

RICCI ρ -SOLITONS ON 3-DIMENSIONAL η -EINSTEIN ALMOST KENMOTSU MANIFOLDS

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ABSTRACT. The notion of quasi-Einstein metric in theoretical physics and in relation with string theory is equivalent to the notion of Ricci soliton in differential geometry. Quasi-Einstein metrics or Ricci solitons serve also as solution to Ricci flow equation, which is an evolution equation for Riemannian metrics on a Riemannian manifold. Quasi-Einstein metrics are subject of great interest in both mathematics and theoretical physics. In this paper the notion of Ricci ρ -soliton as a generalization of Ricci soliton is defined. We are motivated by the Ricci-Bourguignon flow to define this concept. We show that if a 3-dimensional almost Kenmotsu Einstein manifold M is a ρ -soliton, then M is a Kenmotsu manifold of constant sectional curvature -1 and the ρ -soliton is expanding with $\lambda = 2$.

1. Introduction

Ricci flow and other geometric flows are active subjects of current research in physics and mathematics. The notion of Ricci-Bourguignon flow as a generalization of Ricci flow has been introduced in [5]. The Ricci-Bourguignon flow is an evolutionary equation for Riemannian metrics on a manifold M^n as follows.

$$(1) \quad \frac{\partial g}{\partial t} = -2(\text{Ric} - \rho Rg), \quad g(0) = g_0,$$

where Ric is the Ricci curvature tensor, R is the scalar curvature with respect to g and ρ is a real non-zero constant. Short time existence and uniqueness for the solution of this geometric flow has been proved in [6]. In fact, for sufficiently small t the equation has a unique solution for $\rho < 1/2(n-1)$.

On the other hand, quasi Einstein metrics or Ricci solitons serve as a solution to Ricci flow equation. This motivates a more general type of Ricci soliton by considering the Ricci-Bourguignon flow. In fact, a Riemannian manifold (M, g)

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of dimension $n \geq 3$ is said to be Ricci ρ -soliton if

$$(2) \quad \frac{1}{2} \mathcal{L}_V g + \text{Ric} + (\lambda + \rho R)g = 0,$$

where \mathcal{L}_V denotes the Lie derivative operator along vector field V and λ is an arbitrary real constant. Similar to Ricci solitons, a Ricci ρ soliton is called expanding if $\lambda > 0$, steady if $\lambda = 0$ and shrinking if $\lambda < 0$. If the vector field V is the gradient of a smooth function $f \in C^\infty(M)$, then (M, g) is called a gradient ρ -soliton. Hence, (2) reduces to the form

$$(3) \quad \text{Hess}f + \text{Ric} + (\lambda + \rho R)g = 0.$$

Recently, Ricci solitons and gradient Ricci solitons on some kinds of three dimensional almost contact metric manifolds have been studied by many authors. For instances, Ricci solitons and gradient Ricci solitons on three-dimensional normal almost contact metric manifolds are investigated in [8] and three-dimensional trans-Sasakian manifolds are considered in [14]. Moreover, a complete classification of Ricci solitons on three-dimensional Kenmotsu manifolds is given (see [7] and [10]). Also, in [15] Wang and Liuva showed that if the metric g of a three-dimensional η -Einstein almost Kenmotsu manifold M is a Ricci soliton, then M is a Kenmotsu manifold of constant sectional curvature -1 and the soliton is expanding. Generalizing some corresponding results of the paper [15], the present paper is devoted to investigating Ricci ρ -solitons on a type of almost Kenmotsu manifolds of dimension three, namely, η -Einstein almost Kenmotsu manifolds.

2. Preliminaries

In this section we summarize some basic definitions on contact manifolds with emphasis on those aspects that will be needed in the next section. For more details one can consult [4].

Definition. An almost contact structure on a $(2n + 1)$ -dimensional smooth manifold M is a triple (ϕ, ξ, η) , where ϕ is a $(1, 1)$ -type tensor field, ξ is a global vector field and η a 1-form, such that

$$(4) \quad \phi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where id denotes the identity mapping, which imply that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\text{rank}(\phi) = 2n$. Generally, ξ is called the characteristic vector field or the Reeb vector field.

