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GRADIENT EINSTEIN-TYPE CONTACT METRIC MANIFOLDS

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Abstract. Consider a gradient Einstein-type metric in the setting of K-contact manifolds and (κ, μ)-contact manifolds. First, it is proved that, if a complete K-contact manifold admits a gradient Einstein-type metric, then M is compact, Einstein, Sasakian and isometric to the unit sphere $\mathbb{S}^{2n+1}$. Next, it is proved that, if a non-Sasakian (κ, μ)-contact manifolds admits a gradient Einstein-type metric, then it is flat in dimension 3, and for higher dimension, M is locally isometric to the product of a Euclidean space $\mathbb{E}^{n+1}$ and a sphere $\mathbb{S}^n(4)$ of constant curvature $+4$.

1. Introduction

Let (M, g) be a smooth Riemannian manifold of dimension ≥ 3. We say that (M, g) is an Einstein-type manifold or that (M, g) supports an Einstein-type structure if there exist a vector field V on M and a smooth function $\lambda : M \to \mathbb{R}$ such that

$$\alpha S + \frac{\beta}{2}\pounds_V g + \nu V^{\#} \otimes V^{\#} = \gamma g = (\rho r + \lambda) g \tag{1}$$

for some constants $\alpha, \beta, \nu, \rho \in \mathbb{R}$ with $(\alpha, \beta, \nu) \neq 0$. Here $\pounds$ and $V^{\#}(X) = g(V, X)$ stand for the Lie derivative and the 1-form metrically dual to the vector field V, respectively. If $V = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, we say that (M, g) is a gradient Einstein-type manifold. In this case, the equation (1) can be written as

$$\alpha S + \beta \nabla^2 f + \nu df \otimes df = \gamma g, \tag{2}$$

where S is Ricci tensor and ∇^2 stands for the Hessian of f. We refer to f as the potential function.

The concept of Einstein-type manifold was studied and introduced by Catino et al. as a generalization of Einstein spaces [8]. In case f is constant we say that the Einstein-type structure is trivial. Notice that, an Einstein-type structure

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on a Riemannian manifold (M, g) unifies several particular cases well studied in the literature, such as Ricci solitons [15,22], Ricci almost solitons [23], gradient Ricci solitons, Yamabe solitons [6,10], Yamabe quasi-solitons [16], conformal gradient solitons [25], m-quasi-Einstein manifolds [7], (m, ρ)-quasi-Einstein manifold [17] and ρ-Einstein solitons [9].

There has been a growing interest in the study of Einstein condition and its various generalizations in the setting of contact metric manifolds in a recent years. In [4], Boyer-Galicki studied Einstein and η-Einstein K-contact manifolds and they proved that any compact K-contact Einstein manifold is Sasakian. In [24], the author generalizing Boyer-Galicki result proved that if a complete K-contact metric represents a gradient Ricci soliton, then it is compact Einstein and Sasakian. Extending these for gradient Ricci almost solitons, the author [11] proved that if a compact K-contact metric represents a gradient Ricci almost soliton, then it is isometric to a unit sphere $\mathbb{S}^{2n+1}$. Recently, Ghosh studied m-quasi-Einstein, generalized m-quasi-Einstein and (m, ρ)-quasi-Einstein metric within the background of contact geometry respectively in [13], [14] and [12]. These works of Ghosh inspires us to study the gradient Einstein-type condition within the background of K-contact manifolds and (κ, μ)-contact manifolds.

In this paper, we confine our study to the gradient Einstein-type metric within the framework of K-contact and (κ, μ)-contact manifolds. In Section 2, we gathered some preliminary definitions and formulas on contact manifolds. In Section 3, we prove that if complete K-contact manifolds admit a gradient Einstein-type metric, then M is compact, Einstein, Sasakian and isometric to the unit sphere $\mathbb{S}^{2n+1}$. In Section 4, we consider (κ, μ)-contact manifold which admits a gradient Einstein-type metric and we prove that if a non-Sasakian (κ, μ)-contact manifold supports a gradient Einstein-type structure, then for $n = 1$, M is flat, and for $n > 1$, M is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$.

