Commun. Korean Math. Soc. **35** (2020), No. 2, pp. 625–637
https://doi.org/10.4134/CKMS.c190121
pISSN: 1225-1763 / eISSN: 2234-3024

# $*$-RICCI SOLITONS AND $*$-GRADIENT RICCI SOLITONS ON 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to characterize 3-dimensional trans-Sasakian manifolds of type $(\alpha, \beta)$ admitting $*$-Ricci solitons and $*$-gradient Ricci solitons. Under certain restrictions on the smooth functions $\alpha$ and $\beta$, we have proved that a trans-Sasakian 3-manifold of type $(\alpha, \beta)$ admitting a $*$-Ricci soliton reduces to a $\beta$-Kenmotsu manifold and admitting a $*$-gradient Ricci soliton is either flat or $*$-Einstein or it becomes a $\beta$-Kenmotsu manifold. Also an illustrative example is presented to verify our results.

## 1. Introduction

The study of Ricci solitons is a very interesting topic in differential geometry and physics. The notion of Ricci soliton was introduced by Hamilton [11] as a natural generalization of Einstein metrics. A Ricci soliton $(g, V, \lambda)$ is defined on a Riemannian manifold $(M, g)$ by

$$\frac{1}{2}\pounds_V g + S = \lambda g, \tag{1}$$

where $\pounds_V g$ denotes the Lie derivative of the Riemannian metric $g$ in the direction of the vector field $V$, $S$ is the Ricci tensor associated to $g$ and $\lambda$ is a constant. The Ricci soliton is said to be expanding, steady or shrinking according as $\lambda$ is negative, zero or positive, respectively. Ricci solitons have been studied by Wang ([20, 21]) and many others on certain class of almost contact metric manifolds.

In 1959, Tachibana [17] introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds. Later in [10], Hamada defined the $*$-Ricci tensor of real hypersurfaces in a non-flat complex space form by

$$S^*(X, Y) = g(Q^*X, Y) = \frac{1}{2}(trace\{\phi \circ R(X, \phi Y)\}), \tag{2}$$

Received April 1, 2019; Revised September 1, 2019; Accepted September 26, 2019.
2010 *Mathematics Subject Classification.* Primary 53D15, 53C25.
*Key words and phrases.* Trans-Sasakian manifolds, $*$-Ricci soliton, $*$-gradient Ricci soliton, $*$-Einstein manifold.

where $Q^*$ is the $(1,1)$ $*$-Ricci operator for any vector fields $X$, $Y$ on $M$. The $*$-scalar curvature is denoted by $r^*$ and is defined by $r^* = trace(Q^*)$.

**Definition.** A Riemannian manifold $(M, g)$ of dimension $n > 2$ is called $*$-Einstein, if the $*$-Ricci tensor $S^*$ satisfies the relation

$$S^* = \mu g,$$

where $\mu$ is a constant.

In 2014, Kaimakamis and Panagiotidou [12] introduced the notion of $*$-Ricci soliton in non-flat complex space forms just by replacing the Ricci tensor $S$ in (1) with the $*$-Ricci tensor $S^*$, which is given by

$$\frac{1}{2}\pounds_V g + S^* = \lambda g. \tag{3}$$

They have proved that for real hypersurfaces in $\mathbb{C}P^n$, $n \geq 2$, $(g, \xi, \lambda)$ cannot be a $*$-Ricci soliton. Further they proved that a real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, admitting a $*$-Ricci soliton with the potential vector field $\xi$ is locally congruent to a geodesic hypersurface. This notion has been studied by Ghosh and Patra [9].