As mentioned, contact manifolds are endowed with extra structures rather than differential structure, so it is natural to consider special metrics on these manifold in which some conditions of compatibility are requested for them.

Definition. A Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ, ξ, η) if for every $X, Y \in \mathcal{X}(M)$, we have

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. Also, the fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by

$$\Phi(X, Y) = g(X, \phi(Y))$$

for any vector fields X, Y on M^{2n+1} .

Definition. If (M, ϕ, ξ, η, g) is an almost contact metric structure, then there is a well known deformation of contact forms which is named D -homothetic deformation and is defined by

$$\bar{\eta} = a\eta, \quad \bar{\phi} = \phi, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{g} = ag + a(a - 1)\eta \otimes \eta,$$

where a is a positive constant.

Also, we have the following definitions and concepts in contact manifolds.

Definition. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Also, an almost Kenmotsu manifold is said to be a β -Kenmotsu manifold if for all vector field X and Y on M , we have

$$(\nabla_X \phi)Y = \beta[g(\phi(X), Y)\xi - \eta(Y)\phi(X)],$$

where ∇ is the Levi-Civita connection with respect to g and β is a smooth function on M . If $\beta = 1$, the definition of Kenmotsu manifold is obtained.

Local structure of Kenmotsu manifolds is determined in [11].

Theorem 2.1 ([11]). *A Kenmotsu manifold M^{2n+1} is locally isometric to a warped product $I \times_{\theta} M^{2n}$, where M^{2n} is a Kahlerian manifold, I is an open interval with coordinate t and the warping function $\theta = ce^t$ for some positive constant c .*

Definition. On an almost contact metric manifold M , if the Ricci operator satisfies

$$(5) \quad \text{Ric} = \alpha g + \beta \eta \otimes \eta,$$

where Ric is the Ricci curvature tensor and both α and β are smooth functions on M , then M is said to be an η -Einstein manifold.

Obviously, an η -Einstein manifold with vanishing β and α a constant is an Einstein manifold. An η -Einstein manifold is said to be proper η -Einstein if $\beta \neq 0$.

Finally, remind that there are two natural tensor fields (with respect to metric contact structure) on an almost metric contact manifold. Set

$$(6) \quad h = \frac{1}{2}\mathcal{L}_{\xi}\phi, \quad \ell = R(\cdot, \xi)(\xi),$$

where R denotes the Riemannian curvature tensor with respect to g . One can easily check that both h and ℓ are symmetric tensor fields and satisfy the following equations.

$$\ell(\xi) = 0, \quad h(\xi) = 0, \quad \text{tr}(h) = 0, \quad \text{tr}(h\phi) = 0, \quad h \circ \phi + \phi \circ h = 0.$$

Also, the following identities are proven in [4].

$$(7) \quad \nabla_X \xi = \phi^2(\xi) + h'X,$$

$$(8) \quad \phi\ell\phi - \ell = 2(h^2 - \phi^2),$$

$$(9) \quad \text{tr}(\ell) = \text{Ric}(\xi, \xi) = g(\text{Rc}(\xi), \xi) = -2n - \text{tr}(h^2),$$

$$(10) \quad R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y,$$

where $h' = h \circ \phi$ and Rc is Ricci operator with respect to g .

3. Main results

In this section (M, g) is a three-dimensional almost Kenmotsu manifold. If the characteristic vector field ξ of M belongs to generalized k -nullity distribution defined by

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

then Proposition 3.1 of [13] guarantees that M is an η -Einstein manifold and vice versa. Moreover, the function k in the above formula can be expressed by $k = (\alpha + \beta)/2$.

If the Reeb vector field ξ belongs to the generalized k -nullity distribution in the manifold (M, g) , then according to [12] the following formulas hold.

$$(11) \quad \begin{aligned} h^2 &= h'^2 = (k+1)\phi^2, \\ \text{Rc}(\xi) &= 2k\xi. \end{aligned}$$

Then the above equation follows that $k \leq -1$ everywhere on M . Moreover, $k = -1$ holds if and only if $h = h' = 0$. If $k < -1$, we denote the two non-zero eigenvalues of h by ν and $-\nu$ respectively, where $\nu = \sqrt{-1-k} > 0$. Furthermore, by Proposition 3.1 of [8] we also have

$$\nabla_\xi h' = -2h'.$$

We need the following results from [15] for proving our main theorem.

