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L^p ($p \geq 1$) SOLUTIONS OF MULTIDIMENSIONAL BSDES WITH TIME-VARYING QUASI-HÖLDER CONTINUITY GENERATORS IN GENERAL TIME INTERVALS

Xiao Lishun and Fan Shengjun

Abstract. The objective of this paper is solving multidimensional backward stochastic differential equations with general time intervals, in L^p ($p \geq 1$) sense, where the generator g satisfies a time-varying Osgood condition in y, a time-varying quasi-Hölder continuity condition in z, and its ith component depends on the ith row of z. Our result strengthens some existing works even for the case of finite time intervals.

1. Introduction

Let k and d be two given positive integers, $(\Omega, \mathcal{F}, \mathbf{P})$ a completed probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$, and $(\mathcal{F}_t)_{t\geq 0}$ the natural σ-algebra filtration generated by $(B_t)_{t\geq 0}$. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and complete and that $\mathcal{F}_T = \mathcal{F}$ for a given terminal time T satisfying $0 \leq T \leq \infty$. In this paper, we are concerned with the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$(1) \qquad y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s\,\mathrm{d}B_s, \quad t \in [0, T],$$

where the terminal condition ξ is an $\mathcal{F}_T$-measurable and k-dimensional random vector, and the generator $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \mapsto \mathbf{R}^k$ is $(\mathcal{F}_t)$-progressively measurable for each (y, z). The solution $(y_t, z_t)_{t\in[0,T]}$ is a pair of $(\mathcal{F}_t)$-progressively measurable processes. We also denote by BSDE (ξ, T, g) the BSDE with parameters (ξ, T, g).

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Pardoux and Peng [20] initially introduced the nonlinear multidimensional BSDEs, proving an existence and uniqueness result for square-integrable solutions when the generator g is Lipschitz continuous in (y, z) uniformly with respect to t, ξ and $\{g(t, 0, 0)\}_{t\in[0,T]}$ are square-integrable, and the terminal time T is a finite constant. Since then, a considerable amount of literature on this field have been published, and many applications of BSDEs in mathematical finance, stochastic control, partial differential equations and so on (See El Karoui, Peng and Quenez [8] for details) have been explored.

Generally, there are three main directions to extend the existence and uniqueness result: (i) weakening the conditions of the generator g in (y, z); (ii) studying non-square integrable solutions or reformulating a more general solution space; (iii) improving the time interval from the finite case to the general case. Other ways we refer to Buckdahn, Engelbert and Ruascanu [4], Bouchard, Elie and Reveillac [1]. We note that different ways above require different tools and techniques.

Concerning the first direction, we would like to mention the following works: Mao [19], Lepeltier and San Martin [18], Kobylanski [17], Hamadène [16], Briand, Lepetier and San Mrtin [3] and Fan, Jiang and Davison [11], see also the references therein. Particularly, by virtue of some results on deterministic backward differential equations, Hamadène [16] proved the existence for solutions of multidimensional BSDEs when the generator g satisfies a Osgood condition in y, a uniform continuity condition in z and the ith component of g depends on the ith row of z. Furthermore, by establishing an estimate of a linear-growth function, Fan, Jiang and Davison [11] obtained the uniqueness result under the same assumptions as those in Hamadène [16]. These works dealt only with square-integrable parameters or L^2 solutions.

To our knowledge, along the direction (ii), El Karoui, Peng and Quenez [8] first studied the existence and uniqueness result for L^p $(p > 1)$ solutions of multidimensional BSDEs when ξ and $\{g(t, 0, 0)\}_{t\in[0,T]}$ are p-integrable, see also Briand, Delyon, Hu, Pardoux and Stoica [2], Chen [5], Fan [9], etc. Particularly, Briand, Delyon, Hu, Pardoux and Stoica [2] also investigated the existence and uniqueness for L^1 solutions of multidimensional BSDEs when the generator g is monotonic in y, Lipschitz continuous and of sublinear growth in z. Based on this result, Fan and Liu [13] established an existence and uniqueness for L^1 solutions of one dimensional BSDEs whose generator g is Lipschitz continuous in y and Hölder continuous in z; and Tian, Jiang and Shi [21] further extended this result, in which the generator g satisfies a Osgood condition in y and a quasi-Hölder continuity condition in z. All the aforementioned works, however, dealt only with the BSDEs with finite time intervals.

Many works have also been made along the direction (iii), see for example Chen and Wang [6], Fuhrman and Tessitore [15]. Currently, many researchers concentrate on synthesizing the three directions, such as Fan, Jiang and Tian [12], Fan and Jiang [10], Fan and Dong [7], Wang, Liao and Fan [22]. In particular, Xiao, Fan and Xu [23] studied the existence and uniqueness of

L^p ($p \geq 1$) solutions of multidimensional BSDEs with general time intervals, which extends the result of Briand, Delyon, Hu, Pardoux and Stoica [2] to the general time interval case. And Fan, Wang and Xiao [14] obtained the existence and uniqueness for L^2 solutions of multidimensional BSDEs with uniformly continuous generators in general time intervals, improving the results of Hamadène [16] and Fan, Jiang and Davison [11].

Motivated by these results, in this paper we establish an existence and uniqueness result of L^p ($p \geq 1$) solutions for multidimensional BSDEs with general time intervals, where the generator g satisfies a time-varying Osgood condition in y, a time-varying quasi-Hölder continuity condition in z and the ith component of g depends on the ith row of z (See Theorem 3.2 in Section 3). The whole idea can be viewed as a synthesis of directions (i) – (iii). This result generalizes some known works including Hamadène [16], Fan, Jiang and Davison [11], Fan and Liu [13], Tian, Jiang and Shi Tian, Jiang and Shi [21] and Fan, Wang and Xiao [14], even for the case of finite time intervals.

The remainder of this paper is organized as follows. Section 2 contains some usual notations and useful propositions. Section 3 is mainly devoted to the statement of the existence and uniqueness result of L^p ($p \geq 1$) solutions and some examples are also provided. Section 4 gives the proof of our main result.

2. Notations and preliminaries

In this paper, the Euclidean norm of a vector $y \in \mathbf{R}^k$ will be defined by $|y|$, and for a $k \times d$ matrix z, we define $|z| = \sqrt{Tr(zz^*)}$, where and hereafter z^* represents the transpose of z. Let $\langle x, y\rangle$ represent the scalar product of x, $y \in \mathbf{R}^k$. For each real number $p > 0$, let $L^p(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^k)$ (or $L^p(\mathcal{F}_T; \mathbf{R}^k)$ for simplicity) be the set of $\mathbf{R}^k$-valued and $\mathcal{F}_T$-measurable random variables ξ such that $\|\xi\|_{L^p}^p := \mathbf{E}[|\xi|^p] < \infty$ and let $\mathcal{S}^p(0, T; \mathbf{R}^k)$ (or $\mathcal{S}^p$ for simplicity) denote the set of $\mathbf{R}^k$-valued, $(\mathcal{F}_t)$-adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbf{E}\left[\sup_{t\in[0,T]} |Y_t|^p\right]\right)^{1\wedge 1/p} < \infty.$$

If $p \geq 1$, $\|\cdot\|_{\mathcal{S}^p}$ is a norm on $\mathcal{S}^p$ and if $p \in (0, 1)$, $(Y, Y') \mapsto \|Y - Y'\|_{\mathcal{S}^p}$ defines a distance on $\mathcal{S}^p$. Under this metric, $\mathcal{S}^p$ is complete. Moreover, for $p > 0$, let $\mathrm{M}^p(0, T; \mathbf{R}^{k\times d})$ (or M^p) denote the set of (equivalent classes of) $(\mathcal{F}_t)$-progressively measurable $\mathbf{R}^{k\times d}$-valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{\mathrm{M}^p} := \left\{\mathbf{E}\left[\left(\int_0^T |Z_t|^2\,\mathrm{d}t\right)^{p/2}\right]\right\}^{1\wedge 1/p} < \infty.$$

For any $p \geq 1$, M^p is a Banach space endowed with this norm and for any $p \in (0, 1)$, M^p is a complete metric space with the resulting distance. Besides, we say that a continuous process $(Y_t)_{t\in[0,T]}$ belongs to class (D) if the family $\{Y_\tau : \tau \in \Sigma_T\}$ is uniformly integrable, where Σ_T stands for the set of all $(\mathcal{F}_t)$-stopping times τ such that $\tau \leq T$. For a process $(Y_t)_{t\in[0,T]}$ belonging to class

(D), we define $\|Y\|_1 := \sup\{\mathbf{E}[|Y_\tau|] : \tau \in \Sigma_T\}$. The space of $(\mathcal{F}_t)$-progressively measurable continuous processes which belong to class (D) is complete under this norm.