Let $(M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold [1]. Then, the product $\overline{M} = M \times \mathbb{R}$ has a natural almost complex structure $J$, which makes $(\overline{M}, G)$ an almost Hermitian manifold, where $G$ is the product metric. The geometry of the almost Hermitian manifold $(\overline{M}, J, G)$ dictates the geometry of the almost contact metric manifold $(M, \phi, \xi, \eta, g)$ and gives different structures on $M$ like Sasakian structure, quasi-Sasakian structure, Kenmotsu structure and others (see [1], [3] and [13]). It is known that there are sixteen different types of structures on the almost Hermitian manifold $(\overline{M}, J, G)$ [8], and recently, using the structure in the class $\mathcal{W}_4$ on $(\overline{M}, J, G)$ a structure $(\phi, \xi, \eta, g, \alpha, \beta)$ on $M$ called trans-Sasakian structure was introduced [15], which generalizes Sasakian structure and Kenmotsu structure on almost contact metric manifolds ([3], [13]), where $\alpha$, $\beta$ are smooth functions defined on $M$. Since the introduction of trans-Sasakian manifolds, important contributions of Blair and Oubiña [3] and Marrero [14] have appeared to study the geometry of trans-Sasakian manifolds. In general, a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$ and trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called a cosymplectic, an $\alpha$-Sasakian and a $\beta$-Kenmotsu manifolds, respectively, provided $\alpha$, $\beta \in \mathbb{R}$ [19]. Marrero [14] has shown that a trans-Sasakian manifold of dimension $\geq 5$ is either a cosymplectic manifold, an $\alpha$-Sasakian manifold or a $\beta$-Kenmotsu manifold. Since then there is an attention on studying the geometry of 3-dimensional trans-Sasakian manifolds only. In ([4–6]), authors have studied 3-dimensional trans-Sasakian manifolds with some restrictions on the smooth functions $\alpha$, $\beta$.

Throughout the paper we assume that the smooth functions $\alpha$ and $\beta$ satisfy the condition

$$\phi\,\mathrm{grad}\,\alpha = \mathrm{grad}\,\beta. \tag{4}$$

Then it follows that

$$X\beta + (\phi X)\alpha = 0 \tag{5}$$

and hence $\xi\beta = 0$.

**Example 1.1.** In [7], the authors have constructed an example of a 3-dimensional trans-Sasakian manifold. They consider $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}),\ \ e_2 = e^{-z}\frac{\partial}{\partial y},\ \ e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of $M$. Let $g$ be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0.$$

Let $\eta$ be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field $X$. Let $\phi$ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_2,\ \ \phi(e_2) = -e_1,\ \ \phi(e_3) = 0.$$

Then we have

$$\phi^2(X) = -X + \eta(X)e_3,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields $X$, $Y$. For $e_3 = \xi$, they have shown that $(M, \phi, \xi, \eta, g)$ forms a trans-Sasakian manifold of type $(\alpha, \beta)$, where $\alpha = \frac{1}{2}e^{-2z}$ and $\beta = 1$. Then it follows that $\phi\,\mathrm{grad}\,\alpha = -e^{-2z}\phi e_3 = 0 = \mathrm{grad}\,\beta$. Thus the existence of trans-Sasakian manifolds of type $(\alpha, \beta)$ satisfying (4) is verified.

Since trans-Sasakian manifolds generalize a large class of almost contact metric manifolds, we consider the notion of $*$-Ricci soliton and $*$-gradient Ricci soliton in the framework of 3-dimensional trans-Sasakian manifolds of type $(\alpha, \beta)$. Finally an illustrative example is presented to verify our results.

## 2. Preliminaries

Let $(M, \phi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold, where $\phi$ is a $(1, 1)$-tensor field, $\xi$ a unit vector field and $\eta$ the smooth 1-form dual to $\xi$ with respect to the Riemannian metric $g$ satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \ \eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \eta \circ \phi = 0 \tag{6}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$

$X, Y \in \chi(M)$, where $\chi(M)$ denotes the Lie algebra of smooth vector fields on $M$ [1]. If there are smooth functions $\alpha$, $\beta$ on an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ satisfying

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{8}$$

$X, Y \in \chi(M)$, then it is said to be a trans-Sasakian manifold, where $\nabla$ is the Levi-Civita connection with respect to the metric $g$ ([3], [14], [16]). We shall denote the trans-Sasakian manifold by $(M, \phi, \xi, \eta, g, \alpha, \beta)$ and it is called trans-Sasakian manifold of type $(\alpha, \beta)$. From (8) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{9}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{10}$$

A trans-Sasakian manifold is said to be

- cosymplectic or co-Kaehler if $\alpha = \beta = 0$,
- quasi-Sasakian if $\beta = 0$ and $\xi(\alpha) = 0$,
- $\alpha$-Sasakian if $\alpha$ is a non-zero constant and $\beta = 0$,
- $\beta$-Kenmotsu if $\alpha = 0$ and $\beta$ is a non-zero constant.