Lemma 3.1 ([15]). *Let (M, g) be an almost Kenmotsu manifold of dimension 3 such that the Reeb vector field belongs to the generalized k -nullity distribution. Then we have*

$$\vec{\nabla}k = -4(k+1)\xi,$$

where $\vec{\nabla}$ denotes the gradient operator with respect to g .

Lemma 3.2 ([15]). *Let (M, g) be a three-dimensional almost Kenmotsu manifold such that the characteristic vector field belongs to the generalized k -nullity distribution. Then either $k = -1$ identically or $k < -1$ everywhere on M .*

Now, we are ready to present our main theorem.

Theorem 3.3. *Let the metric g of a three-dimensional η -Einstein almost Kenmotsu manifold (M, g) be a Ricci ρ -soliton. Then M is a Kenmotsu manifold of constant sectional curvature -1 and the soliton is expanding with $\lambda = 2$.*

Proof. According to previous lemma, we prove the theorem, which are the cases of $k = -1$ identically and $k < -1$ everywhere on M .

Case 1: Suppose that we have $k < -1$ everywhere on M which is equivalent to $h \neq 0$. Putting relation (5) into (2) we obtain

$$(12) \quad \mathcal{L}_V g = -2(\alpha + \rho R + \lambda)g - 2\beta\eta \otimes \eta.$$

Taking the covariant differentiation from both sides of the above formula along an arbitrary vector field X we obtain the following equality for any vector fields Y and Z on M .

$$(13) \quad \begin{aligned} (\nabla_X \mathcal{L}_V g)(Y, Z) &= -2(X(\alpha) + \rho X(R))g(Y, Z) - 2X(\beta)\eta(Y)\eta(Z) \\ &\quad - 2\beta g(X + h'X, Y)\eta(Z) - 2\beta g(X + h'X, Z)\eta(Y) \\ &\quad + 2\beta\eta(X)\eta(Y)\eta(Z). \end{aligned}$$

But we know the following formula from Yano [16],

$$\begin{aligned} &(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V, X]}g)(Y, Z) \\ &= -g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y). \end{aligned}$$

Since ∇ is the Levi-Civita connection of M we have $\nabla g = 0$ and then the above formula becomes

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y).$$

One can easily check that the operator $(\mathcal{L}_V \nabla)$ is a symmetric tensor field of type $(1, 2)$, i.e., $(\mathcal{L}_V \nabla)(X, Y) = (\mathcal{L}_V \nabla)(Y, X)$. In fact, this symmetry is a consequence of Jacobi identity in the Lie algebra of smooth real function on M . Hence, a simple combinatorial argument shows that

$$(14) \quad \begin{aligned} &g((\mathcal{L}_V \nabla)(X, Y), Z) \\ &= \frac{1}{2}(\nabla_X \mathcal{L}_V g)(Y, Z) + \frac{1}{2}(\nabla_Y \mathcal{L}_V g)(Z, X) - \frac{1}{2}(\nabla_Z \mathcal{L}_V g)(X, Y). \end{aligned}$$

Using (14) and (13) the following formula is obtained,

$$(15) \quad \begin{aligned} (\mathcal{L}_V \nabla)(X, Y) &= -(X(\alpha) + \rho X(R))Y - (Y(\alpha) + \rho Y(R))X + g(X, Y)\vec{\nabla}\alpha \\ &\quad + \rho g(X, Y)\vec{\nabla}R + \eta(X)\eta(Y)\vec{\nabla}\beta - [X(\beta)\eta(Y) \\ &\quad + 2\beta g(X + h'X, Y) - 2\beta\eta(X)\eta(Y) + Y(\beta)\eta(Y)]\xi. \end{aligned}$$

