{aligned} \varphi(Y) &= a_0 + a_1 Y + a_2 Y^2 + \frac{a_3}{Y} + \frac{a_4}{Y^2}, \\ \Phi(Y) &= b_0 + b_1 Y + \frac{b_2}{Y}, \end{aligned}$$

where $Y = \tanh(\Omega)$, which satisfied (7), and $a_i\,(i = 0,\,1,\,2,\,3,\,4)$, $b_i\,(i = 0,\,1,\,2)$ are constants to be determined from the solution. Substituting (33) into (32) using (7), we obtained a system of equations in terms of $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ b_0,\ b_1,\ b_2,\ k, c(t)$, and whose solutions are as follows,
Case I:

$$\text{(34a)} \qquad \begin{aligned} a_0 &= 0,\ a_1 = -\frac{6q^2}{25pr},\ a_2 = -\frac{6q^2}{25pr},\ a_3 = a_4 = 0, \\ b_0 &= \pm\, i\frac{6q^2}{25pr},\ b_1 = \pm i\frac{6q^2}{25pr},\ b_2 = 0,\ k = \frac{q}{5r}, \\ c(t) &= -\frac{6q^2 - 25\int f(t)\,\mathrm{d}t}{25r}. \end{aligned}$$

Case II:

$$\text{(34b)} \qquad \begin{aligned} a_0 &= 0,\ a_1 = +\frac{6q^2}{25pr},\ a_2 = -\frac{6q^2}{25pr},\ a_3 = a_4 = 0, \\ b_0 &= \mp\, i\frac{6q^2}{25pr},\ b_1 = \pm i\frac{6q^2}{25pr},\ b_2 = 0,\ k = -\frac{q}{5r}, \\ c(t) &= -\frac{6q^2 - 25\int f(t)\,\mathrm{d}t}{25r}. \end{aligned}$$

Case III:

$$\text{(34c)} \qquad \begin{aligned} a_0 &= \frac{12q^2}{25pr},\ a_1 = -\frac{6q^2}{25pr},\ a_2 = -\frac{6q^2}{25pr},\ a_3 = a_4 = 0, \\ b_0 &= \pm\, i\frac{6q^2}{25pr},\ b_1 = \pm i\frac{6q^2}{25pr},\ b_2 = 0,\ k = \frac{q}{5r}, \\ c(t) &= \frac{6q^2 + 25\int f(t)\,\mathrm{d}t}{25r}. \end{aligned}$$

Case IV:

$$\text{(34d)} \qquad \begin{aligned} a_0 &= \frac{12q^2}{25pr},\ a_1 = +\frac{6q^2}{25pr},\ a_2 = -\frac{6q^2}{25pr},\ a_3 = 0,\ a_4 = 0, \\ b_0 &= \mp\, i\frac{6q^2}{25pr},\ b_1 = \pm i\frac{6q^2}{25pr},\ b_2 = 0,\ k = -\frac{q}{5r}, \\ c(t) &= \frac{6q^2 + 25pr\int f(t)\,\mathrm{d}t}{25r}. \end{aligned}$$

Introducing (34a)-(34d) into (33), we have

$$\text{(35a)}\quad \begin{aligned} \varphi(\Omega) &= -\frac{6q^2}{25pr}\left(\tanh\Omega + \tanh^2\Omega\right), \\ \Phi(\Omega) &= \pm\, i\frac{6q^2}{25pr}\left(1 + \tanh\Omega\right), \\ \Omega &= \frac{q}{5r}\left(x + \int \frac{6q^2 - 25pr\int f(t)\ \mathrm{d}t}{25r}\ \mathrm{d}t\right), \end{aligned}$$

$$\text{(35b)}\quad \begin{aligned} \varphi(\Omega) &= \frac{6q^2}{25pr}\left(\tanh\Omega - \tanh^2\Omega\right), \\ \Phi(\Omega) &= \ i\frac{6q^2}{25pr}\left(\mp 1 \pm \tanh\Omega\right), \\ \Omega &= -\frac{q}{5r}\left(x + \int \frac{6q^2 - 25\int f(t)\ \mathrm{d}t}{25r}\ \mathrm{d}t\right), \end{aligned}$$