Therefore, trans-Sasakian manifolds generalize a large class of almost contact manifolds. From [4] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0. \tag{11}$$

If $M$ satisfies the condition (2), then from [4] we have

$$S(X, Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X, Y) - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y), \tag{12}$$

$$\begin{aligned} R(X, Y)Z = {} & (\frac{r}{2} - 2(\alpha^2 - \beta^2))(g(Y, Z)X - g(X, Z)Y) \\ & - g(Y, Z)(\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ & + g(X, Z)(\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ & - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)X \\ & + (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)Y. \end{aligned} \tag{13}$$

From (12) we get

$$S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X) \tag{14}$$

and from (13) it follows that

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \tag{15}$$

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X). \tag{16}$$

## 3. $*$-Ricci soliton

In this section we consider the notion of $*$-Ricci soliton in the framework of 3-dimensional trans-Sasakian manifold. Before proving our main Theorem in this section, we first state and prove the following:

**Lemma 3.1.** *In a 3-dimensional trans-Sasakian manifold satisfying* (4)*, the $*$-Ricci tensor $S^*$ is given by*

$$S^*(X,Y) = (\frac{r}{2} - 2(\alpha^2-\beta^2))(g(X,Y) - \eta(X)\eta(Y)) \tag{17}$$

*for any vector fields $X$, $Y$ on $M$.*

*Proof.* Substituting $Y = \phi Y$ and $Z = \phi Z$ in (13) we obtain

$$R(X,\phi Y)\phi Z = (\frac{r}{2} - 2(\alpha^2-\beta^2))(g(Y,Z)X - \eta(Y)\eta(Z)X - g(X,\phi Z)\phi Y). \tag{18}$$

Now taking the inner product of the above equation with $W$ and then contracting $Z$ and $W$ we complete the proof. □

*Remark* 3.2. It is important to note that the $*$-Ricci tensor is not necessarily symmetric. From Lemma 3.1, it is clear that the $*$-Ricci tensor $S^*$ of a 3-dimensional trans-Sasakian manifold satisfying (4) is symmetric, that is, $S^*(X,Y) = S^*(Y,X)$ for all vector fields $X$, $Y$ on $M$. Therefore, the Ricci soliton equation is consistent in this setting.

**Theorem 3.3.** *A 3-dimensional trans-Sasakian manifold $(M,\phi,\xi,\eta,g,\alpha,\beta)$ satisfying the condition* (4) *and admitting $*$-Ricci soliton reduces to a $\beta$-Kenmotsu manifold, provided $(\xi r - 4\xi(\alpha^2-\beta^2)) - \beta(r - 4(\alpha^2-\beta^2)) \neq 0$.*

*Proof.* Substituting the value of $S^*$ from (17) in (3) we have

$$\begin{aligned}(\pounds_V g)(X,Y) &= (2\lambda - r + 4(\alpha^2-\beta^2))g(X,Y)\\ &\quad + (r - 4(\alpha^2-\beta^2))\eta(X)\eta(Y).\end{aligned} \tag{19}$$

Differentiating the above equation covariantly along any vector field $Z$ and using (10) we obtain

$$\begin{aligned}(\nabla_Z \pounds_V g)(X,Y) &= (-Zr + 4Z(\alpha^2-\beta^2))(g(X,Y) - \eta(X)\eta(Y))\\ &\quad + (r - 4(\alpha^2-\beta^2))[-\alpha g(\phi Z,X)\eta(Y) + \beta g(X,Z)\eta(Y)\\ &\quad - \alpha\eta(X)g(\phi Z,Y) + \beta\eta(X)g(Z,Y) - 2\beta\eta(X)\eta(Y)\eta(Z)].\end{aligned} \tag{20}$$