We have borrowed some ideas and arguments from [21], but our goals and main results are different from [21].

2. Preliminaries

Let us recall the basic concepts and formulas of contact metric manifolds. A $(2n+1)$-dimensional smooth manifold M is said to be contact if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. This 1-form is called a contact 1-form. For a contact 1-form η, there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ for all vector field X and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact sub-bundle $\mathcal{D}$ (defined by $\eta = 0$), we obtain a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 X = -X + \eta(X)\xi \tag{3}$$

for all $X, Y \in TM$. It can also be deduced from these equations:

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{4}$$

The structure (φ, ξ, η, g) on M is known as a contact metric structure and the metric g is called an associated metric. A Riemannian manifold M together with the structure (φ, ξ, η, g) is said to be a contact metric manifold and we denote it by $(M, \varphi, \xi, \eta, g)$. On a contact metric manifold (see [1])

$$\nabla_X \xi = -\varphi X - \varphi h X, \qquad h\varphi + \varphi h = 0, \tag{5}$$

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)\varphi Y = 2g(Y, X)\xi - \eta(Y)(X + hX + \eta(Z)\xi), \tag{6}$$

for any vector field X, Y on M and ∇ denotes the operator of covariant differentiation of g. If the vector field ξ is Killing (equivalently, $h = 0$) with respect to g, then the contact metric manifold M is said to be K-contact. On a K-contact (Sasakian) manifold the following formulas are known [1]

$$\nabla_X \xi = -\varphi X, \tag{7}$$

$$Q\xi = 2n\xi, \tag{8}$$

$$(\nabla_X \varphi)Y = R(\xi, X)Y, \tag{9}$$

where Q and R denote the Ricci operator and the Riemann curvature tensor of g, respectively. A contact metric manifold is said to be Sasakian if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X. \tag{10}$$

On a Sasakian manifold the curvature tensor satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{11}$$

Also, the contact metric structure on M is said to be Sasakian if the almost Käehler structure on the metric cone $(M \times R^+, r^2 g + dr^2)$ over M, is Käehler [1]. Any Sasakian manifold is K-contact, and the converse only holds when the dimension is 3. See [1] and [5] for more information about it.

3. *K*-contact manifold satisfying the gradient Einstein-type metrics

Here, consider a K-contact metric as a gradient Einstein-type metric. The following will be needed to prove our main result.

Lemma 3.1. *If $(M, g, \alpha, \beta, \nu, \gamma)$ is a gradient Einstein-type contact metric manifold, then the curvature tensor R has the expression*

$$\begin{aligned} \beta R(X, Y)Df = {} & \alpha[(\nabla_Y Q)X - (\nabla_X Q)Y] + \frac{\nu\gamma}{\beta}[(Xf)Y - (Yf)X] \\ & + \frac{\nu\alpha}{\beta}[(Yf)QX - (Xf)QY] + [(X\gamma)Y - (Y\gamma)X] \end{aligned} \tag{12}$$

for any vector fields X, Y on M.

Proof. The gradient Einstein-type equation (2) can be expressed as

$$\alpha QY + \beta \nabla_Y Df + \nu g(Y, Df)Df = \gamma Y, \tag{13}$$

where D is the gradient operator of g. Differentiate (13) covariantly along X, we obtain

$$\begin{aligned}&\alpha(\nabla_X Q)Y+\alpha Q\nabla_X Y+\beta\nabla_X\nabla_Y Df+\nu g(\nabla_X Y,Df)Df\\ &+\nu g(Y,\nabla_X Df)Df+\nu g(Y,Df)\nabla_X Df=(X\gamma)Y+\gamma\nabla_X Y.\end{aligned}\tag{14}$$

Then the required result follows by applying this equation and (13) to the well known expression $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. □