$$\text{(35c)}\quad \begin{aligned} \varphi(\Omega) &= \frac{12q^2}{25pr} - \frac{6q^2}{25pr}\tanh\Omega - \frac{6q^2}{25pr}\tanh^2\Omega, \\ \Phi(\Omega) &= \pm\, i\frac{6q^2}{25pr}\left(1 + \tanh\Omega\right), \\ \Omega &= \frac{q}{5r}\left(x - \int \frac{6q^2 + 25pr\int f(t)\ \mathrm{d}t}{25r}\ \mathrm{d}t\right), \end{aligned}$$

$$\text{(35d)}\quad \begin{aligned} \varphi(\Omega) &= \frac{12q^2}{25pr} + \frac{6q^2}{25pr}\tanh\Omega - \frac{6q^2}{25pr}\tanh^2\Omega, \\ \Phi(\Omega) &= \ i\frac{6q^2}{25pr}\left(\mp 1 \pm \tanh\Omega\right), \\ \Omega &= -\frac{q}{5r}\left(x - \int \frac{6q^2 + 25pr\int f(t)\ \mathrm{d}t}{25r}\ \mathrm{d}t\right). \end{aligned}$$

Substituting (35a)-(35d) into (30), equation (3) which is a solution to (8) gives the following complex solutions to the KdVB equation with forcing term in the form

$$\text{(36a)}\quad \begin{aligned} u_{9,10} = {} & \frac{6q^2}{25pr}\left(-1 - \tanh\Omega + \operatorname{sech}^2\Omega\right) \pm i\frac{6q^2}{25pr}\left(1 + \tanh\Omega\right)\operatorname{sech}\Omega \\ & + \int f(t)\ \mathrm{d}t, \quad \Omega = \frac{q}{5r}\left(x + \int \frac{6q^2 - 25pr\int f(t)\ \mathrm{d}t}{25r}\mathrm{d}t\right), \end{aligned}$$

$$\text{(36b)}\quad \begin{aligned} u_{11,12} = {} & \frac{6q^2}{25pr}\left(-1 + \tanh\Omega + \operatorname{sech}^2\Omega\right) + i\frac{6q^2}{25pr}\left(\mp 1 \pm \tanh\Omega\right)\operatorname{sech}\Omega \\ & + \int f(t)\ \mathrm{d}t, \quad \Omega = -\frac{q}{5r}\left(x + \int \frac{6q^2 - 25pr\int f(t)\ \mathrm{d}t}{25r}\mathrm{d}t\right), \end{aligned}$$

$$u_{13,14} = \frac{6q^2}{25pr}\left(1-\tanh\Omega+\operatorname{sech}^2\Omega\right) \pm i\frac{6q^2}{25pr}\left(1+\tanh\Omega\right)\operatorname{sech}\Omega + \int f(t)\,\mathrm{d}t, \quad \Omega = \frac{q}{5r}\left(x-\int\frac{6q^2+25pr\int f(t)\,\mathrm{d}t}{25r}\mathrm{d}t\right), \tag{36c}$$

$$u_{15,16} = \frac{6q^2}{25pr}\left(1+\tanh\Omega+\operatorname{sech}^2\Omega\right) + i\frac{6q^2}{25pr}\left(\mp 1\pm\tanh\Omega\right)\operatorname{sech}\Omega + \int f(t)\,\mathrm{d}t, \quad \Omega = -\frac{q}{5r}\left(x-\int\frac{6q^2+25pr\int f(t)\,\mathrm{d}t}{25r}\mathrm{d}t\right). \tag{36d}$$

The solutions $u_{9,10}$, $u_{11,12}$, $u_{13,14}$ and $u_{15,16}$ form the exact complex travelling wave solutions to KdVB equation with forcing term. A complex solutions for one dimensional KdVB equation with a positive coefficient dissipative term without forcing term is given in [13], while the complex line solutions for two dimensional KdVB equation without forcing term is derived in [17]. All the complex travelling wave solutions to the KdVB equation with forcing term are new.