It is well known that ([22, p. 23])

$$\begin{aligned}&(\pounds_V \nabla_X g - \pounds_X \nabla_V g - \nabla_{[V,X]} g)(Y,Z)\\ &= -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y).\end{aligned}$$

Since $g$ is parallel with respect to the Levi-Civita connection $\nabla$, then the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{21}$$

Since $\pounds_V\nabla$ is symmetric, then it follows from (21) that

$$\begin{aligned} g((\pounds_V\nabla)(X,Y),Z) = \frac{1}{2}(\nabla_X\pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y\pounds_V g)(X,Z) \\ -\frac{1}{2}(\nabla_Z\pounds_V g)(X,Y). \end{aligned} \tag{22}$$

Using (19) in (22) we have

$$\begin{aligned} 2g((\pounds_V\nabla)(X,Y),Z) = {} & (-Xr+4X(\alpha^2-\beta^2))(g(Y,Z)-\eta(Y)\eta(Z)) \\ & + (r-4(\alpha^2-\beta^2))(-\alpha g(\phi X,Y)\eta(Z) \\ & + \beta g(X,Y)\eta(Z) - \alpha g(\phi X,Z)\eta(Y) \\ & + \beta g(X,Z)\eta(Y) - 2\beta\eta(X)\eta(Y)\eta(Z)) \\ & + (-Yr+4Y(\alpha^2-\beta^2))(g(X,Z)-\eta(X)\eta(Z)) \\ & + (r-4(\alpha^2-\beta^2))(-\alpha g(\phi Y,X)\eta(Z) \\ & + \beta g(X,Y)\eta(Z) - \alpha g(\phi Y,Z)\eta(X) \\ & + \beta g(Y,Z)\eta(X) - 2\beta\eta(X)\eta(Y)\eta(Z)) \\ & - (-Zr+4Z(\alpha^2-\beta^2))(g(X,Y)-\eta(X)\eta(Y)) \\ & - (r-4(\alpha^2-\beta^2))(-\alpha g(\phi Z,X)\eta(Y) \\ & + \beta g(Z,X)\eta(Y) - \alpha g(\phi Z,Y)\eta(X) \\ & + \beta g(Y,Z)\eta(X) - 2\beta\eta(X)\eta(Y)\eta(Z)), \end{aligned} \tag{23}$$

which implies

$$\begin{aligned} 2(\pounds_V\nabla)(X,Y) = {} & (Dr-4D(\alpha^2-\beta^2))(g(X,Y)-\eta(X)\eta(Y)) \\ & - (Xr-4X(\alpha^2-\beta^2))(Y-\eta(Y)\xi) \\ & - (Yr-4Y(\alpha^2-\beta^2))(X-\eta(X)\xi) \\ & + (r-4(\alpha^2-\beta^2))(2\beta g(X,Y)\xi - 2\alpha\eta(X)\phi Y \\ & - 2\alpha\eta(Y)\phi X - 2\beta\eta(X)\eta(Y)\xi), \end{aligned} \tag{24}$$

where $D$ denotes the gradient operator. Now, replacing $Y$ by $\xi$ in the foregoing equation we get

$$(\pounds_V\nabla)(X,\xi) = -\frac{1}{2}(\xi r-4\xi(\alpha^2-\beta^2))(X-\eta(X)\xi) - \alpha(r-4(\alpha^2-\beta^2))\phi X. \tag{25}$$

Differentiating (25) covariantly with respect to $Y$ yields

$$\begin{aligned} \nabla_Y(\pounds_V\nabla)(X,\xi) = {} & -\frac{1}{2}(Y(\xi r)-4Y(\xi(\alpha^2-\beta^2)))(X-\eta(X)\xi) \\ & -\frac{1}{2}(\xi r-4\xi(\alpha^2-\beta^2))(\nabla_Y X-(\nabla_Y\eta(X))\xi-\eta(X)\nabla_Y\xi) \\ & -(Y\alpha)(r-4(\alpha^2-\beta^2))\phi X - \alpha(Yr-4Y(\alpha^2-\beta^2))\phi X \\ & -\alpha(r-4(\alpha^2-\beta^2))\nabla_Y\phi X. \end{aligned} \tag{26}$$