Finally, let $\mathbf{S}$ be the set of all non-decreasing linear-growth continuous functions $\rho(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with $\rho(0) = 0$ and $\rho(x) > 0$ for all $x > 0$, where and hereafter $\mathbf{R}^+ := [0, \infty)$. We denote the linear-growth constant for $\rho(\cdot) \in \mathbf{S}$ by $A > 0$, i.e., $\rho(x) \le A(1 + x)$ for all $x \in \mathbf{R}^+$. In this paper, we will use the following definition.

Definition. A pair of processes $(y_t, z_t)_{t\in[0,T]}$ taking values in $\mathbf{R}^k \times \mathbf{R}^{k\times d}$ is called a solution of BSDE (1), if $(y_t, z_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)$-progressively measurable and satisfies that $\mathrm{d}\mathbf{P}$-a.s., $t \mapsto y_t$ is continuous, $t \mapsto z_t$ belongs to $L^2(0, T)$, $t \mapsto g(t, y_t, z_t)$ belongs to $L^1(0, T)$ and BSDE (1) holds for each $t \in [0, T]$.

Next we present an a priori estimate for solutions of multidimensional BSDEs, which will play an important role in the proof of our main result. To state it, the following assumption is necessary, where $0 \le T \le \infty$ and $p > 0$.

(H) There exist two non-negative functions $\mu(\cdot)$, $\lambda(\cdot) : [0, T] \mapsto \mathbf{R}^+$ with $\int_0^T \left(\mu(t) + \lambda^2(t)\right) \mathrm{d}t < \infty$ such that $\mathrm{d}\mathbf{P} \times \mathrm{d}t$- a.e., for each $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,

$$\langle y, g(t, y, z)\rangle \le \mu(t)|y|^2 + \lambda(t)|y||z| + f_t|y|,$$

where $(f_t)_{t\in[0,T]}$ is a non-negative and $(\mathcal{F}_t)$-progressively measurable process with $\mathbf{E}[(\int_0^T f_t \mathrm{d}t)^p] < \infty$.

Proposition 2.1. *Assume that $0 \le T \le \infty$, g satisfies assumption* (H), *$(y_t, z_t)_{t\in[0,T]}$ is a solution of BSDE* (1) *such that $(y_t)_{t\in[0,T]}$ belongs to $\mathcal{S}^p$ with some $p > 0$. Then there exist two constants $C_p^1 > 0$ and $C_p^2 > 0$, where C_p^1 depends on p, $\int_0^T \mu(t)\,\mathrm{d}t$ and $\int_0^T \lambda^2(t)\,\mathrm{d}t$, and C_p^2 depends only on p, such that* $\mathrm{d}\mathbf{P}$-a.s., *for each $0 \le r \le t \le T$,*

$$\mathbf{E}\left[\left(\int_t^T |z_s|^2\,\mathrm{d}s\right)^{p/2}\middle|\mathcal{F}_r\right] \le C_p^1 \mathbf{E}\left[\sup_{s\in[t,T]} |y_s|^p \middle|\mathcal{F}_r\right] + C_p^2 \mathbf{E}\left[\left(\int_t^T f_s\,\mathrm{d}s\right)^p\middle|\mathcal{F}_r\right]. \tag{2}$$

Furthermore, if $p > 1$, then there exists a constant $C_p^3 > 0$ depending on p, $\int_0^T \mu(t)\,\mathrm{d}t$ and $\int_0^T \lambda^2(t)\,\mathrm{d}t$, such that $\mathrm{d}\mathbf{P}$-a.s., *for each $0 \le r \le t \le T$,*

$$\begin{aligned}&\mathbf{E}\left[\sup_{s\in[t,T]} |y_s|^p \middle|\mathcal{F}_r\right] + \mathbf{E}\left[\left(\int_t^T |z_s|^2\,\mathrm{d}s\right)^{p/2}\middle|\mathcal{F}_r\right]\\ &\le C_p^3\left(\mathbf{E}\left[|\xi|^p\middle|\mathcal{F}_r\right] + \mathbf{E}\left[\left(\int_t^T f_s\,\mathrm{d}s\right)^p\middle|\mathcal{F}_r\right]\right).\end{aligned} \tag{3}$$

Remark 2.2. Proposition 2.1 comes from Lemmas 2.3 and 2.4 in Xiao, Fan and Xu [23] with the only difference lying on the constants appearing in (2). So here we omit its proof. The fact that C_p^2 depends only on p will be used in the proof of our main result.

In the sequel, we introduce the following assumptions, where $0 \le T \le \infty$ and $p \ge 1$.

(A1) $\mathbf{E}\big[|\xi|^p + \big(\int_0^T |g(t,0,0)|\,\mathrm{d}t\big)^p\big] < \infty$;

(A2) $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each $z \in \mathbf{R}^{k\times d}$, $y \mapsto g(t,y,z)$ is continuous;

(A3) g has a general growth in y, i.e., for each $r \in \mathbf{R}^+$, $\mathbf{E}\big[\int_0^T \psi_r(t)\,\mathrm{d}t\big] < \infty$, where

$$\psi_r(t) := \sup_{|y|\le r} |g(t,y,0) - g(t,0,0)|;$$

(A4) There exists a deterministic function $\overline{u}(\cdot) : [0,T] \mapsto \mathbf{R}^+$ with $\int_0^T \overline{u}(t)\,\mathrm{d}t < \infty$ such that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each y_1, $y_2 \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,

$$\langle y_1 - y_2, g(t,y_1,z) - g(t,y_2,z)\rangle \le \overline{u}(t)|y_1 - y_2|^2;$$

(A5) There exists a deterministic function $\overline{v}(\cdot) : [0,T] \mapsto \mathbf{R}^+$ with $\int_0^T \overline{v}^2(t)\,\mathrm{d}t < \infty$ such that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each $y \in \mathbf{R}^k$ and z_1, $z_2 \in \mathbf{R}^{k\times d}$,

$$|g(t,y,z_1) - g(t,y,z_2)| \le \overline{v}(t)|z_1 - z_2|;$$

(A6) There exist a constant $\alpha \in (0,1)$ and a deterministic function $\gamma(t) : [0,T] \mapsto \mathbf{R}^+$ with $\int_0^T \big(\gamma(t)+\gamma^{2/(2-\alpha)}(t)\big)\,\mathrm{d}t < \infty$ such that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,

$$|g(t,y,z) - g(t,y,0)| \le \gamma(t)(1 + |z|^\alpha).$$

Using a similar argument to that in the proofs of Theorems 3.1 and 4.1 in Xiao, Fan and Xu [23] with some small changes due to the difference of the growth condition of g in z, we can deduce the following existence and uniqueness result, whose proof is omitted here.

Proposition 2.3. *Assume that $0 \le T \le \infty$, $p \ge 1$ and* (A1)–(A5) *hold. We have that*

(i) *if $p > 1$, then BSDE* (1) *admits a unique solution $(y_t, z_t)_{t\in[0,T]}$ in $\mathcal{S}^p \times \mathrm{M}^p$;*

(ii) *if $p = 1$ and g also satisfies* (A6)*, then BSDE* (1) *admits a solution $(y_t, z_t)_{t\in[0,T]} \in \bigcap_{\beta\in(0,1)}(\mathcal{S}^\beta \times \mathrm{M}^\beta)$ such that $(y_t)_{t\in[0,T]}$ belongs to class* (D)*, which is unique in $\mathcal{S}^\beta \times \mathrm{M}^\beta$ for each $\beta \in (\alpha,1)$.*

3. Existence and uniqueness result for L^p $(p \ge 1)$ solutions

In this section we will state the existence and uniqueness result for solutions of BSDE (1) and give two examples to compare it with some existing works. Let us first introduce the following assumptions with respect to the generator g of BSDE (1), where $0 \le T \le \infty$.