Considering an orthonormal local frame $\{e_i\}_{i=1}^3$ on M and replacing X and Y by e_i and summing over $i = 1, 2, 3$, we have

$$(16) \quad \sum_{i=1}^3 (\mathcal{L}_V \nabla)(e_i, e_i) = \vec{\nabla}\alpha + \vec{\nabla}\beta + \rho\vec{\nabla}R - 2[\xi(\beta) + 2\beta]\xi.$$

On the other hand, taking the covariant differentiation of the Ricci soliton equation (2) along an arbitrary vector field X we obtain $\nabla_X \mathcal{L}_V g = -2\rho X(R)g - 2\nabla_X \text{Ric}$, putting this relation into (14) we obtain

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z \text{Ric})(X, Y) - (\nabla_X \text{Ric})(Y, Z) - (\nabla_Y \text{Ric})(X, Z) \\ + \rho Z(R)g(X, Y) - \rho X(R)g(Y, Z) - \rho Y(R)g(X, Z).$$

Replacing $X = Y = e_i$ in the above formula and summing over $i = 1, 2, 3$, we obtain $\sum_{i=1}^3 (\mathcal{L}_V \nabla)(e_i, e_i) = \rho \vec{\nabla} R$, and this relation with (16) gives us the following equation

$$(17) \quad \vec{\nabla} \alpha + \vec{\nabla} \beta - 2[\xi(\beta) + 2\beta]\xi = 0.$$

Using the relation (15) and taking the covariant differentiation of $(\mathcal{L}_V \nabla)(Y, Z)$ along an arbitrary vector field X , we may obtain

$$(18) \quad (\nabla_X \mathcal{L}_V \nabla)(Y, Z) \\ = -g(Y, \nabla_X \vec{\nabla} \alpha)Z - g(Z, \nabla_X \vec{\nabla} \alpha)Y - \rho g(Y, \nabla_X \vec{\nabla} R)Z - \rho g(Z, \nabla_X \vec{\nabla} R)Y \\ + \eta(Y)\eta(Z)\nabla_X \vec{\nabla} \beta + g(Y, Z)\nabla_X \vec{\nabla} \alpha + \rho g(Y, Z)\nabla_X \vec{\nabla} R \\ + [g(X + h'X, Z)\eta(Y) + g(X + h'X, Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)]\vec{\nabla} \beta \\ - [Y(\beta)\eta(Z) + 2\beta g(Y + h'Y, Z) - 2\beta\eta(Y)\eta(Z) + \eta(Y)Z(\beta)](X + h'X) \\ - g(Y, \nabla_X \vec{\nabla} \beta)\eta(Z)\xi - g(Z, \nabla_X \vec{\nabla} \beta)\eta(Y)\xi - 2\beta g((\nabla_X h')Y, Z)\xi \\ + 2\beta[g(X + h'X, Y)\eta(Z) + g(X + h'X, Z)\eta(Y) + g(Y + h'Y, Z)\eta(X)]\xi \\ - Y(\beta)[g(X + h'X, Z) - 2\eta(X)\eta(Z)]\xi - Z(\beta)[g(X + h'X, Y) \\ - 2\eta(X)\eta(Y)]\xi - X(\beta)[g(Y + h'Y, Z) - \eta(Y)\eta(Z)]\xi \\ - 6\eta(X)\eta(Y)\eta(Z)\xi.$$

The following tensorial identity is well known (see [16]),

$$(19) \quad (\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z)$$

for any vector fields X, Y , and Z .

Also, note that for any smooth function f on a Riemannian manifold (M, g) we have $g(\nabla_X \vec{\nabla} f, Y) = g(\nabla_Y \vec{\nabla} f, X)$. Applying this fact and using the relations (19) and (18), by a straightforward computation we obtain