Again, using (9) and (25) we obtain

$$\begin{aligned}(\nabla_Y \pounds_V \nabla)(X,\xi) &= \nabla_Y(\pounds_V\nabla)(X,\xi) - (\pounds_V\nabla)(\nabla_Y X,\xi) - (\pounds_V\nabla)(X,\nabla_Y\xi)\\ &= \nabla_Y(\pounds_V\nabla)(X,\xi) + \frac{1}{2}(\xi r - 4\xi(\alpha^2-\beta^2))(\nabla_Y X\\ &\quad - \eta(\nabla_Y X)\xi) + \alpha(r-4(\alpha^2-\beta^2))\phi(\nabla_Y X)\\ &\quad + \alpha(\pounds_V\nabla)(X,\phi Y) - \beta(\pounds_V\nabla)(X,Y)\\ &\quad + \frac{1}{2}\beta\eta(Y)[-(\xi r - 4\xi(\alpha^2-\beta^2))(X-\eta(X)\xi)\\ &\quad - \alpha(r-4(\alpha^2-\beta^2))\phi X],\end{aligned} \tag{27}$$

which implies

$$\begin{aligned}\nabla_Y(\pounds_V\nabla)(X,\xi) &= (\nabla_Y\pounds_V\nabla)(X,\xi) - \alpha(\pounds_V\nabla)(X,\phi Y) + \beta(\pounds_V\nabla)(X,Y)\\ &\quad + \frac{1}{2}\beta\eta(Y)[(\xi r - 4\xi(\alpha^2-\beta^2))(X-\eta(X)\xi)\\ &\quad + \alpha(r-4(\alpha^2-\beta^2))\phi X]\\ &\quad - \frac{1}{2}(\xi r - 4\xi(\alpha^2-\beta^2))(\nabla_Y X - \eta(\nabla_Y X)\xi).\end{aligned} \tag{28}$$

Equating (26) and (28) and then using (9) and (10) we obtain

$$\begin{aligned}(\nabla_Y\pounds_V\nabla)(X,\xi) &= \alpha(\pounds_V\nabla)(X,\phi Y) - \beta(\pounds_V\nabla)(X,Y)\\ &\quad - \frac{1}{2}\beta\eta(Y)[(\xi r - 4\xi(\alpha^2-\beta^2))(X-\eta(X)\xi)\\ &\quad + \alpha(r-4(\alpha^2-\beta^2))\phi X]\\ &\quad - \frac{1}{2}(\xi r - 4\xi(\alpha^2-\beta^2))(\alpha g(\phi Y,X)\xi - \beta g(X,Y)\xi\\ &\quad + 2\beta\eta(X)\eta(Y)\xi + \alpha\eta(X)\phi Y - \beta\eta(X)Y))\\ &\quad - \frac{1}{2}(Y(\xi r) - 4Y(\xi(\alpha^2-\beta^2)))(X-\eta(X)\xi).\end{aligned} \tag{29}$$

Using the foregoing equation in the following formula ([22, p. 23])

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z),$$

and using (25) we infer that

$$\begin{aligned}(\pounds_V R)(X,\xi)\xi &= (\xi r - 4\xi(\alpha^2-\beta^2))(-\alpha\phi X + \beta(X-\eta(X)\xi))\\ &\quad + \alpha^2(r-4(\alpha^2-\beta^2))(X-\eta(X)\xi)\\ &\quad + \frac{1}{2}\alpha\beta(r-4(\alpha^2-\beta^2))\phi X\\ &\quad + \frac{1}{2}(\xi(\xi r) - 4\xi(\xi(\alpha^2-\beta^2)))(X-\eta(X)\xi).\end{aligned} \tag{30}$$