(H1) g satisfies a time-varying Osgood condition in y, i.e., there exist a deterministic function $u(\cdot) : [0,T] \mapsto \mathbf{R}^+$ with $\int_0^T u(t)\,\mathrm{d}t < \infty$ and a function $\rho(\cdot) \in \mathbf{S}$ with $\int_{0^+} 1/\rho(u)\,\mathrm{d}u = \infty$ such that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each $y_1, y_2 \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,

$$|g(\omega,t,y_1,z) - g(\omega,t,y_2,z)| \le u(t)\rho(|y_1-y_2|);$$

(H2)$_\alpha$ g satisfies a time-varying quasi-Hölder continuity condition in z, i.e., there exist a constant $\alpha \in (0,1]$, a deterministic function $v(\cdot) : [0,T] \mapsto \mathbf{R}^+$ with $\int_0^T \big(v(t)+v^2(t)\big)\,\mathrm{d}t < \infty$ and a function $\phi(\cdot) \in \mathbf{S}$ such that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each $y \in \mathbf{R}^k$ and $z_1, z_2 \in \mathbf{R}^{k\times d}$,

$$|g(\omega,t,y,z_1) - g(\omega,t,y,z_2)| \le v(t)\phi(|z_1-z_2|^\alpha);$$

(H3) For any $i = 1,\dots,k$, the ith component of g, denoted by $g_i(\omega,t,y,z)$, depends only on ${}^i z$.

In the remaining of this paper, we put an i at upper left of $y \in \mathbf{R}^k$, $z \in \mathbf{R}^{k\times d}$ to represent the ith component of y and the ith row of z, like ${}^i y$ and ${}^i z$.

Remark 3.1. We provide some remarks with respect to the assumption (H2)$_\alpha$ as follows:

(i) In the case of $\alpha = 1$ and $\phi(x) \le Ax$ for all $x \in \mathbf{R}^+$, we do not need the condition that $\int_0^T v(t)\,\mathrm{d}t < \infty$.

(ii) When $\alpha = 1$, (H2)$_\alpha$ becomes the time-varying uniform continuity condition, which is obviously weaker than (A5). And, when $\phi(x) = x$, (H2)$_\alpha$ becomes the time-varying Hölder continuity condition. This is the reason that we call (H2)$_\alpha$ a time-varying quasi-Hölder continuity condition.

(iii) The larger the α, the weaker the condition (H2)$_\alpha$, i.e., for each $0 < \alpha < \beta \le 1$, (H2)$_\alpha \Rightarrow$ (H2)$_\beta$. Indeed, if (H2)$_\alpha$ holds with $\phi(\cdot) \in \mathbf{S}$, then $\overline{\phi}(\cdot) := \phi(|\cdot|^{\alpha/\beta})$ is non-decreasing and continuous, and satisfies $\overline{\phi}(x) > 0$ for all $x > 0$ and $\overline{\phi}(0) = 0$. On the other hand, since for all $x \in \mathbf{R}^+$, $\overline{\phi}(x) \le Ax^{\alpha/\beta} + A \le Ax + 2A$, then $\overline{\phi}(\cdot)$ is at most of linear-growth. Hence, $\overline{\phi}(\cdot) \in \mathbf{S}$, and (H2)$_\beta$ holds with $\overline{\phi}(\cdot)$ due to the fact that $\overline{\phi}(|z_1-z_2|^\beta) = \phi(|z_1-z_2|^\alpha)$. Clearly, the contrary of the above statement is not true in general.

(iv) For each $\alpha \in (0,1]$ and $\phi(\cdot) \in \mathbf{S}$, we can deduce from (iii) with taking $\beta = 1$ that $\phi(|\cdot|^\alpha) \in \mathbf{S}$, and then it admits an estimate like $\phi(\cdot)$ (see the inequality (4) in Fan, Jiang and Davison [11] for details), i.e., for each $x \in \mathbf{R}^+$ and $n \ge 1$,

$$\phi(x^\alpha) \le (n+2A)x + \phi\left(\left(\frac{2A}{n+2A}\right)^\alpha\right). \tag{4}$$

The main result of this paper is the following Theorem 3.2.

Theorem 3.2. *Assume that* $0 \le T \le \infty$*,* $p \ge 1$*,* $\alpha \in (0,1]$*, and* (A1)*,* (H1)*,* (H2)$_\alpha$ *and* (H3) *are in force. We have*

(i) *if* $p > 1$*, then BSDE* (1) *admits a unique solution* $(y_t, z_t)_{t\in[0,T]}$ *in* $\mathcal{S}^p \times \mathrm{M}^p$;

(ii) *if $p = 1$ and $\alpha \in (0,1)$, then BSDE* (1) *admits a solution $(y_t, z_t)_{t\in[0,T]} \in \bigcap_{\beta\in(0,1)}(\mathcal{S}^\beta \times \mathrm{M}^\beta)$ such that $(y_t)_{t\in[0,T]}$ belongs to class* (D)*, which is unique in $\mathcal{S}^\beta \times \mathrm{M}^\beta$ for each $\beta \in (\alpha, 1)$.*

We provide two examples to illustrate the applications of Theorem 3.2.

Example 3.3. Let $0 \le T < \infty$, $p \ge 1$, $\alpha \in (0, 1]$, $k = 1$ and $\xi \in L^p(\mathcal{F}_T; \mathbf{R})$. For each $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d}$, define the generator g by

$$g(\omega, t, y, z) = |\ln t| h(|y|) + \frac{|z|^\alpha}{\sqrt[4]{t}} + |B_t(\omega)|,$$

where $h(x) := (-x \ln x)\mathbf{1}_{0\le x\le\delta} + \big(h'(\delta-)(x-\delta) + h(\delta)\big)\mathbf{1}_{x>\delta}$, with δ small enough. Since $h(0) = 0$ and $h(\cdot)$ is concave and increasing, we have $h(x_1+x_2) \le h(x_1) + h(x_2)$ for all x_1 and $x_2 \in \mathbf{R}^+$, which implies that $|h(x_1) - h(x_2)| \le h(|x_1-x_2|)$. Moreover, noticing that $\int_{0^+} 1/(-x\ln x)\,\mathrm{d}x = \infty$, we know that the generator g satisfies assumptions (A1), (H1), (H2)$_\alpha$ and (H3) with $u(t) = |\ln t|$, $v(t) = 1/\sqrt[4]{t}$, $\rho(x) = h(x)$ and $\phi(x) = x$. Then by Theorem 3.2 we know that if $p > 1$, then BSDE (1) admits a unique solution $(y_t, z_t)_{t\in[0,T]}$ in $\mathcal{S}^p \times \mathrm{M}^p$; and if $p = 1$ and $\alpha \in (0, 1)$, then BSDE (1) admits a solution $(y_t, z_t)_{t\in[0,T]} \in \bigcap_{\beta\in(0,1)}(\mathcal{S}^\beta \times \mathrm{M}^\beta)$ such that $(y_t)_{t\in[0,T]}$ belongs to class (D), which is unique in $\mathcal{S}^\beta \times \mathrm{M}^\beta$ for each $\beta \in (\alpha, 1)$.

We mention that Hamadène [16], Fan, Jiang and Davison [11] and Fan, Wang and Xiao [14] dealt only with the case of square-integrable parameters, Fan and Liu [13] and Tian, Jiang and Shi [21] dealt only with the case of integrable parameters and $u(t)$ and $v(t)$ being bounded, and the required conditions for g in z of Theorem 4.2 in Briand, Delyon, Hu, Pardoux and Stoica [2] and (i) in Proposition 2.3 are the Lipschitz continuity. Thus, since for each $\alpha \in (0, 1)$, the function $|x|^\alpha$ is only uniformly continuous or α-Hölder continuous rather than Lipschitz continuous on $[0, \infty)$, the existence and uniqueness result in Example 3.3 can not be obtained by these aforementioned works. In addition, Example 3.3 also illustrates that in our framework, $u(t)$ and $v(t)$ appearing in (H1) and (H2)$_\alpha$ may be unbounded and their integrability is the only requirement.