Theorem 3.2. *Let (M,φ,ξ,η,g) be a complete K-contact manifold of dimension $2n+1$. If there exists a gradient Einstein-type structure $(f,\alpha,\beta,\nu,\gamma)$ associated with the contact metric g, then M is compact, Einstein, Sasakian and isometric to the unit sphere $\mathbb{S}^{2n+1}$.*

Proof. Applying the covariant derivative to (8) and then employing (7), we obtain

$$(\nabla_X Q)\xi=Q\varphi X-2n\varphi X.\tag{15}$$

At this point we remember that for a K-contact manifold ξ is Killing, and hence $L_\xi Q=0$. In view of (7) and (8), we obtain $\nabla_\xi Q=Q\varphi-\varphi Q$. Replacing ξ with X in (12) and making use of $\nabla_\xi Q=Q\varphi-\varphi Q$, (15) and (9), we get

$$\begin{aligned}-\beta g((\nabla_Y\varphi)X,Df)=&\ \alpha g(\varphi QY-2n\varphi Y,X)+\left[(\xi\gamma)+\frac{\nu\gamma}{\beta}(\xi f)\right]g(Y,X)\\ &+\left[\frac{\nu(2n\alpha-\gamma)}{\beta}\right](Yf)\eta(X)-(\xi f)\frac{\nu\alpha}{\beta}g(QY,X)\\ &-(Y\gamma)\eta(X).\end{aligned}\tag{16}$$

Replacing X and Y by φX and φY respectively in relation (16), adding the resulting equation with (16) and then using (6) (where $h=0$, as M is K-contact) and (7), we have

$$\begin{aligned}&-2\beta(\xi f)g(Y,X)+\beta(Yf)\eta(X)+\beta(\xi f)\eta(Y)\eta(X)\\ =&\ \alpha g((\varphi Q+Q\varphi)Y,X)+2\left[(\xi\gamma)+\frac{\nu\gamma}{\beta}(\xi f)\right]g(Y,X)\\ &+(\xi f)\frac{\nu\alpha}{\beta}g(\varphi Q\varphi Y-QY,X)-\left[(\xi\gamma)+\frac{\nu\gamma}{\beta}(\xi f)\right]\eta(Y)\eta(X)\\ &+\frac{\nu(2n\alpha-\gamma)}{\beta}(Yf)\eta(X)-4n\alpha g(\varphi Y,X)-(Y\gamma)\eta(X).\end{aligned}\tag{17}$$

Since Q is self-adjoint, anti-symmetrizing the above equation gives

$$\begin{aligned}\beta[(Yf)\eta(X)-(Xf)\eta(Y)]=&\ 2\alpha g((\varphi Q+Q\varphi)Y,X)-8n\alpha g(\varphi Y,X)\\ &+\frac{\nu(2n\alpha-\gamma)}{\beta}[(Yf)\eta(X)-(Xf)\eta(Y)]\\ &+[(X\gamma)\eta(Y)-(Y\gamma)\eta(X)].\end{aligned}\tag{18}$$

Now replacing X by φX and Y by φY in the equation (18) and applying the K-contact condition (8), (3), $\eta \circ \varphi = 0$ and $\varphi\xi = 0$ gives

$$g((\varphi Q + Q\varphi)Y, X) = 4ng(\varphi Y, X)$$

for all vector fields Y, Z on M. It follows from last equation that

$$(\varphi Q + Q\varphi)Y = 4n\varphi Y. \tag{19}$$

In view of above equation, it follows from (18) that

$$\frac{\nu(2n\alpha - \gamma) - \beta^2}{\beta}[(Yf)\eta(X) - (Xf)\eta(Y)] = [(Y\gamma)\eta(X) - (X\gamma)\eta(Y)]. \tag{20}$$

Next, taking $\sigma = \nu(2n\alpha - \gamma) - \beta^2$. So $D\sigma = -\nu D\gamma$. On account of these, (20) can be exhibited as

$$\sigma Df + \frac{\beta}{\nu} D\sigma = \left\{\sigma(\xi f) + \frac{\beta}{\nu}(\xi\sigma)\right\}\xi. \tag{21}$$