3. Conclusion

We employed a modified tanh-coth equation method, combined with Riccati equation to find abundantly many real exact solutions to KdVB equation with forcing term. With the help of secant hyperbolic ansatz, more general complex travelling wave solutions of the KdVB equation with forcing term are obtained. This approach has been successfully applied to obtain some real and complex kink wave solutions to KdVB equation with forcing term and constant coefficients. We noticed that the real part is the sum of the shock wave solution of a Burgers equation and the solitary wave solution of a KdV equation with forcing, while the imaginary part is the product of a shock wave solution of Burgers with a solitary wave travelling solution of KdV equation. The method adopted here gives more solutions than the previous methods. This paper showed that the secant hyperbolic ansatz and tanh-coth method combined with Riccati equation gives a unified approach of constructing complex travelling wave solution to many nonlinear partial differential equations.

References

[1] R. Abazari, *Application of extended Tanh function method on KdV-Burgers equation with forcing term*, Romanian J. Phys. **59** (2014), no. 1-2, 3–11.

[2] M. Abdou, A. Hendi, and H. K. Alanzi, *New exact solutions of kdv equation in an elastic tube filled with a variable viscosity fluid*, Stud. Nonlinear Sci. **3** (2012), no. 2, 62–68.

[3] M. A. Abdou and A. A. Soliman, *Modified extended tanh-function method and its application on nonlinear physical equations*, Phys. Lett. A **353** (2006), no. 6, 487–492.

[4] G. Adomian, *Solving frontier problems of physics: the decomposition method*, Fundam. Theor. Phys. **60**, Kluwer Academic Publishers Group, Dordrecht, 1994. https://doi.org/10.1007/978-94-015-8289-6

[5] M. Ali Akbar, N. Hj. Mohd. Ali, and E. M. E. Zayed, *Generalized and improved* (G'/G)-*expansion method combined with Jacobi elliptic equation*, Commun. Theor. Phys. (Beijing) **61** (2014), no. 6, 669–676. https://doi.org/10.1088/0253-6102/61/6/02

[6] A. H. A. Ali and A. A. Soliman, *New exact solutions of some nonlinear partial differential equations*, Int. J. Nonlinear Sci. **5** (2008), no. 1, 79–88.

[7] A. Bekir, *On traveling wave solutions to combined KdV-mKdV equation and modified Burgers-KdV equation*, Commun. Nonlinear Sci. Numer. Simul. **14** (2009), no. 4, 1038–1042. https://doi.org/10.1016/j.cnsns.2008.03.014

[8] E. Bilyay, B. Ozbahceci, and A. Yalciner, *Extreme waves at filyos, southern black sea*, Nat. Hazards Earth Syst. Sci. **11** (2011), no. 3, 659–666.

[9] Y. B. Chukkol, M. N. Mohamad, and M. I. Muminov, *Exact solutions to the KDV-Burgers equation with forcing term using Tanh-Coth method*, AIP Conf. Proc. **1870** (2017), no. 1, 040024.

[10] M. Dehghan and J. Manafian, *The solution of the variable coefficients fourth-order parabolic partial differential equations by the homotopy perturbation method*, Z. Naturforsch. A **64** (2009), no. 7–8, 420–430.

[11] M. Dehghan, J. Manafian, and A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Numer. Methods Partial Differential Equations **26** (2010), no. 2, 448–479. https://doi.org/10.1002/num.20460

[12] M. Dehghan, J. Manafian Heris, and A. Saadatmandi, *Application of the Exp-function method for solving a partial differential equation arising in biology and population genetics*, Internat. J. Numer. Methods Heat Fluid Flow **21** (2011), no. 6-7, 736–753. https://doi.org/10.1108/09615531111148482

[13] H. Demiray, *A complex travelling wave solution to the kdv–burgers equation*, Phys. Lett. A **344** (2005), no. 6, 418–422.