Example 3.4. Let $0 \le T \le \infty$, $p = 1$ and $\xi \in L^1(\mathcal{F}_T; \mathbf{R}^k)$. For each $i = 1, \dots, k$ and $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d}$, define the generator $g = (g_1, \dots, g_k)$ by

$$g_i(\omega, t, y, z) = t^2 \mathrm{e}^{-t} h(|y|) + \frac{1}{1+t^2}\big(\sqrt[3]{|{}^i z|} + \sqrt[4]{|{}^i z|}\big) + |B_t(\omega)|,$$

where $h(x)$ is defined in Example 3.3. It is not very hard to verify that the generator g fulfills assumptions (A1), (H1), (H2)$_\alpha$ and (H3) with $u(t) = t^2\mathrm{e}^{-t}$, $v(t) = 1/(1 + t^2)$, $\rho(\cdot) = h(\cdot)$, $\alpha = 1/3$ and $\phi(x) := x + x^{3/4}$ for each $x \in \mathbf{R}^+$. It then follows from (ii) in Theorem 3.2 that BSDE (1) admits a solution

$(y_t, z_t)_{t\in[0,T]} \in \bigcap_{\beta\in(0,1)}(\mathcal{S}^\beta \times \mathrm{M}^\beta)$ such that $(y_t)_{t\in[0,T]}$ belongs to class (D), which is unique in $\mathcal{S}^\beta \times \mathrm{M}^\beta$ for each $\beta \in (1/3, 1)$.

It should be pointed out that, to our best knowledge, the existence and uniqueness result for L^1 solutions in Example 3.4 can not be obtained by any existing results including Briand, Delyon, Hu, Pardoux and Stoica [2], Fan and Liu [13], Tian, Jiang and Shi [21] and (ii) in Proposition 2.3. The reason lies in that Theorem 6.3 in Briand, Delyon, Hu, Pardoux and Stoica [2] and Proposition 2.3 require the Lipschitz condition for g in z, Fan and Liu [13] and Tian, Jiang and Shi [21] only consider one dimensional BSDEs with finite time intervals, and Fan and Liu [13] also requires the Lipschitz continuity of g in y.

4. Proof of the main result

In this section we will prove our main result — Theorem 3.2. Before the proof we first establish the following Propositions 4.1 and 4.4 for preparation.

Proposition 4.1. *Assume that* $\alpha \in (0,1]$, $p > 1$, $0 < q \le p$, $(y_t^n, z_t^n)_{t\in[0,T]}$ *is a solution of BSDE* (ξ, T, g^n) *in* $\mathcal{S}^q \times \mathrm{M}^q$ *for each* $n \ge 1$, *and* (H1), $(\mathrm{H2})_\alpha$ *and* (H3) *hold for* g *and* g^n *with the same parameters* $u(\cdot)$, $v(\cdot)$, $\rho(\cdot)$ *and* $\phi(\cdot)$. *If for each* $n \ge 1$, $(y_t^n - y_t^1)_{t\in[0,T]}$ *belongs to* $\mathcal{S}^p$, *and* $\{g^n\}_{n=1}^\infty$ *converges uniformly to* g, *i.e., there exists a non-increasing function sequence* $a_n(\cdot) : [0,T] \mapsto \mathbf{R}^+$ *with* $\int_0^T a_n(t)\,\mathrm{d}t \to 0$ *as* $n \to \infty$ *such that* $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., *for each* $n \ge 1$, $y \in \mathbf{R}^k$ *and* $z \in \mathbf{R}^{k\times d}$,

$$|g^n(t,y,z) - g(t,y,z)| \le a_n(t), \tag{5}$$

then there exists a solution $(y_t, z_t)_{t\in[0,T]} \in \mathcal{S}^q \times \mathrm{M}^q$ *of BSDE* (ξ, T, g) *such that*

$$\lim_{n\to\infty} \left(\|y^n - y\|_{\mathcal{S}^q} + \|z^n - z\|_{\mathrm{M}^q}\right) = 0.$$

Proof. Let the assumptions of Proposition 4.1 hold. For each n, $m \ge 1$, set $\hat{y}_\cdot^{n,m} := y_\cdot^n - y_\cdot^m$ and $\hat{z}_\cdot^{n,m} := z_\cdot^n - z_\cdot^m$. Then $(\hat{y}_t^{n,m}, \hat{z}_t^{n,m})_{t\in[0,T]}$ with $\hat{y}_\cdot^{n,m} \in \mathcal{S}^p$ is a solution of the following BSDE:

$$\hat{y}_t^{n,m} = \int_t^T \hat{g}^{n,m}(s, \hat{y}_s^{n,m}, \hat{z}_s^{n,m})\,\mathrm{d}s - \int_t^T \hat{z}_s^{n,m}\,\mathrm{d}B_s, \quad t \in [0,T],$$

where $\hat{g}^{n,m}(t,y,z) := g^n(t, y + y_t^m, z + z_t^m) - g^m(t, y_t^m, z_t^m)$ for each $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$. It then follows from (H1), $(\mathrm{H2})_\alpha$ and (5) that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$, adding and subtracting the term $g^n(t, y_t^m, z_t^m)$,

$$\begin{aligned} \langle y, \hat{g}^{n,m}(t,y,z)\rangle &\le u(t)|y|\rho(|y|) + v(t)|y|\phi(|z|^\alpha) + |y|(a_n(t) + a_m(t)) \\ &\le Au(t)|y|^2 + Av(t)|y||z| + |y|\big(A(u(t) + 2v(t)) + 2a_1(t)\big), \end{aligned} \tag{6}$$

which indicates that assumption (H) is satisfied by the generator $\hat{g}^{n,m}(t,y,z)$ with $\mu(t) = Au(t)$, $\lambda(t) = Av(t)$ and $f_t = A(u(t) + 2v(t)) + 2a_1(t)$. Thus, Proposition 2.1 leads to the existence of a uniform constant $C_1 > 0$ such that for each n, $m \geq 1$,

$$|y_t^n - y_t^m| \leq C_1, \quad \forall\, t \in [0,T] \quad \text{and} \quad \hat{z}_\cdot^{n,m} \in \mathrm{M}^p. \tag{7}$$

The following proof will be split into three steps.

First Step: In this step we prove the convergence of $\{y_\cdot^n\}_{n=1}^\infty$ in $\mathcal{S}^q$.

Firstly, assumption (H3) and Tanaka's formula give that, for each n, $m \geq 1$, $i = 1, \ldots, k$ and $t \in [0,T]$,

$$|{}^i\hat{y}_t^{n,m}| \leq \int_t^T \operatorname{sgn}({}^i\hat{y}_s^{n,m})\hat{g}_i^{n,m}(s, \hat{y}_s^{n,m}, {}^i\hat{z}_s^{n,m})\,\mathrm{d}s - \int_t^T \langle \operatorname{sgn}({}^i\hat{y}_s^{n,m})({}^i\hat{z}_s^{n,m})^*, \,\mathrm{d}B_s\rangle,$$

where $\hat{g}_i^{n,m}$ represents the ith component of $\hat{g}^{n,m}$. Furthermore, due to (H1), $(\mathrm{H2})_\alpha$ and (5), we can get that, $\mathrm{d}\mathbf{P}\times\mathrm{d}s$ - a.e.,

$$|\hat{g}_i^{n,m}(s, \hat{y}_s^{n,m}, {}^i\hat{z}_s^{n,m})| \leq u(s)\rho(|\hat{y}_s^{n,m}|) + v(s)\phi(|{}^i\hat{z}_s^{n,m}|^\alpha) + a_n(s) + a_m(s).$$

Thus, combining the previous two inequalities and (4) yields that for each n, m, $l \geq 1$, $i = 1, \ldots, k$ and $t \in [0,T]$,

$$\begin{aligned}|{}^i\hat{y}_t^{n,m}| \leq \tilde{a}_{n,m}^l &+ \int_t^T \big(u(s)\rho(|\hat{y}_s^{n,m}|) + (l+2A)v(s)|{}^i\hat{z}_s^{n,m}|\big)\,\mathrm{d}s \\ &- \int_t^T \langle \operatorname{sgn}({}^i\hat{y}_s^{n,m})({}^i\hat{z}_s^{n,m})^*, \,\mathrm{d}B_s\rangle,\end{aligned}$$

where

$$\tilde{a}_{n,m}^l := \int_0^T (a_n(s) + a_m(s))\,\mathrm{d}s + \mathbf{1}_D\phi\left(\left(\frac{2A}{l+2A}\right)^\alpha\right)\int_0^T v(s)\,\mathrm{d}s,$$

and here and later on $\mathbf{1}_D = 0$ if $\alpha = 1$ and $\phi(x) \leq Ax$ for all $x \in \mathbf{R}^+$, otherwise, $\mathbf{1}_D = 1$.