Differentiating (21) in the direction of X and utilization of (7) provides

$$\begin{aligned}(X\sigma)Df + \sigma\nabla_X Df + \frac{\beta}{\nu}\nabla_X D\sigma = X\left\{\sigma(\xi f) + \frac{\beta}{\nu}(\xi\sigma)\right\}\xi \\ - \left\{\sigma(\xi f) + \frac{\beta}{\nu}(\xi\sigma)\right\}\varphi X.\end{aligned} \tag{22}$$

Taking inner product of (22) with Y and then anti-symmetrizing the resulting equation, we obtain

$$\begin{aligned}(X\sigma)(Yf) - (Y\sigma)(Xf) = X\left\{\sigma(\xi f) + \frac{\beta}{\nu}(\xi\sigma)\right\}\eta(Y) \\ - Y\left\{\sigma(\xi f) + \frac{\beta}{\nu}(\xi\sigma)\right\}\eta(X) \\ - 2\left\{\sigma(\xi f) + \frac{\beta}{\nu}(\xi\sigma)\right\}g(\varphi X, Y).\end{aligned} \tag{23}$$

Now we can write the equation (21) as

$$(X\sigma) = \left\{\frac{\sigma\nu}{\beta}(\xi f) + (\xi\sigma)\right\}\eta(X) - \frac{\sigma\nu}{\beta}(Xf). \tag{24}$$

Substituting (24) in (23), inserting X by φX, Y by φY in the resulting equation and noting that $g(\varphi X, Y) \neq 0$ for any contact metric manifold, we obtain

$$\sigma(\xi f) + \frac{\beta}{\nu}(\xi\sigma) = 0. \tag{25}$$

Making use of (25) in (21), we get

$$(\nu(2n\alpha - \gamma) - \beta^2)(Xf) = -\frac{\beta}{\nu}(X\sigma). \tag{26}$$

On the other hand, taking the trace of (12) over X we obtain

$$\left[\frac{\beta^2+\nu\alpha}{\beta}\right]S(Y,Df)=\frac{\alpha}{2}(Yr)+\frac{\nu(r\alpha-2n\gamma)}{\beta}(Yf)-2n(Y\gamma). \tag{27}$$

Let $\{e_i,\varphi e_i,\xi\}$, $i=1,2,3,\dots,n$ be an orthonormal φ-basis of M such that $Qe_i=\rho_i e_i$. Thus, we have $\varphi Qe_i=\rho_i\varphi e_i$. Substituting e_i for Y in (19), we obtain $Q\varphi e_i=(4n-\rho_i)\varphi e_i$. Using the φ-basis and (8), the scalar curvature r is given by

$$\begin{aligned} r&=g(Q\xi,\xi)+\sum_{i=1}^{n}[g(Qe_i,e_i)+g(Q\varphi e_i,\varphi e_i)]\\ &=g(Q\xi,\xi)+\sum_{i=1}^{n}[\rho_i g(e_i,e_i)+(4n-\rho_i)g(\varphi e_i,\varphi e_i)]\\ &=2n(2n+1). \end{aligned}$$

Making use of the constancy of r, $D\sigma=-\nu D\gamma$ and (26) in (27), it follows that $QDf=2nDf$. Differentiating this along X and recalling (13) and $QDf=2nDf$, we obtain

$$(\nabla_X Q)Df-\frac{\alpha}{\beta}Q^2X+\frac{(\gamma+2n\alpha)}{\beta}QX-\frac{2n\gamma}{\beta}X=0.$$

Contracting the foregoing equation over X and observing that $r=2n(2n+1)$, we get

$$\sum_{i=1}^{2n+1}g((\nabla_{e_i}Q)Df,e_i)-\frac{\alpha}{\beta}|Q|^2+r\frac{(\gamma+2n\alpha)}{\beta}-\frac{r\gamma}{\beta}=0. \tag{28}$$