Now, substituting $Y=\xi$ in (19) we obtain

$$(\pounds_V g)(X,\xi) = 2\lambda\eta(X). \tag{31}$$

Now Lie-differentiating $g(X,\xi) = \eta(X)$ along $V$ and using the above equation we get

$$(\pounds_V \eta)X - g(X, \pounds_V \xi) - 2\lambda\eta(X) = 0. \tag{32}$$

From the foregoing equation, we can easily obtain that $\eta(\pounds_V \xi) = -\lambda$ and $(\pounds_V \eta)\xi = \lambda$. Again from (15) we have

$$R(X,\xi)\xi = (\alpha^2 - \beta^2)(X - \eta(X)\xi). \tag{33}$$

Again, we have

$$(\pounds_V R)(X,\xi)\xi = \pounds_V R(X,\xi)\xi - R(\pounds_V X,\xi)\xi - R(X,\pounds_V \xi)\xi - R(X,\xi)\pounds_V \xi.$$

Using (15), (32) and (33) in the above equation we deduce that

$$(\pounds_V R)(X,\xi)\xi = (V(\alpha^2 - \beta^2) + 2\lambda(\alpha^2 - \beta^2))(X - \eta(X)\xi). \tag{34}$$

Now, equating the value of $(\pounds_V R)(X,\xi)\xi$ from (30) and (34) and then taking the inner product with $\phi X$, where $X$ is not collinear with $\xi$, we get

$$\alpha g(\phi X, \phi X)[(\xi r - 4\xi(\alpha^2 - \beta^2)) - \beta(r - 4(\alpha^2 - \beta^2))] = 0, \tag{35}$$

which implies $\alpha = 0$, since $(\xi r - 4\xi(\alpha^2 - \beta^2)) - \beta(r - 4(\alpha^2 - \beta^2)) \neq 0$ by hypothesis. If $\alpha = 0$, then from (5) it follows that $\beta = \textit{constant}$ and hence the manifold is $\beta$-Kenmotsu. This completes the proof. □

## 4. ∗-gradient Ricci soliton

If the potential vector field $V$ is gradient of some smooth function $f$, i.e., $V = Df$, then the ∗-Ricci soliton is called ∗-gradient Ricci soliton and the equation (3) reduces to

$$\nabla^2 f = \lambda g - S^*. \tag{36}$$

The proof of our Theorem in this section relies on the following Lemmas:

**Lemma 4.1** ([2])**.** *A contact metric manifold $M^{2n+1}$ satisfying the condition $R(X,Y)\xi = 0$ for all $X$, $Y$ is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

**Lemma 4.2.** *In a 3-dimensional trans-Sasakian manifold satisfying (4) the following relation holds*

$$\begin{aligned}(\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y = & -(\frac{r}{2} - 2(\alpha^2 - \beta^2))(-\alpha\phi Y + \beta(Y - \eta(Y)\xi)) \\ & -(\frac{\xi r}{2} - 2\xi(\alpha^2 - \beta^2))(Y - \eta(Y)\xi).\end{aligned} \tag{37}$$

*Proof.* From Lemma 3.1 we can write

$$Q^* X = (\frac{r}{2} - 2(\alpha^2 - \beta^2))(X - \eta(X)\xi). \tag{38}$$

Differentiating the above equation covariantly with respect to $Y$ we get

$$\begin{aligned}\nabla_Y Q^* X = &(\frac{Yr}{2} - 2Y(\alpha^2 - \beta^2))(X - \eta(X)\xi)\\ &+ (\frac{r}{2} - 2(\alpha^2 - \beta^2))(\nabla_Y X - (\nabla_Y \eta(X))\xi - \eta(X)\nabla_Y \xi).\end{aligned} \tag{39}$$

Therefore, using (38) and (39) we have

$$\begin{aligned}(\nabla_Y Q^*)X &= \nabla_Y Q^* X - Q^*(\nabla_Y X)\\ &= (\frac{Yr}{2} - 2Y(\alpha^2 - \beta^2))(X - \eta(X)\xi)\\ &\quad + (\frac{r}{2} - 2(\alpha^2 - \beta^2))(\nabla_Y X - (\nabla_Y \eta(X))\xi - \eta(X)\nabla_Y \xi)\\ &\quad - (\frac{r}{2} - 2(\alpha^2 - \beta^2))(\nabla_Y X - \eta(\nabla_Y X)\xi).\end{aligned} \tag{40}$$