Now for each n, m, $l \geq 1$, $i = 1, 2, \ldots, k$ and $t \in [0,T]$, let

$${}^ie_t^{n,m,l} := (l+2A)\frac{\operatorname{sgn}({}^i\hat{y}_t^{n,m})({}^i\hat{z}_t^{n,m})^*}{|{}^i\hat{z}_t^{n,m}|}\mathbf{1}_{|{}^i\hat{z}_t^{n,m}|\neq 0}.$$

Then, $({}^ie_t^{n,m,l})_{t\in[0,T]}$ is an $\mathbf{R}^d$-valued, bounded and $(\mathcal{F}_t)$-progressively measurable process. It follows from Girsanov's theorem that ${}^iB_t^{n,m,l} := B_t - \int_0^t {}^ie_s^{n,m,l}v(s)\,\mathrm{d}s$, $t \in [0,T]$, is a d-dimensional Brownian motion under the probability $\mathbf{P}_{l,i}^{n,m}$ on $(\Omega, \mathcal{F}_T)$ defined by

$$\frac{\mathrm{d}\mathbf{P}_{l,i}^{n,m}}{\mathrm{d}\mathbf{P}} = \exp\left\{\int_0^T v(s)\langle({}^ie_s^{n,m,l})^*, \,\mathrm{d}B_s\rangle - \frac{1}{2}\int_0^T v^2(s)|{}^ie_s^{n,m,l}|^2\,\mathrm{d}s\right\}.$$

Thus, for each n, m, $l \geq 1$, $i = 1, \ldots, k$ and $t \in [0,T]$,

$$|{}^i\hat{y}_t^{n,m}| \leq \tilde{a}_{n,m}^l + \int_t^T u(s)\rho(|\hat{y}_s^{n,m}|)\,\mathrm{d}s - \int_t^T \langle \operatorname{sgn}({}^i\hat{y}_s^{n,m})({}^i\hat{z}_s^{n,m})^*, \,\mathrm{d}^iB_s^{n,m,l}\rangle. \tag{8}$$

Moreover, since $\hat{z}_\cdot^{n,m} \in \mathrm{M}^p$ due to (7), the process

$$\left(\int_0^t \langle \mathrm{sgn}({}^i\hat{y}_s^{n,m})({}^i\hat{z}_s^{n,m})^*,\, \mathrm{d}^i B_s^{n,m,l} \rangle \right)_{t\in[0,T]}$$

is an $(\mathcal{F}_t, \mathbf{P}_{l,i}^{n,m})$-martingale by the Burkholder-Davis-Gundy inequality and Hölder's inequality. Thus, for each n, m, $l \geq 1$, $i = 1, \ldots, k$ and $0 \leq r \leq t \leq T$, by taking the conditional mathematical expectation $\mathbf{E}_{l,i}^{n,m}[\,\cdot\,|\mathcal{F}_r]$ with respect to $\mathcal{F}_r$ under $\mathbf{P}_{l,i}^{n,m}$ in both sides of (8), we can get that

$$\mathbf{E}_{l,i}^{n,m}\left[|{}^i\hat{y}_t^{n,m}|\big|\mathcal{F}_r\right] \leq \tilde{a}_{n,m}^l + \mathbf{E}_{l,i}^{n,m}\left[\int_t^T u(s)\rho(|\hat{y}_s^{n,m}|)\,\mathrm{d}s\bigg|\mathcal{F}_r\right]. \tag{9}$$

Next, in an analogous way to that in the proof of the uniqueness part of Theorem 7 in Fan, Wang and Xiao [14] we can deduce the desired result of this step. For readers' convenience, we provide a sketch of this proof highlighting the difference. For each $l \geq 1$, define $\rho_l(\cdot) : \mathbf{R} \mapsto \mathbf{R}^+$ by

$$\rho_l(x) := \sup_{y\in\mathbf{R}}\{\rho(|y|) - l|x - y|\}.$$

It follows from Lemma 1 in Lepeltier and San Martin [18] that $\rho_l(\cdot)$ is well defined for $l \geq A$, Lipschitz continuous, decreasing in l and converges to $\rho(|\cdot|)$. For each $l \geq A$, let $f_t^{n,m,l}$ be the unique solution of the following deterministic backward differential equation (see Proposition 3 in Fan, Wang and Xiao [14]),

$$f_t^{n,m,l} = \tilde{a}_{n,m}^l + \int_t^T \big(u(s)\rho_l(k\cdot f_s^{n,m,l})\big)\,\mathrm{d}s, \quad t \in [0,T]. \tag{10}$$

Noticing that ρ_l is decreasing in l, and $\tilde{a}_{n,m}^l$ is respectively decreasing in n, m and l, we can deduce that $f_t^{n,m,l}$ is also decreasing in n, m and l respectively, which implies that $f_t^{n,m,l}$ converges point wisely to a function f_t as n, m, $l \to \infty$ (see Proposition 5 in Fan, Wang and Xiao [14]). Thus, by sending n, m, $l \to \infty$ in (10), the Lebesgue dominated convergence theorem leads to that

$$f_t = \int_t^T \big(u(s)\rho(|k\cdot f_s|)\big)\,\mathrm{d}s = \int_t^T \big(u(s)\rho(k\cdot f_s)\big)\,\mathrm{d}s, \quad t \in [0,T].$$

Recalling that $\rho(\cdot) \in \mathbf{S}$ and $\int_{0^+} 1/\rho(u)\,\mathrm{d}u = \infty$, we can derive that $f_t \equiv 0$ (see Proposition 6 in Fan, Wang and Xiao [14]).

Now, for each $l \geq A$, n, m, $j \geq 1$ and $t \in [0,T]$, let $f_t^{n,m,l,j}$ be defined recursively as follows:

$$f_t^{n,m,l,1} := C_1; \qquad f_t^{n,m,l,j+1} := \tilde{a}_{n,m}^l + \int_t^T \big(u(s)\rho_l(k\cdot f_s^{n,m,l,j})\big)\,\mathrm{d}s, \tag{11}$$

where C_1 is defined in (7). Noticing that ρ_l is Lipschitz continuous, by Proposition 4 in Fan, Wang and Xiao [14] we know that $f_t^{n,m,l,j}$ converges point wisely to $f_t^{n,m,l}$ as $j \to \infty$.

On the other hand, it is easy to verify by induction that for each $l \geq A$, n, m, $j \geq 1$ and $i = 1, \ldots, k$,

$$|{}^i\hat{y}_t^{n,m}| \leq f_t^{n,m,l,j} \leq f_0^{n,m,l,j}, \quad t \in [0,T]. \tag{12}$$

Indeed, (12) holds true for $j = 1$ due to (7). Suppose that (12) holds true for $j \geq 1$. Then, for each $t \in [0,T]$,

$$u(t)\rho(|\hat{y}_t^{n,m}|) \leq u(t)\rho(k \cdot f_t^{n,m,l,j}) \leq u(t)\rho_l(k \cdot f_t^{n,m,l,j}).$$

In view of (9) with $r = t$ as well as (11), we deduce that (12) holds for $j+1$.

Finally, by taking supermum with respect to t and then sending j, l, n, $m \to \infty$ successively in (12), we obtain that for each $i = 1, 2, \ldots, k$,

$$\lim_{n,m\to\infty} \sup_{t\in[0,T]} |{}^i y_t^n - {}^i y_t^m| = 0, \tag{13}$$

which together with (7) implies that $\{(y_\cdot^n - y_\cdot^1)\}_{n=1}^\infty$ converges in the space $\mathcal{S}^p$ to some $(Y_t)_{t\in[0,T]}$. Then we can deduce that $\{y_\cdot^n\}_{n=1}^\infty$ converges to $y_\cdot := Y_\cdot + y_\cdot^1$ in the space $\mathcal{S}^q$ since $y_\cdot^1$ belongs to it.