Using that the scalar curvature is constant, the first term vanishes because $\mathrm{div}Q=\frac{1}{2}dr$ (this follows from the contraction of Bianchi's second identity). From (28), we deduce $|Q|^2=2nr$. Then, since $r=2n(2n+1)$, we get

$$\begin{aligned} \left|Q-\frac{r}{2n+1}I\right|^2&=|Q|^2+\frac{r^2}{2n+1}-\frac{2r^2}{2n+1}\\ &=2nr-\frac{r^2}{2n+1}\\ &=4n^2(2n+1)-4n^2(2n+1)=0. \end{aligned}$$

Since the symmetric tensor $Q-\frac{r}{2n+1}I$ is of length zero, we get

$$Q=\frac{r}{2n+1}I=2nI.$$

This shows that M is Einstein with Einstein constant $2n$. Since M is complete, compactness of M follows from Myers' theorem [19]. Applying the result of

Boyer-Galicki [4], we can conclude that M is Sasakian. Consequently, (13) reduces to

$$\nabla_Y Df = \frac{(\gamma - 2n\alpha)}{\beta} Y - \frac{\nu}{\beta} g(Y, Df) Df. \tag{29}$$

Now consider a smooth function $u = e^{\frac{\nu}{\beta} f}$ on M. From this we have the following relation (see Gomes [18]);

$$Du = \frac{\nu}{\beta} u Df, \tag{30}$$

$$\nabla_Y Df + \frac{\nu}{\beta} g(Y, Df) Df = \frac{\beta}{\nu u} \nabla_Y Du. \tag{31}$$

Comparing (29) and (31), we get

$$\nabla_Y Du = \frac{(\gamma - 2n\alpha)\nu u}{\beta^2} Y. \tag{32}$$

As M is Einstein with constant scalar curvature $r = 2n(2n+1)$, the equation (27) takes the form $(\gamma\nu - 2n\nu\alpha + \beta^2)Df = -\beta D\gamma$. Using (30) in the foregoing equation we immediately infer that

$$(\gamma\nu - 2n\nu\alpha + \beta^2)Du = -\nu u D\gamma.$$

From this we can write $\gamma\nu Du + \nu u D\gamma = (2n\nu\alpha - \beta^2)Du$, which is equivalent to $D(\gamma\nu u) = (2n\nu\alpha - \beta^2)Du$. In other words, $\gamma\nu u = (2n\nu\alpha - \beta^2)u + k$, where k is a constant. This together with (32) gives

$$\nabla_Y Du = \left(-u + \frac{k}{\beta^2}\right) Y. \tag{33}$$

As a result of Theorem 2 of Tashiro [25] it follows that M is isometric to unit sphere $\mathbb{S}^{2n+1}$. This completes the proof. □

Corollary 3.3. *Let $(M, g, \alpha, \beta, \nu, \gamma)$ be a complete gradient Einstein-type manifold. If g represents a Sasakian metric, then it is compact, Einstein and isometric to the unit sphere $\mathbb{S}^{2n+1}$.*

Proof. This follows with the same proof as Corollary 3.1 in [21]. □

Further, we remark that our Theorem 3.2 generalizes the results of Ghosh [11, 12, 14] on K-contact manifold admitting Ricci almost soliton, (m, ρ)-quasi-Einstein metric and generalized m-quasi-Einstein metric.

4. (κ, μ)-contact manifold satisfying gradient Einstein-type metrics

Blair et al. [2] introduced a (κ, μ)-contact manifold which is a contact metric manifold $(M, \varphi, \xi, \eta, g)$ whose curvature tensor satisfies

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \tag{34}$$

for all vector fields X, Y on M and for some real numbers (κ, μ). Later on, Boeckx [3] classified these manifolds completely. This type of manifold is obtained by applying the D-homothetic deformation to a contact metric manifold that satisfies $R(X,Y)\xi = 0$. This class contains Sasakian manifolds (for $\kappa = 1$) and the trivial sphere bundle $E^{n+1} \times S^n(4)$ (for $\kappa = \mu = 0$). Examples of non-Sasakian (κ, μ)-contact metric manifolds are the unit tangent bundles of Riemannian manifolds of constant curvature $\neq 1$. A lot of examples of (κ, μ)-contact structures can be constructed because of a D-homothetic deformation preserves (κ, μ)-contact structures (see [2]). On non-Sasakian (κ, μ)-contact manifolds, the following formulas are also true [2]:

$$\begin{aligned} QX = {} & [2(n-1) - n\mu]X + [2(n-1) + \mu]hX \\ & + [2(1-n) + n(2\kappa + \mu)]\eta(X)\xi, \end{aligned} \tag{35}$$

$$Q\xi = 2n\kappa\xi, \tag{36}$$

$$h^2 = (\kappa - 1)\varphi^2, \quad \kappa < 1. \tag{37}$$

For the non-Sasakian case, i.e., $\kappa < 1$, the equation (34) determines the curvature of M completely. As a result of this, it is proved that a non-Sasakian (κ, μ)-contact manifold is locally homogeneous and hence analytic [3]. Moreover, the scalar curvature r of such manifold is given

$$r = 2n(2(n-1) + \kappa - n\mu), \tag{38}$$

which is constant. On a (κ, μ)-contact manifold we have

$$(\nabla_\xi Q)X = \mu(2(n-1) + \mu)h\varphi X \tag{39}$$

for any vector field X on M.

Here we intend to examine the existence of gradient Einstein-type metric on (κ, μ)-contact manifold, and prove the following fruitful outcome.

Theorem 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian (κ, μ)-contact manifold. If there exists a gradient Einstein-type structure $(f, \alpha, \beta, \nu, \gamma)$ associated with the metric g, then for $n = 1$, M is flat, and for $n > 1$, M is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$.*

Proof. First, differentiate (36) covariantly along an arbitrary vector field X and utilization of (5), we obtain

$$(\nabla_X Q)\xi = Q(\varphi + \varphi h)X - 2nk(\varphi + \varphi h)X. \tag{40}$$

Thus, taking the scalar product of (12) with ξ and using (36), the equation (40) gives

$$\begin{aligned} g(R(X,Y)Df, \xi) = {} & \frac{\alpha}{\beta}\{g(Q\varphi Y + \varphi QY, X) + g(Q\varphi hY + h\varphi QY, X) \\ & - 4nkg(\varphi Y, X)\} + \frac{\nu(2nk\alpha - \gamma)}{\beta^2}\{(Yf)\eta(X) - (Xf)\eta(Y)\} \end{aligned}$$

$$+\frac{1}{\beta}\{(X\gamma)\eta(Y)-(Y\gamma)\eta(X)\}. \tag{41}$$

Replacing X with φX, and Y with φY in (41), and observing that $R(\varphi X,\varphi Y)\xi = 0$ (this follows from (34)), we get

$$\alpha\{Q\varphi Y+\varphi QY-\varphi QhY-hQ\varphi Y-4nk\varphi Y\}=0.$$

Since $\alpha\neq 0$, the above equation gives

$$Q\varphi Y+\varphi QY-\varphi QhY-hQ\varphi Y-4nk\varphi Y=0. \tag{42}$$

As a result of (35), one can get

$$(Q\varphi+\varphi Q)Y=2[2(n-1)-n\mu]\varphi Y.$$

Inserting X by φX in (35) and then applying h to the resulting equation, and utilization of (37) implies that

$$hQ\varphi X=[2(n-1)-n\mu]h\varphi X-(\kappa-1)[2(n-1)+\mu]\varphi X.$$

Moreover, applying φ to (35) and then using hX instead of X in the resulting equation, and using (37), we get

$$\varphi QhX=[2(n-1)-n\mu]\varphi hX-(\kappa-1)[2(n-1)+\mu]\varphi X.$$

Utilization of last three equations in the equation (42) yields

$$\kappa(\mu-2)=\mu(n+1). \tag{43}$$

Substituting ξ instead of X in (41), recalling (34) and (36) we obtain

$$\mu\beta hDf=\left[\frac{\nu(2nk\alpha-\gamma)}{\beta}-\kappa\beta\right]\{Df-(\xi f)\xi\}+\{(\xi\gamma)\xi-D\gamma\}. \tag{44}$$