Replacing $X$ by $\xi$ in the above equation and using the fact that $\eta(\nabla_Y \xi) = 0$ and (9) we have

$$(\nabla_Y Q^*)\xi = -(\frac{r}{2} - 2(\alpha^2 - \beta^2))(-\alpha\phi Y + \beta(Y - \eta(Y)\xi)). \tag{41}$$

Again substituting $Y = \xi$ in (40) and using (10) we get

$$(\nabla_\xi Q^*)Y = (\frac{\xi r}{2} - 2\xi(\alpha^2 - \beta^2))(Y - \eta(Y)\xi). \tag{42}$$

Now, substracting (42) from (41), the result follows. □

**Lemma 4.3.** *In a* 3*-dimensional trans-Sasakian manifold satisfying* (4) *admitting a* $*$*-gradient Ricci soliton, the following relation holds*

$$R(X,Y)Df = (\nabla_Y Q^*)X - (\nabla_X Q^*)Y. \tag{43}$$

*Proof.* From (36) we can write

$$\nabla_Y Df = \lambda Y - Q^* Y. \tag{44}$$

Differentiating the above equation covariantly along any vector field $X$ we obtain

$$\nabla_X \nabla_Y Df = \lambda \nabla_X Y - \nabla_X Q^* Y. \tag{45}$$

Interchanging $X$ and $Y$ in (45) we get

$$\nabla_Y \nabla_X Df = \lambda \nabla_Y X - \nabla_Y Q^* X. \tag{46}$$

Again from (44) we have

$$\nabla_{[X,Y]} Df = \lambda(\nabla_X Y - \nabla_Y X) - Q^*(\nabla_X Y - \nabla_Y X). \tag{47}$$

Therefore, the result follows by using (45)-(47) in

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df.$$ □

**Theorem 4.4.** *A 3-dimensional trans-Sasakian manifold* $(M, \phi, \xi, \eta, g, \alpha, \beta)$ *satisfying the condition* (4) *and admitting a* $*$*-gradient Ricci soliton is either flat or* $*$*-Einstein or reduces to a* $\beta$*-Kenmotsu manifold.*

*Proof.* Substituting $X = \xi$ in (43) we have

$$R(\xi, Y)Df = (\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y. \tag{48}$$

Taking the inner product of the above equation with $\xi$ and using Lemma 4.2 we get

$$g(R(\xi, Y)Df, \xi) = 0.$$

Using (15) the above equation reduces to

$$(\alpha^2 - \beta^2)(\eta(Y)(\xi f) - Yf) = 0. \tag{49}$$

Case 1: If $\alpha^2 - \beta^2 = 0$, then from (15) we have $R(X, Y)\xi = 0$. Therefore, it follows from Lemma 4.1 that the manifold is flat.
Case 2: If $Yf = \eta(Y)(\xi f)$, then we have $Df = (\xi f)\xi$. Substituting the value of $Df$ in (44) and using (9) we infer that

$$(Y(\xi f))\xi + (\xi f)(-\alpha\phi Y + \beta(Y - \eta(Y)\xi)) = \lambda Y - Q^*Y,$$

which implies

$$Q^*Y = (\lambda - \beta(\xi f))Y + (\beta(\xi f)\eta(Y) - Y(\xi f))\xi + \alpha(\xi f)\phi Y. \tag{50}$$