Second Step: In this step, we show the convergence of $\{z_\cdot^n\}_{n=1}^\infty$ in M^q.

It follows from (6) and (4) that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each n, m, $l \geq 1$, $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,

$$\langle y, \hat{g}^{n,m}(t,y,z)\rangle \leq (l+2A)u(t)|y|^2 + (l+2A)v(t)|y||z| + |y|H_l^{n,m}(t),$$

where

$$H_l^{n,m}(t) := u(t)\rho\left(\frac{2A}{l+2A}\right) + \mathbf{1}_D v(t)\phi\left(\left(\frac{2A}{l+2A}\right)^\alpha\right) + a_n(t) + a_m(t).$$

Here we note that $H_l^{n,m}(\cdot)$ is non-increasing with respect to n, m and l, respectively. Hence, the generator $\hat{g}^{n,m}$ satisfies assumption (H) with $\mu(t) = (l+2A)u(t)$, $\lambda(t) = (l+2A)v(t)$ and $f_t = H_l^{n,m}(t)$. Then by (2) in Proposition 2.1 we know that there exist two constants $C_l^1 > 0$ depending on q, l, A, $\int_0^T u(t)\,\mathrm{d}t$ and $\int_0^T v^2(t)\,\mathrm{d}t$ and $C^2 > 0$ depending only on q such that

$$\mathbf{E}\left[\left(\int_0^T |z_t^n - z_t^m|^2\,\mathrm{d}t\right)^{q/2}\right] \leq C_l^1 \mathbf{E}\left[\sup_{t\in[0,T]} |y_t^n - y_t^m|^q\right] + C^2\left(\int_0^T H_l^{n,m}(t)\,\mathrm{d}t\right)^q.$$

Recalling that $\{y_\cdot^n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{S}^q$ and $\rho(\cdot)$, $\phi(\cdot) \in \mathbf{S}$, by sending first n, $m \to \infty$ then $l \to \infty$ in the previous inequality we deduce that

$$\lim_{n,m\to\infty} \mathbf{E}\left[\left(\int_0^T |z_t^n - z_t^m|^2\,\mathrm{d}t\right)^{q/2}\right] = 0.$$

That is to say, $\{z_\cdot^n\}_{n=1}^\infty$ is a Cauchy sequence in the space M^q. We denote the limit by $(z_t)_{t\in[0,T]}$.

Third Step: This step aims to show $(y_t, z_t)_{t\in[0,T]}$ is a solution of BSDE (ξ, T, g) in the space $\mathcal{S}^q \times \mathrm{M}^q$.

We have known that the sequence $\{y_t^n\}_{n=1}^\infty$ and $\{\int_t^T z_s^n \,\mathrm{d}B_s\}_{n=1}^\infty$ converge in $L^q(\mathcal{F}_T; \mathbf{R}^k)$, uniformly with respect to t, toward to y_t and $\int_t^T z_s \,\mathrm{d}B_s$, respectively. Next let us check the limit of $g^n(s, y_s^n, z_s^n)$ in BSDE (ξ, T, g^n). It is clear that there exists a positive constant c_q depending only on q such that

$$\begin{aligned}
&\mathbf{E}\left[\sup_{t\in[0,T]}\left(\int_t^T |g^n(s, y_s^n, z_s^n) - g(s, y_s, z_s)| \,\mathrm{d}s\right)^q\right] \\
&\le c_q \mathbf{E}\left[\left(\int_0^T |g^n(s, y_s^n, z_s^n) - g(s, y_s^n, z_s^n)| \,\mathrm{d}s\right)^q\right] \\
&\quad + c_q \mathbf{E}\left[\left(\int_0^T |g(s, y_s^n, z_s^n) - g(s, y_s, z_s)| \,\mathrm{d}s\right)^q\right]. \qquad (14)
\end{aligned}$$

It follows from (5) that the first term on the right hand side of (14) converges to 0 as $n \to \infty$. Furthermore, by (H1), $(\mathrm{H2})_\alpha$ and (4) we see that for each n, $l \ge 1$,

$$\begin{aligned}
&\mathbf{E}\left[\left(\int_0^T |g(s, y_s^n, z_s^n) - g(s, y_s, z_s)| \,\mathrm{d}s\right)^q\right] \\
&\le c_q (l + 2A)^q \mathbf{E}\left[\left(\int_0^T \big(u(s)|y_s^n - y_s| + v(s)|z_s^n - z_s|\big) \,\mathrm{d}s\right)^q\right] \\
&\quad + c_q \left(\int_0^T \left[u(s)\rho\left(\frac{2A}{l + 2A}\right) + \mathbf{1}_D v(s)\phi\left(\left(\frac{2A}{l + 2A}\right)^\alpha\right)\right] \mathrm{d}s\right)^q. \qquad (15)
\end{aligned}$$

Note that the second term on the right hand side of (15) converges to 0 as $l \to \infty$. On the other hand, it follows from Hölder's inequality that

$$\begin{aligned}
&\mathbf{E}\left[\left(\int_0^T \big(u(s)|y_s^n - y_s| + v(s)|z_s^n - z_s|\big) \,\mathrm{d}s\right)^q\right] \\
&\le c_q \left(\int_0^T u(s) \,\mathrm{d}s\right)^q \mathbf{E}\left[\sup_{t\in[0,T]} |y_t^n - y_t|^q\right] \\
&\quad + c_q \left(\int_0^T v^2(s) \,\mathrm{d}s\right)^{q/2} \mathbf{E}\left[\left(\int_0^T |z_s^n - z_s|^2 \,\mathrm{d}s\right)^{q/2}\right].
\end{aligned}$$

Thus, by virtue of the fact that $\{(y_\cdot^n, z_\cdot^n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{S}^q \times \mathrm{M}^q$, letting $n \to \infty$ then $l \to \infty$ in (15) and letting $n \to \infty$ in (14) yield that

$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{t\in[0,T]} \left|\int_t^T g^n(s, y_s^n, z_s^n) \,\mathrm{d}s - \int_t^T g(s, y_s, z_s) \,\mathrm{d}s\right|^q\right] = 0.$$

Subsequently, by passing to the limit in BSDE (ξ, T, g^n) under the sense of uniform convergence in probability we can conclude the desired result. □

Remark 4.2. We would like to mention that the constant q in Proposition 4.1 may satisfy $0 < q < 1$ or $q > 1$. This allows Proposition 4.1 to handle both L^p $(p > 1)$ solutions and L^1 solutions.

The following corollary of Proposition 4.1 interprets the uniqueness for solutions of the BSDEs, which is useful to the proof of our main result.

Corollary 4.3. *Assume that* $\alpha \in (0,1]$, $\overline{p} > 1$, $0 < q \leq \overline{p}$, (H1), $(\mathrm{H2})_\alpha$ *and* (H3) *hold for* g, *and* $(y_t, z_t)_{t\in[0,T]}$ *and* $(y'_t, z'_t)_{t\in[0,T]}$ *are two solutions of BSDE* (ξ, T, g) *in* $\mathcal{S}^q \times \mathrm{M}^q$. *If* $(y_t - y'_t)_{t\in[0,T]}$ *belongs to* $\mathcal{S}^{\overline{p}}$, *then we have* $y_\cdot = y'_\cdot$ *and* $z_\cdot = z'_\cdot$.

Proof. Let assumptions hold and (H1), $(\mathrm{H2})_\alpha$ and (H3) hold for g with parameters $u(\cdot)$, $v(\cdot)$, $\rho(\cdot)$ and $\phi(\cdot)$. By letting $g^n \equiv g$, $y^{2n-1}_\cdot = y'_\cdot$, $z^{2n-1}_\cdot = z'_\cdot$, $y^{2n}_\cdot = y_\cdot$ and $z^{2n}_\cdot = z_\cdot$ for each $n \geq 1$ in Proposition 4.1, we deduce that $(y^{2n}_\cdot - y^{2n-1}_\cdot)$ and $(z^{2n}_\cdot - z^{2n-1}_\cdot)$ converge to 0 respectively in $\mathcal{S}^q$ and M^q as $n \to \infty$, which means that $y_\cdot = y'_\cdot$ and $z_\cdot = z'_\cdot$. $\square$

Next we introduce the following proposition, which is actually a direct corollary of Lemma 12 in Fan, Wang and Xiao [14] in view of Remark 3.1.