Since the scalar curvature is constant, it follows from (27) that

$$D\gamma=\frac{\nu(r\alpha-2n\gamma)}{2n\beta}Df-\frac{(\beta^2+\nu\alpha)}{2n\beta}QDf. \tag{45}$$

This in combination with (44) implies that

$$\mu hDf=\left[\frac{4n^2\kappa\nu\alpha-2nk\beta^2-\nu r\alpha}{2n}\right]Df+\left[\frac{r\nu\alpha-2nk\nu\alpha-4n^2\kappa\nu\alpha}{2n}\right](\xi f)\xi$$
$$+\frac{\beta^2+\nu\alpha}{2n}QDf. \tag{46}$$

In a (κ,μ)-contact manifold, the following relationship is well established (see [2])

$$(\nabla_\xi h)=\mu h\varphi. \tag{47}$$

From (13), we have

$$\nabla_\xi Df=\frac{\gamma-2nk\alpha}{\beta}\xi-\frac{\nu}{\beta}(\xi f)Df. \tag{48}$$

Differentiating (46) along ξ and taking into account (39), (46)-(48) we ultimately obtain

$$\mu^2 h\varphi Df = \frac{\nu b}{\beta}(\xi f)\xi + \left[\frac{a(\gamma - 2nk\alpha) + b(\gamma - 2nk\alpha - \nu(\xi f)^2)}{\beta}\right]\xi + \frac{\beta^2 + \nu\alpha}{2n}[\mu(2(n-1) + \mu)h\varphi Df], \tag{49}$$

where we used $g(\nabla_\xi Df, \xi) = \xi(\xi f)$. Here,

$$a = \frac{4n^2\kappa\nu\alpha - 2nk\beta^2 - \nu r\alpha}{2n} \text{ and } b = \frac{r\nu\alpha - 2nk\nu\alpha - 4n^2\kappa\nu\alpha}{2n}.$$

Applying φ to the above equation, we obtain

$$\left\{\mu^2 - \mu\left(\frac{\beta^2 + \nu\alpha}{2n}\right)[2(n-1) + \mu]\right\} hDf = 0. \tag{50}$$

Furthermore, operating the preceding equation by h and using (37), it follows that

$$\mu\left[\mu(2n - (\beta^2 + \nu\alpha)) - 2(\beta^2 + \nu\alpha)(n-1)\right](\kappa - 1)\varphi^2 Df = 0. \tag{51}$$

Since M is non-Sasakian, we have either (i) $\mu = 0$ or (ii) $\varphi^2 Df = 0$ or (iii) $\mu = \frac{2(\beta^2+\nu\alpha)(n-1)}{2n-(\beta^2+\nu\alpha)}$.

Case (i). Here, it follows from (43) that $\kappa = 0$ because of $\mu = 0$. Hence $R(X, Y)\xi = 0$, according to the result of Blair [1] we obtain that M is flat in dimension 3 and in higher dimensions it is locally isometric to the trivial bundle $\mathbb{E}^{(n+1)} \times \mathbb{S}^n(4)$.

Case (ii). Making use of (3) in $\varphi^2 Df = 0$ yields $Df = (\xi f)\xi$. Differentiating this along X, employing (3) gives that $\nabla_X Df = X(\xi f)\xi - (\xi f)(\varphi X + \varphi hX)$. As a result of Poincare lemma $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$, the last equation provides