$$(20) \quad (\mathcal{L}_V R)(X, Y)Z \\ = g(Z, \nabla_Y \vec{\nabla} \alpha)X - g(Z, \nabla_X \vec{\nabla} \alpha)Y + \rho g(Z, \nabla_Y \vec{\nabla} R)X - \rho g(Z, \nabla_X \vec{\nabla} R)Y \\ + [g(X + h'X, Z)\eta(Y) - g(Y + h'Y, Z)\eta(X)]\vec{\nabla} \beta + g(Y, Z)\nabla_X \vec{\nabla} \alpha \\ + \eta(Z)[\eta(Y)\nabla_X \vec{\nabla} \beta - \eta(X)\nabla_Y \vec{\nabla} \beta] - g(X, Z)\nabla_Y \vec{\nabla} \alpha \\ + \rho g(Y, Z)\nabla_X \vec{\nabla} R - \rho g(X, Z)\nabla_Y \vec{\nabla} R \\ + [X(\beta)\eta(Z) + 2\beta g(X + h'X, Z) - 2\beta\eta(X)\eta(Z) + \eta(X)Z(\beta)](Y + h'Y) \\ - [Y(\beta)\eta(Z) + 2\beta g(Y + h'Y, Z) - 2\beta\eta(Y)\eta(Z) + \eta(Y)Z(\beta)](X + h'X)$$

$$\begin{aligned}
 & - X(\beta)g(Y + h'Y, Z)\xi + Y(\beta)g(X + h'X, Z)\xi - 2\beta g((\nabla_X h')Y, Z)\xi \\
 & - [g(Z, \nabla_X \vec{\nabla}\beta)\eta(Y) - g(Z, \nabla_Y \vec{\nabla}\beta)\eta(X)]\xi + 2\beta g((\nabla_Y h')X, Z)\xi
 \end{aligned}$$

for any vector fields X, Y , and Z .

Consider again the local orthonormal frame $\{e_i\}_{i=1}^3$, remind that for any smooth function f on the Riemannian manifold (M, g) , the Laplace operator Δ acts on f by

$$\Delta(f) = - \sum_{i=1}^3 g(\nabla_{e_i} \vec{\nabla}f, e_i).$$

Contracting the tonsorial relation (20) over X , then a straightforward computation shows

$$\begin{aligned}
 (21) \quad & (\mathcal{L}_V \text{Ric})(Y, Z) \\
 & = -g(Y, Z)\Delta\alpha - \rho g(Y, Z)\Delta R - \eta(Y)\eta(Z)\Delta\beta + 2g(Z, \nabla_Y \vec{\nabla}\alpha) \\
 & \quad + 2\rho g(Z, \nabla_Y \vec{\nabla}R) - 2\xi(\beta)g(Y + h'Y, Z) + \eta(Y)g(Z + h'Z, \vec{\nabla}\beta) \\
 & \quad + \eta(Z)g(Y + h'Y, \vec{\nabla}\beta) - \eta(Y)\eta(\nabla_Z \vec{\nabla}\beta) - \eta(Z)\eta(\nabla_Y \vec{\nabla}\beta) - 4\beta g(Y, Z) \\
 & \quad - 2\beta g(h'Y, Z) + 4\beta\eta(Y)\eta(Z) - 2Z(\beta)\eta(Y) - 2Y(\beta)\eta(Z).
 \end{aligned}$$

Moreover, keeping in mind that M is an η -Einstein manifold, by (5) and a straightforward calculation we obtain that

$$\begin{aligned}
 (22) \quad & (\mathcal{L}_V \text{Ric})(Y, Z) \\
 & = [V(\alpha) - 2\alpha(\lambda + \alpha + \rho R)]g(Y, Z) + [V(\beta) - 2\alpha\beta - 2\beta\eta(V)]\eta(Y)\eta(Z) \\
 & \quad + \beta[g(Z + h'Z, V) + \eta(\nabla_Z Y)]\eta(Y) + \beta[g(Y + h'Y, V) + \eta(\nabla_Y Z)]\eta(Z)
 \end{aligned}$$

for any vector fields $Y, Z \in \mathcal{X}(M)$.