Proposition 4.4. *Assume that* $\alpha \in (0,1]$ *and the generator* g *satisfies* (H1), $(\mathrm{H2})_\alpha$ *and* (H3) *with parameters* $u(\cdot)$, $v(\cdot)$, $\rho(\cdot)$ *and* $\phi(\cdot)$. *Then there exists a generator sequence* $\{g^n\}_{n=1}^{\infty}$ *such that* (H3) *is fulfilled for each* g^n, *and* $\mathrm{d}\mathbf{P} \times \mathrm{d}t$ - a.e., *for each* $n \geq 1$, y, y_1, $y_2 \in \mathbf{R}^k$ *and* z, z_1, $z_2 \in \mathbf{R}^{k\times d}$, *we have*

$$|g^n(t,y,z)| \leq |g(t,0,0)| + kA(u(t) + 2v(t)) + kAu(t)|y| + kAv(t)|z|, \tag{16}$$

$$|g^n(t,y_1,z_1) - g^n(t,y_2,z_2)| \leq ku(t)\rho(|y_1 - y_2|) + kv(t)\phi(|z_1 - z_2|^\alpha), \tag{17}$$

$$|g^n(t,y_1,z_1) - g^n(t,y_2,z_2)| \leq k(n + A)\big(u(t)|y_1 - y_2| + v(t)|z_1 - z_2|\big). \tag{18}$$

Moreover, there exists a non-increasing function sequence $b_n(\cdot) : [0,T] \mapsto \mathbf{R}^+$ *with* $\int_0^T b_n(t)\,\mathrm{d}t \to 0$ *as* $n \to \infty$ *such that* $\mathrm{d}\mathbf{P} \times \mathrm{d}t$ - a.e., *for each* $y \in \mathbf{R}^k$ *and* $z \in \mathbf{R}^{k\times d}$,

$$|g^n(t,y,z) - g(t,y,z)| \leq b_n(t). \tag{19}$$

Now we can start to prove our main result — Theorem 3.2.

Proof of Theorem 3.2. Let $0 \leq T \leq \infty$, $p \geq 1$, $\alpha \in (0,1]$ and (A1), (H1), $(\mathrm{H2})_\alpha$ and (H3) hold for g with parameters $u(\cdot)$, $v(\cdot)$, $\rho(\cdot)$ and $\phi(\cdot)$.

Assertion (i). Let $p > 1$. Setting $q = \overline{p} = p$ in Corollary 4.3 yields the uniqueness result. Now, we consider the existence part. By Proposition 4.4 we can find a generator sequence $\{g^n\}_{n=1}^{\infty}$ satisfying (H3) and (16) – (19). It follows from (16) – (18) that g^n satisfies (A1), (A2), (A4) and (A5) for each $n \geq 1$. And from (17) we derive that $\mathrm{d}\mathbf{P} \times \mathrm{d}t$ - a.e., for each $n \geq 1$ and $y \in \mathbf{R}^k$,

$$|g^n(t,y,0) - g^n(t,0,0)| \leq ku(t)\rho(|y|),$$

which means that (A3) is fulfilled by g^n. Then it follows from (i) in Proposition 2.3 that for each $n \geq 1$, the following BSDE:

$$y^n_t = \xi + \int_t^T g^n(s, y^n_s, z^n_s)\,\mathrm{d}s - \int_t^T z^n_s\,\mathrm{d}B_s, \quad t \in [0,T],$$

admits a unique solution $(y_t^n, z_t^n)_{t\in[0,T]}$ in $\mathcal{S}^p \times \mathrm{M}^p$. Thus, taking $q = p$ in Proposition 4.1 and noting that (17), (H3) and (19) hold for g^n, we can conclude that $\{(y_\cdot^n, z_\cdot^n)\}_{n=1}^\infty$ converges to a pair of $(y_t, z_t)_{t\in[0,T]}$ in $\mathcal{S}^p \times \mathrm{M}^p$, which is the desired solution of BSDE (1).

Assertion (ii). Let $p = 1$ and $\alpha \in (0, 1)$. We first treat the uniqueness part. Assume that $(y_t, z_t)_{t\in[0,T]}$ and $(y_t', z_t')_{t\in[0,T]}$ are two solutions of BSDE (1) such that for some $\beta \in (\alpha, 1)$, both $(y_t)_{t\in[0,T]}$ and $(y_t')_{t\in[0,T]}$ are of class (D), and $(z_t)_{t\in[0,T]}$ and $(z_t')_{t\in[0,T]}$ belong to M^β. According to Corollary 4.3, in order to prove the uniqueness, it suffices to show that $(y_t - y_t')_{t\in[0,T]} \in \mathcal{S}^{\overline{p}}$ for some $\overline{p} > 1$.

In fact, let us fix $n \geq 1$ and denote τ_n the $(\mathcal{F}_t)$-stopping time

$$\tau_n := \inf\left\{t \geq 0 : \int_0^t (|z_s|^2 + |z_s'|^2)\,\mathrm{d}s \geq n\right\} \wedge T.$$

Tanaka's formula gives that for each $n \geq 1$ and $t \in [0, T]$, setting $\hat{y}_\cdot := y_\cdot - y_\cdot'$ and $\hat{z}_\cdot := z_\cdot - z_\cdot'$,

$$\begin{aligned} |\hat{y}_{t\wedge\tau_n}| \leq |\hat{y}_{\tau_n}| &+ \int_{t\wedge\tau_n}^{\tau_n} |\hat{y}_s|^{-1}\mathbf{1}_{|\hat{y}_s|\neq 0}\langle \hat{y}_s, g(s, y_s, z_s) - g(s, y_s', z_s')\rangle\,\mathrm{d}s \\ &- \int_{t\wedge\tau_n}^{\tau_n} |\hat{y}_s|^{-1}\mathbf{1}_{|\hat{y}_s|\neq 0}\langle \hat{y}_s, \hat{z}_s\,\mathrm{d}B_s\rangle. \end{aligned} \tag{20}$$

It follows from (H1) and $(\mathrm{H2})_\alpha$ that, $\mathrm{d}\mathbf{P}\times\mathrm{d}s$ - a.e.,

$$|\hat{y}_s|^{-1}\mathbf{1}_{|\hat{y}_s|\neq 0}\langle \hat{y}_s, g(s, y_s, z_s) - g(s, y_s', z_s')\rangle \leq u(s)\rho(|\hat{y}_s|) + v(s)\phi(|\hat{z}_s|^\alpha).$$

Plugging the previous inequality into (20) and taking the conditional mathematical expectation with respect to $\mathcal{F}_t$, we deduce that for each $n \geq 1$ and $t \in [0, T]$,

$$|\hat{y}_{t\wedge\tau_n}| \leq \mathbf{E}\left[|\hat{y}_{\tau_n}| + \int_{t\wedge\tau_n}^{\tau_n} \big(u(s)\rho(|\hat{y}_s|) + v(s)\phi(|\hat{z}_s|^\alpha)\big)\,\mathrm{d}s\middle|\mathcal{F}_t\right].$$

Now sending $n \to \infty$ and noticing that $\tau_n \to T$, $(\hat{y}_t)_{t\in[0,T]}$ belongs to class (D) and $\mathrm{d}\mathbf{P}$-a.s., $\hat{y}_T = 0$, we can derive that for each $t \in [0, T]$,

$$\begin{aligned} |\hat{y}_t| &\leq \mathbf{E}\left[\int_t^T \big(u(s)\rho(|\hat{y}_s|) + v(s)\phi(|\hat{z}_s|^\alpha)\big)\,\mathrm{d}s\middle|\mathcal{F}_t\right] \\ &\leq A\int_0^T \big(u(s) + v(s)\big)\,\mathrm{d}s + A\mathbf{E}\left[\int_t^T u(s)|\hat{y}_s|\,\mathrm{d}s\middle|\mathcal{F}_t\right] \\ &\quad + A\mathbf{E}\left[\int_t^T v(s)|\hat{z}_s|^\alpha\,\mathrm{d}s\middle|\mathcal{F}_t\right]. \end{aligned}$$