$$X(\xi f)\eta(Y) - Y(\xi f)\eta(X) + 2(\xi f)g(X, \varphi Y) = 0.$$

Replacing X and Y with φX and φY, respectively, in the above equation furnishes $\xi f = 0$, where we applied $g(X, \varphi Y) \neq 0$ for any contact metric structure. By virtue of this, we have $Df = 0$, i.e., f is constant and consequently (13) shows that M is Einstein, i.e., $QX = \frac{\gamma}{\alpha}X = 2n\kappa X$ by (48). Contracting this over X we find that the scalar curvature $r = 2n\kappa(2n+1)$. It shows $n\mu = 2(n-1) - 2n\kappa$ in combination with (38). On the other hand, we can easily find $[2(n-1) + \mu]h = 0$ from (35) on the basis of last equation and $QX = 2n\kappa X$. Since M is non-Sasakian, we must have $2(n-1) + \mu = 0$. So it follows for dimension 3 that $\mu = 0 = \kappa$, and by applying Blair's result [1] we obtain that M is flat. Again, for higher dimension it follows from $\mu = 2(1-n)$ and (43) that $\kappa = n - \frac{1}{n} > 1$, contradicting our assumption.

Case (iii). Since $\mu = \frac{2(\beta^2+\nu\alpha)(n-1)}{2n-(\beta^2+\nu\alpha)}$, it follows from (43) that

$$\kappa = \frac{(\beta^2+\nu\alpha)(n^2-1)}{n(\beta^2+\nu\alpha)-2n}.$$

For $n = 1$, it follows that $\mu = \kappa = 0$ and hence flat. For $n > 1$, making use of (35) in (46) provides

$$[4n^2\kappa\nu\alpha - 2n\kappa\beta^2 - \nu r\alpha + (\beta^2+\nu\alpha)\{2(n-1) - n\mu\}]\{Df - (\xi f)\xi\} + [2(\beta^2+\nu\alpha)(n-1) + \mu((\beta^2+\nu\alpha) - 2n)]hDf = 0.$$

By virtue of $\mu = \frac{2(\beta^2+\nu\alpha)(n-1)}{2n-(\beta^2+\nu\alpha)}$, the above equation entails that

$$[4n^2\kappa\nu\alpha - 2n\kappa\beta^2 - \nu r\alpha + (\beta^2+\nu\alpha)\{2(n-1) - n\mu\}]\{Df - (\xi f)\xi\} = 0.$$

If $Df - (\xi f)\xi = 0$, then proceeding as in Case (ii) it follows that, for $n > 1$, a contraction. Therefore, we only have $4n^2\kappa\nu\alpha - 2n\kappa\beta^2 - \nu r\alpha + (\beta^2+\nu\alpha)\{2(n-1) - n\mu\} = 0$. This together with (38) entails that

$$((2n-1)\nu\alpha - \beta^2)[2(1-n) + n(2\kappa+\mu)] = 0,$$

which implies that either $\beta^2 = (2n-1)\nu\alpha$, or $2(1-n) + n(2\kappa+\mu) = 0$. The former case shows that $\kappa > 1$, a contradiction. For later case, utilization of $\mu = \frac{2(\beta^2+\nu\alpha)(n-1)}{2n-(\beta^2+\nu\alpha)}$ and $\kappa = \frac{(\beta^2+\nu\alpha)(n^2-1)}{n(\beta^2+\nu\alpha)-2n}$, the last equation transforms into

$$\beta^2 + \nu\alpha = \frac{2n-2n^2}{n^3-2n^2+1}.$$

Making use of this in $\kappa = \frac{(\beta^2+\nu\alpha)(n^2-1)}{n(\beta^2+\nu\alpha)-2n}$, we obtain $\kappa = 1$, and this leads to a contradiction as M is non-Sasakian. This establishes the proof. □

It is known [18] that *a compact Riemannian manifold admitting a nontrivial gradient Einstein-type metric with constant scalar curvature is isometric to the standard sphere*. But a contact metric manifold of constant curvature is a Sasakian manifold of constant curvature in dimension > 3 [20]. On the other hand, in dimension 3, it is either flat or Sasakian manifold of constant curvature 1 (see Blair [1]). From (38) we see that the scalar curvature of a (κ,μ)-space is constant. Thus, for a compact (κ,μ)-contact manifold we have the following:

Corollary 4.2. *If a compact (κ,μ)-contact manifold admits a gradient Einstein-type metric, then in dimension 3 it is either flat or Sasakian and for higher dimensions it is isometric to a unit sphere $\mathbb{S}^{2n+1}$.*

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