[14] E. Fan, *Extended tanh-function method and its applications to nonlinear equations*, Phys. Lett. A **277** (2000), no. 4-5, 212–218. https://doi.org/10.1016/S0375-9601(00)00725-8

[15] ———, *Two new applications of the homogeneous balance method*, Phys. Lett. A **265** (2000), no. 5-6, 353–357. https://doi.org/10.1016/S0375-9601(00)00010-4

[16] E. Fan and Y. C. Hon, *Applications of extended tanh method to "special" types of nonlinear equations*, Appl. Math. Comput. **141** (2003), no. 2-3, 351–358. https://doi.org/10.1016/S0096-3003(02)00260-6

[17] E. Fan, J. Zhang, and B. Y. C. Hon, *A new complex line soliton for the two-dimensional KdV-Burgers equation*, Phys. Lett. A **291** (2001), no. 6, 376–380. https://doi.org/10.1016/S0375-9601(01)00707-1

[18] Z. Feng and Y. Huang, *Approximate solution of the Burgers-Korteweg-de Vries equation*, Commun. Pure Appl. Anal. **6** (2007), no. 2, 429–440. https://doi.org/10.3934/cpaa.2007.6.429

[19] B. Ibis and M. Bayram, *Approximate solutions some nonlinear evolutions equations by using the reduced differential transform method*, Int. J. Appl. Math. Res. **3** (2012), no. 1, 288–302.

[20] H. Jia and W. Xu, *Solitons solutions for some nonlinear evolution equations*, Appl. Math. Comput. **217** (2010), no. 4, 1678–1687. https://doi.org/10.1016/j.amc.2009.09.061

[21] R. S. Johnson, *A non-linear equation incorporating damping and dispersion*, J. Fluid Mech. **42** (1970), 49–60. https://doi.org/10.1017/S0022112070001064

[22] N. A. Kudryashov and D. I. Sinelshchikov, *Extended models of non-linear waves in liquid with gas bubbles*, Int. J. Non Linear Mech. **63** (2014), 31–38.

[23] H. Leblond and D. Mihalache, *Ultrashort light bullets described by the two-dimensional sine-gordon equation*, Phys. Rev. A **81** (2010), no. 6, 063815.

[24] W. Malfliet, *Solitary wave solutions of nonlinear wave equations*, Amer. J. Phys. **60** (1992), no. 7, 650–654. https://doi.org/10.1119/1.17120

[25] Z. Odibat, *A Riccati equation approach and travelling wave solutions for nonlinear evolution equations*, Int. J. Appl. Comput. Math. **3** (2017), no. 1, 1–13. https://doi.org/10.1007/s40819-015-0085-z

[26] A. H. Salas S and C. A. Gómez S, *Exact solutions for a third-order KdV equation with variable coefficients and forcing term*, Math. Probl. Eng. **2009** (2009), Art. ID 737928, 13 pp. https://doi.org/10.1155/2009/737928

[27] A. M. Wazwaz, *The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations*, Appl. Math. Comput. **184** (2007), no. 2, 1002–1014. https://doi.org/10.1016/j.amc.2006.07.002

[28] ———, *Partial differential equations and solitary waves theory*, Nonlinear Physical Science, Higher Education Press, Beijing, 2009. https://doi.org/10.1007/978-3-642-00251-9

[29] ———, *Partial differential equations and solitary waves theory*, Nonlinear Physical Science, Higher Education Press, Beijing, 2009. https://doi.org/10.1007/978-3-642-00251-9

[30] L. Wazzan, *A modified tanh-coth method for solving the KdV and the KdV-Burgers' equations*, Commun. Nonlinear Sci. Numer. Simul. **14** (2009), no. 2, 443–450. https://doi.org/10.1016/j.cnsns.2007.06.011

[31] L. V. Wijngaarden, *On the equations of motion for mixtures of liquid and gas bubbles*, J. Fluid Mech. **33** (1968), no. 3, 465–474.

[32] N. Zahibo, E. Pelinovsky, and A. Sergeeva, *Weakly damped KdV soliton dynamics with the random force*, Chaos Solitons Fractals **39** (2009), no. 4, 1645–1650.

[33] S. Zhang, *Exp-function method exactly solving the KdV equation with forcing term*, Appl. Math. Comput. **197** (2008), no. 1, 128–134. https://doi.org/10.1016/j.amc.2007.07.041

[34] S. Zhang and J. Li, *Soliton solutions and dynamical evolutions of a generalized akns system in the framework of inverse scattering transform*, Optik **137** (2017), 228–237.

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