Subtracting (21) from (22) gives an equation, substituting Y and Z with ϕY and ϕZ respectively in the resulting equation, we may obtain

$$\begin{aligned}
 (23) \quad & g(\phi Y, \phi Z)\Delta\alpha + \rho g(\phi Y, \phi Z)\Delta R - 2g(\phi Z, \nabla_{\phi Y} \vec{\nabla}\alpha) - 2\rho g(Z, \nabla_{\phi Y} \vec{\nabla}R) \\
 & \quad + 2\xi(\beta)g(\phi Y, Z) + [V(\alpha) - 2\alpha(\lambda + \alpha + \rho R) + 4\beta]g(\phi Y, \phi Z) \\
 & \quad - 2\beta g(h'Y, Z) + 2\xi(\beta)g(hY, Z) = 0.
 \end{aligned}$$

The above formula holds for any $Y, Z \in \mathcal{X}(M)$, so interchanging Y and Z of relation (23) yields a new equation, subtracting the resulting equation from (23) and applying the relation $g(\nabla_X \vec{\nabla}f, Y) = g(\nabla_Y \vec{\nabla}f, X)$ again we may obtain $\xi(\beta)g(\phi Y, Z) = 0$ for any vector fields Y and Z on M , then it follows that

$$(24) \quad \xi(\beta) = 0.$$

Using above equality in relation (17) we get $\vec{\nabla}\alpha + \vec{\nabla}\beta - 4\beta\xi = 0$, taking the inner product of this relation with ξ we obtain $\xi(\beta) = 4\beta$. Recall that M is an η -Einstein almost Kenmotsu manifold of dimension 3 if and only if ξ belongs to the generalized k -nullity distribution with $k = (\alpha + \beta)/2$, then by applying

Lemma 3.1, we obtain $\xi(k) = -4(k+1)$, making use of $k = (\alpha + \beta)/2$, $\xi(\beta) = 0$ and $\xi(\beta) = 4\beta$ in this relation we obtain

$$(25) \quad \alpha + 2\beta + 2 = 0.$$

It is easy to obtain from (24) and (25) that $\beta = 0$ and $\alpha = -2$. However, in fact, in this context we have $k = (\alpha + \beta)/2 = -1$, which contradicts the assumption $k < -1$ everywhere on M . Thus the Case 1 never happens.

Case 2: In the case where $k = -1$, from (11) we get that $h = 0$ and hence M is a Kenmotsu manifold (see Proposition 3 of [9]). Also, according to Lemma 1 of ([10]) the Ricci curvature tensor of (M, g) can be written as follows.

$$(26) \quad \text{Ric}(X, Y) = (1 + \frac{R}{2})g(X, Y) - (3 + \frac{R}{2})\eta(X)\eta(Y)$$

which means $\alpha = 1 + R/2$ and $\beta = -(3 + R/2)$. By replacing these equalities in (17), we may obtain

$$(27) \quad \xi(R) + 2(6 + R) = 0.$$

Also, relation (15) can be rewritten as follows

$$(28) \quad \begin{aligned} & 2(\mathcal{L}_V \nabla)(X, Y) \\ &= (1 + 2\rho)[g(X, Y)\vec{\nabla}R - X(R)Y - Y(R)X] + X(R)\eta(Y)\xi \\ & \quad + Y(R)\eta(X)\xi - \eta(X)\eta(Y)\vec{\nabla}R + 2(6 + R)[g(X, Y)\xi - \eta(X)\eta(Y)\xi]. \end{aligned}$$

Hence, we can write

$$(29) \quad 2(\mathcal{L}_V \nabla)(Y, \xi) = \xi(R)[\phi^2 - 2\rho Y] - 2\rho Y(R)\xi.$$

By differentiation of (29) along an arbitrary vector field X , we get

$$(30) \quad \begin{aligned} & 2(\nabla_X \mathcal{L}_V \nabla)(Y, \xi) + 2(\mathcal{L}_V \nabla)(Y, X) \\ &= X(\xi(R))\phi^2(Y) + \xi(R)[g(X, Y)\xi + \eta(Y)X - \eta(X)Y - \eta(X)\eta(Y)\xi] \\ & \quad + 2\rho[g(X, Y)\vec{\nabla}R - X(R)Y - Y(R)X] \\ & \quad - 2\rho[(\nabla_X dR)(\xi)Y + (\nabla_X dR)(Y)\xi]. \end{aligned}$$

With the help of the above formula and (19) we can write

$$(31) \quad \begin{aligned} & 2(\mathcal{L}_V \nabla)(X, Y)Z \\ &= X(\xi(R))\phi^2(Y) - Y(\xi(R))\phi^2(X) + 2\xi(R)[\eta(Y)X - \eta(X)Y]. \end{aligned}$$