Thus, from Fubini's theorem we get that for each $r \in [t, T]$,

$$\mathbf{E}\left[|\hat{y}_r||\mathcal{F}_t\right] \leq A\int_0^T \big(u(s) + v(s)\big)\,\mathrm{d}s + A\int_r^T u(s)\mathbf{E}\left[|\hat{y}_s||\mathcal{F}_t\right]\,\mathrm{d}s$$

$$+ A\mathbf{E}\left[\int_r^T v(s)|\hat{z}_s|^\alpha \,\mathrm{d}s \middle| \mathcal{F}_t\right].$$

Hence, Gronwall's inequality induces that for each $0 \le r \le t \le T$,

$$\mathbf{E}\left[|\hat{y}_r||\mathcal{F}_t\right] \le A\mathrm{e}^{A\int_0^T u(s)\,\mathrm{d}s}\left(\int_0^T \big(u(s)+v(s)\big)\,\mathrm{d}s + \mathbf{E}\left[\int_0^T v(s)|\hat{z}_s|^\alpha \,\mathrm{d}s \middle| \mathcal{F}_t\right]\right).$$

Moreover, taking $r = t$ and noticing that $\beta/\alpha > 1$, Doob's and Jensen's inequalities yield that there exists a positive constant C_β^α depending on α, β and A such that

$$\mathbf{E}\left[\sup_{t\in[0,T]} |\hat{y}_t|^{\beta/\alpha}\right] \le C_\beta^\alpha + C_\beta^\alpha \mathbf{E}\left[\left(\int_0^T v(s)|\hat{z}_s|^\alpha \,\mathrm{d}s\right)^{\beta/\alpha}\right].$$

And Hölder's inequality yields that

$$\int_0^T v(s)|\hat{z}_s|^\alpha \,\mathrm{d}s \le \left(\int_0^T v^{\frac{2}{2-\alpha}}(s)\,\mathrm{d}s\right)^{\frac{2-\alpha}{2}} \left(\int_0^T |\hat{z}_s|^2 \,\mathrm{d}s\right)^{\alpha/2}.$$

Note that $\int_0^T (v(s) + v^2(s))\,\mathrm{d}s < \infty$ implies $\int_0^T v^{2/(2-\alpha)}(s)\,\mathrm{d}s < \infty$. Since $(\hat{z}_t)_{t\in[0,T]}$ belongs to M^β, from the previous two inequalities we can deduce that $(\hat{y}_t)_{t\in[0,T]}$ belongs to the space $\mathcal{S}^{\beta/\alpha}$. Thus, taking $q = \beta$ and $\overline{p} = \beta/\alpha$ in Corollary 4.3 yields the uniqueness result.

Now we tackle the existence part. Firstly, thanks to Proposition 4.4, there exists a generator sequence $\{g^n\}_{n=1}^\infty$ satisfying (H3) and (16) – (19). Similar to the proof in assertion (i), we can verify that g^n fulfills assumptions (A1) – (A5) for each $n \ge 1$. Moreover, it follows from (17) that $\mathrm{d}\mathbf{P}\times\mathrm{d}t$ - a.e., for each $n \ge 1$, $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k\times d}$,

$$|g^n(t,y,z) - g^n(t,y,0)| \le kv(t)\phi(|z|^\alpha) \le kAv(t)(1+|z|^\alpha),$$

which indicates that g^n fulfills assumption (A6) for each $n \ge 1$. Then (ii) in Proposition 2.3 yields that for each $n \ge 1$, the following BSDE

$$y_t^n = \xi + \int_t^T g^n(s,y_s^n,z_s^n)\,\mathrm{d}s - \int_t^T z_s^n \,\mathrm{d}B_s, \quad t\in[0,T],$$

admits a solution $(y_t^n, z_t^n)_{t\in[0,T]} \in \bigcap_{\beta\in(0,1)}(\mathcal{S}^\beta \times \mathrm{M}^\beta)$ such that $(y_t^n)_{t\in[0,T]}$ belongs to class (D). Furthermore, it follows from (17) and (19) that $\mathrm{d}\mathbf{P}\times \mathrm{d}s$ - a.e., for each $n \ge 1$, setting $\hat{y}_\cdot^{n,1} := y_\cdot^n - y_\cdot^1$ and $\hat{z}_\cdot^{n,1} := z_\cdot^n - z_\cdot^1$,

$$\begin{aligned}&|\hat{y}_s^{n,1}|^{-1}\mathbf{1}_{|\hat{y}_s^{n,1}|\neq 0}\langle \hat{y}_s^{n,1}, g^n(s,y_s^n,z_s^n) - g^1(s,y_s^1,z_s^1)\rangle\\ &\le ku(s)\rho(|\hat{y}_s^{n,1}|) + kv(s)\phi(|\hat{z}_s^{n,1}|^\alpha) + b_n(s) + b_1(s).\end{aligned}$$

Hence, by the aid of the stopping time technique, arguing as in the above proof of the uniqueness part and in view of the fact that $b_n(\cdot)$ is non-increasing with respect to n, we can deduce that for each $n \ge 1$ and $0 \le r \le t \le T$,

$$\mathbf{E}\left[|\hat{y}_r^{n,1}||\mathcal{F}_t\right] \le kA\mathrm{e}^{A\int_0^T ku(s)\,\mathrm{d}s}\int_0^T (u(s)+v(s)+2b_1(s))\,\mathrm{d}s$$

$$+ kA\mathrm{e}^{A\int_0^T ku(s)\,\mathrm{d}s}\mathbf{E}\left[\int_0^T v(s)|\hat{z}_s^{n,1}|^\alpha\,\mathrm{d}s\middle|\mathcal{F}_t\right]. \tag{21}$$

And we can also prove that $I^{n,1} := \int_0^T v(s)|\hat{z}_s^{n,1}|^\alpha\,\mathrm{d}s$ belongs to $L^\gamma(\mathcal{F}_T;\mathbf{R}^k)$ as soon as $\alpha\gamma < 1$ with $\gamma > 1$. Moreover, for some $\gamma > 1$ with $\alpha\gamma < 1$, taking $r = t$ in (21) and Doob's inequality give that there exists a positive constant c_γ depending only on γ, k and A such that

$$\mathbf{E}\left[\sup_{t\in[0,T]}|\hat{y}_t^{n,1}|^\gamma\right] \leq c_\gamma + c_\gamma\mathbf{E}[|I^{n,1}|^\gamma] < \infty,$$

which indicates that $(\hat{y}_t^{n,1})_{t\in[0,T]}$ belongs to the space $\mathcal{S}^\gamma$ for some $\gamma > 1$. Then in view of (17), (H3) and (19), we can apply Proposition 4.1 with taking $q = \beta$ and $p = \gamma$ to obtain that $\{(y_\cdot^n, z_\cdot^n)\}_{n=1}^\infty$ converges to some $(y_t, z_t)_{t\in[0,T]}$ in $\mathcal{S}^\beta \times \mathrm{M}^\beta$ for each $\beta \in (0,1)$, which solves BSDE (1). Moreover, note by (13) that $\{(y_\cdot^n - y_\cdot^1)\}_{n=1}^\infty$ converges in $\mathcal{S}^\gamma$. Since $(y_t^n)_{t\in[0,T]}$ belongs to class (D) and the convergence in $\mathcal{S}^\gamma$ with $\gamma > 1$ is stronger than the convergence under the norm $\|\cdot\|_1$, we can get that $\{y_\cdot^n\}_{n=1}^\infty$ also converges to $(y_t)_{t\in[0,T]}$ under $\|\cdot\|_1$. Therefore, $(y_t, z_t)_{t\in[0,T]}$ is exactly the desired solution of BSDE (1), which belongs to $\bigcap_{\beta\in(0,1)}(\mathcal{S}^\beta \times \mathrm{M}^\beta)$ and $(y_t)_{t\in[0,T]}$ is of class (D). □

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Xiao Lishun
Department of Epidemiology and Biostatistics
Xuzhou Medical University
Xuzhou, Jiangsu 221004, P. R. China
Email address: xiaolishun@xzhmu.edu.cn

Fan Shengjun
School of Mathematics
China University of Mining and Technology
Xuzhou, Jiangsu, 221116, P. R. China
Email address: f_s_j@126.com