Comparing (38) and (50) we obtain

$$\lambda - \beta(\xi f) = (\frac{r}{2} - 2(\alpha^2 - \beta^2)), \tag{51}$$

$$\beta(\xi f)\eta(Y) - Y(\xi f) = -(\frac{r}{2} - 2(\alpha^2 - \beta^2))\eta(Y) \tag{52}$$

and

$$\alpha(\xi f) = 0. \tag{53}$$

The foregoing equation implies that either $\alpha = 0$ or $\xi f = 0$. If $\alpha = 0$, then from (5) it follows that the manifold is $\beta$-Kenmotsu. If $\xi f = 0$, then from $Yf = \eta(Y)(\xi f)$ we get $Yf = 0$ for any vector field $Y$ and hence $f$ is constant. In this case the potential vector field $V$ being gradient of $f$ becomes a null vector. Therefore, from (3) we get $S^* = \lambda g$, i.e., the manifold is $*$-Einstein. This completes the proof. □

## 5. Example

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of $M$. Let $g$ be the Riemannian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1,$$
$$g(e_1,e_2)=g(e_2,e_3)=g(e_1,e_3)=0.$$

Let $\eta$ be the 1-form defined by $\eta(X)=g(X,e_3)$ for any vector field $X$. Let $\phi$ be the $(1,1)$ tensor field defined by

$$\phi(e_1)=-e_2,\ \ \phi(e_2)=e_1,\ \ \phi(e_3)=0.$$

Then we have

$$\phi^2(X)=-X+\eta(X)e_3,$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$$

for any vector fields $X$, $Y$. Hence the structure $(\phi,\xi,\eta,g)$ defines an almost contact metric structure on $M$, where $e_3=\xi$. Now, after calculating we have

$$[e_1,e_3]=-e_1,\ \ [e_1,e_2]=0\ \text{ and }\ [e_2,e_3]=-e_2.$$

The Riemannian connection $\nabla$ of the metric $g$ is given by the Koszul's formula

$$\begin{aligned}2g(\nabla_X Y,Z)&=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\&\quad-g(X,[Y,Z])-g(Y,[X,Z])-g(Z,[X,Y]).\end{aligned}$$

By Koszul's formula we get

$$\nabla_{e_1}e_1=e_3,\ \ \nabla_{e_1}e_2=0,\ \ \nabla_{e_1}e_3=-e_1,$$
$$\nabla_{e_2}e_1=0,\ \ \nabla_{e_2}e_2=e_3,\ \ \nabla_{e_2}e_3=-e_2,$$
$$\nabla_{e_3}e_1=0,\ \ \nabla_{e_3}e_2=0,\ \ \nabla_{e_3}e_3=0.$$

From the above we found that $\alpha=0$, $\beta=-1$ and $M^3(\phi,\xi,\eta,g)$ is a trans-Sasakian manifold. This example is given in [18].

The Riemannian curvature tensor is given by

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

Therefore, we have

$$R(e_1,e_2)e_1=e_2,\ \ R(e_1,e_2)e_2=-e_1,\ \ R(e_1,e_2)e_3=0,$$
$$R(e_2,e_3)e_1=0,\ \ R(e_2,e_3)e_2=e_3,\ \ R(e_2,e_3)e_3=-e_2,$$
$$R(e_1,e_3)e_1=e_3,\ \ R(e_1,e_3)e_2=0,\ \ R(e_1,e_3)e_3=-e_1.$$

Now, it is easy to see that

$$S^*(e_1,e_1)=-1,\ \ S^*(e_2,e_2)=-1\ \text{ and }\ S^*(e_3,e_3)=0.$$

Also,

$$(\pounds_{e_3}g)(e_1,e_1)=-2,\ \ (\pounds_{e_3}g)(e_2,e_2)=-2\ \text{ and }\ (\pounds_{e_3}g)(e_3,e_3)=0.$$

Therefore, tracing (3) we get $\lambda=-\frac{3}{4}$. Hence $(g,e_3,-\frac{3}{4})$ is a $*$-Ricci soliton on $M$ and the manifold is $\beta$-Kenmotsu. Thus, Theorem 3.3 is verified.

**Acknowledgements.** The authors would like to thank the anonymous referee for his/her careful reading and valuable suggestions that have improved the paper. The author Dibakar Dey is thankful to the Council of Scientific and Industrial Research, India (File no: 09/028(1010)/2017-EMR-1) for their assistance.

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