On the other hand, the equality $R(X, Y)\xi = \eta(X)Y - \eta(Y)\xi$ holds in any Kenmotsu manifolds and by differentiation both sides of this equality along the vector field V and making use of (12) we obtain

$$(32) \quad \begin{aligned} & (\mathcal{L}_V R)(X, Y)\xi + R(X, Y)\mathcal{L}_V \xi \\ &= (4 - 2\lambda)[\eta(X)Y - \eta(Y)X] + g(X, \mathcal{L}_V \xi)Y - g(Y, \mathcal{L}_V \xi)X. \end{aligned}$$

Comparing (31) and (32) yields an equation and then contracting the result equation over X and making use of (26) again, we get

$$(33) \quad \begin{aligned} &(6 + R)g(Y, \mathcal{L}_V \xi) - (6 + R)\eta(Y)\eta(\mathcal{L}_V \xi) \\ &= -Y(\xi(R)) - (\xi(\xi(R)))\eta(Y) - 4(4 - 2\lambda + \xi(R))\eta(Y). \end{aligned}$$

If we set $Y = \xi$ in the above formula then, by (27) we get $\lambda = 2$ which shows the soliton is expanding. Now, by Theorem 1 of [9] we have completed the proof. \square

4. Example

In what follows we consider $M = \mathbb{R} \times_\gamma N$, where, N is a Riemannian surface with constant negative sectional curvature (a Kahler manifold), \mathbb{R} is real line and $\gamma = \gamma(t)$ is warp function. In fact, we consider the following warped metric on M

$$g = \frac{h}{\gamma^2(t)} + dt^2,$$

where h is a Riemannian metric with constant curvature. So, M is a β -Kenmotsu manifold with $\beta = \gamma'(t)/\gamma(t)$ (see [2]). Suppose that R stands for scalar curvature of M . Then, an argument analogous to that of Example 2.10 in [3] shows that g is a ρ -soliton with vector field $V = -\mu + f \frac{\partial}{\partial t}$ if and only if

$$(\ln \gamma)'' - \gamma^2 K^N = 0,$$

where K^N denotes the Gaussian curvature of N and,

$$f = \frac{\gamma'' + (\lambda + \rho R)\gamma + K^N \gamma^3}{\gamma'^2} - \frac{3\gamma'}{\gamma}.$$

If we just restrict attention to the case in which $K^N = -1$, then this leads us to the following ordinary differential equation

$$(\ln \gamma)'' + \gamma^2 = 0.$$

The curve $\gamma = \frac{1}{\cosh t}$ is a particular solution for the above equation and for which we have

$$g = (\cosh t)^2 h + dt^2.$$

Hence, M is a β -Kenmotsu manifold with $\beta = \tanh t$ (see [2]). By a D -homothetic transformation we derive a Kenmotsu metric on M . Let

$$g^* = \sigma g + (1 - \sigma)\eta \otimes \eta,$$

where σ is a positive function which depends only on $\xi = \frac{\partial}{\partial t}$. Using Lemma 4.1 in the paper [1], first we derive a β -Kenmotsu manifold $(M^*, \phi, \xi, \eta, g^*)$ with $\beta^* = \beta + \frac{\xi(\sigma)}{2\sigma}$. Now, we wish to choose β^* such that the smooth manifold M^* is a Kenmotsu manifold. It is sufficient to set $1 = \beta + \frac{\xi(\sigma)}{2\sigma}$, which leads us to

$$\frac{\partial}{\partial t}(\ln \sigma) = 2(1 - \beta).$$

The curve $\sigma = \frac{e^{2t}}{(\cosh t)^2}$ satisfies the above equation and so the metric $g^* = e^{2t}h + dt^2$ is the desired Kenmotsu metric.

5. Conclusion

In this paper, we showed that if the metric of a three dimensional almost Kenmotsu manifold is a ρ -soliton, then the underlying manifold is a Kenmotsu manifold with constant sectional curvature -1 and the soliton is expanding. Of course, we have considered 3-dimensional manifolds and extending the results of this paper to higher dimensional spaces will be a good project.

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