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# ON ALMOST QUASI RICCI SYMMETRIC MANIFOLDS

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Abstract. The purpose of this note is to introduce a type of Riemannian manifold called an almost quasi Ricci symmetric manifold and investigate the several properties of such a manifold on which some geometric conditions are imposed. And the existence of such a manifold is ensured by a proper example.

## 1. Introduction

In [2] the author studied a weakly symmetric structure on a Riemannian manifold, namely, a quasi Ricci symmetric manifold defined as follows: A Riemannian manifold $(M^n, g)(n \geq 3)$ is said to be quasi Ricci symmetric if its Ricci curvature $Ric$ satisfies the condition

$$(\nabla_X Ric)(Y, Z) = 2A(X)Ric(Y, Z) - A(Y)Ric(X, Z) - A(Z)Ric(X, Y) \tag{1}$$

for any nonzero 1-form $A$, where $\nabla$ denotes the covariant differentiation with respect to the metric tensor $g$ and $X, Y, Z \in TM^n$. As an extended class of quasi Ricci symmetric manifolds, we introduce another weakly symmetric structure on a Riemannian manifold. More precisely, a Riemannian manifold $(M^n, g)(n \geq 3)$ is said to be almost quasi Ricci symmetric if its Ricci curvature $Ric$ fulfills the condition

$$(\nabla_X Ric)(Y, Z) = [A(X) + B(X)]Ric(Y, Z) - A(Y)Ric(X, Z) - A(Z)Ric(X, Y) \tag{2}$$

for two 1-forms $A, B$, where $B$ is a nowhere vanishing 1-form. In particular, if $A = B$, then the relation (2) becomes the relation (1) which represents a quasi Ricci symmetric manifold. In the next section, we study the several properties of such a manifold on which some geometric conditions are imposed. Finally the existence of such a manifold is ensured by a proper example.

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## 2. Main results

A Riemannian manifold $(M^n, g)$ is said to be Einstein if its Ricci tensor $Ric$ is proportional to the metric $g$, i.e.,

$$Ric = \frac{s}{n} g. \tag{3}$$

Note that the scalar curvature $s$ of an Einstein manifold of dimension larger than 2 is constant [1]. In this note we call a Riemannian manifold $(M^n, g)$ an Einstein and almost quasi Ricci symmetric manifold if the manifold is both an Einstein manifold and an almost quasi Ricci symmetric manifold. Concerning an Einstein and almost quasi Ricci symmetric manifold, we have:

**Theorem 2.1.** *Let $(M^n, g)(n > 2)$ be an Einstein and almost quasi Ricci symmetric manifold. Then either $(M^n, g)$ is Ricci-flat or its associated 1-forms $A$ and $B$ in (2) are equal.*

*Proof.* From (1) it follows that

$$(\nabla_X Ric)(Y, Z) - (\nabla_Z Ric)(X, Y) = B(X)Ric(Y, Z) - B(Z)Ric(X, Y). \tag{4}$$

Contracting (4) on $Y$ and $Z$, we obtain

$$ds(X) - (\delta Ric)(X) = B(X)s - Ric(X, B^\sharp), \tag{5}$$

where $B^\sharp$ denotes the associated vector field of 1-form $B$ in (1), i.e., $g(X, B^\sharp) = B(X)$. By virtue of the second Bianchi identity, the relation (5) reduces to

$$ds(X) = 2sB(X) - 2Ric(X, B^\sharp). \tag{6}$$

On the other hand, contracting (2) on $Y$ and $Z$, we get

$$ds(X) = [A(X) + B(X)]s - 2Ric(X, A^\sharp). \tag{7}$$

Therefore considering (6) and (7) we obtain

$$[A(X) - B(X)]s = 2[Ric(X, A^\sharp) - Ric(X, B^\sharp)]. \tag{8}$$

From the Einstein condition (3) and (8), it follows that

$$[A(X) - B(X)]s = \frac{2}{n} s[A(X) - B(X)],$$

which implies either $s = 0$ (hence $Ric = 0$) or $A = B$ under our assumption $n > 2$. This completes the proof. □

The Ricci tensor $Ric$ of $(M^n, g)$ is said to be of Codazzi type if it satisfies the relation:

$$(\nabla_X Ric)(Y, Z) = (\nabla_Y Ric)(X, Z). \tag{9}$$

Now we can state the following result:

**Theorem 2.2.** *Let $(M^n, g)$ be an almost quasi Ricci symmetric manifold. If its Ricci tensor Ric is of Codazzi type, then the manifold has*

$$Ric(X, Y) = fU(X)U(Y),$$

*where $U = 2A + B$ and $f$ is a smooth function.*

*Proof.* By virtue of (2), we have

$$\begin{aligned}&(\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z)\\ &= [2A(X) + B(X)]Ric(Y, Z) - [2A(Y) + B(Y)]Ric(X, Z),\end{aligned}$$

which yields from (9)

$$[2A(X) + B(X)]Ric(Y, Z) = [2A(Y) + B(Y)]Ric(X, Z).$$

Considering the last relation we have

$$Ric(X, Y) = fU(X)U(Y),$$

where $U = 2A + B$ and $f$ is a smooth function. This completes the proof. □

The Ricci tensor $Ric$ of $(M^n, g)$ is said to be cyclic if it satisfies the relation:

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y) = 0. \tag{10}$$

Concerning an almost quasi Ricci symmetric manifold with cyclic Ricci tensor, we have:

**Theorem 2.3.** *Let $(M^n, g)$ be an almost quasi Ricci symmetric manifold. If its Ricci tensor Ric is definite and cyclic, then the manifold is quasi Ricci symmetric.*

*Proof.* Taking account of (2), we get

$$\begin{aligned}&(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y)\\ &= [B(X) - A(X)]Ric(Y, Z) + [B(Y) - A(Y)]Ric(X, Z)\\ &\quad + [B(Z) - A(Z)]Ric(X, Y) = 0.\end{aligned} \tag{11}$$

By virtue of (10), the relation (11) reduces to

$$D(X)Ric(Y, Z) + D(Y)Ric(X, Z) + D(Z)Ric(X, Y) = 0, \tag{12}$$

where $D = B - A$. Putting $X = Y = Z = D$ in (12), we have

$$3||D||^2 Ric(D^\sharp, D^\sharp) = 0,$$

which yields from the definiteness of Ricci tensor

$$D = 0,$$

showing that the manifold is quasi Ricci symmetric. This completes the proof. □

A Riemannian manifold $(M^n, g)(n > 3)$ is said to be conformally flat [1] if its curvature tensor $R$ satisfies the relation:

$$\begin{aligned} R(X,Y,Z,W) = & \frac{1}{n-2}(Ric(Y,Z)g(X,W) - Ric(Y,W)g(X,Z) \\ & + g(Y,Z)Ric(X,W) - g(Y,W)Ric(X,Z)) \\ & - \frac{s}{(n-1)(n-2)}(g(Y,Z)g(X,W) - g(Y,W)g(X,Z)). \end{aligned} \tag{13}$$

It is well known [1] that a conformally flat manifold satisfies the relation:

$$(\nabla_X Ric)(Y,Z) - (\nabla_Z Ric)(X,Y) = \frac{1}{2(n-1)}[g(Y,Z)ds(X) - g(X,Y)ds(Z)]. \tag{14}$$

A Riemannian manifold $(M^n, g)(n > 3)$ is said to be a conformally flat and almost quasi Ricci symmetric manifold if the manifold is both a conformally flat manifold and an almost quasi Ricci symmetric manifold. Concerning a conformally flat and almost quasi Ricci symmetric manifold, we have:

**Theorem 2.4.** *Let $(M^n, g)(n > 3)$ be a conformally flat and almost quasi Ricci symmetric manifold. Then the manifold is quasi-Einstein.*

*Proof.* From (4), (6) and (14) it follows that

$$\begin{aligned} & B(X)Ric(Y,Z) - B(Z)Ric(X,Y) \\ = & \frac{s}{(n-1)}[B(X)g(Y,Z) - B(Z)g(X,Y)] \\ & - \frac{1}{(n-1)}[Ric(X,B^\sharp)g(Y,Z) - Ric(Z,B^\sharp)g(X,Y)]. \end{aligned} \tag{15}$$

Putting $Y = B^\sharp$ in (15) we obtain

$$B(X)Ric(B^\sharp, Z) = B(Z)Ric(X, B^\sharp),$$

which leads to

$$Ric(X, B^\sharp) = tB(X). \tag{16}$$

By virtue of (6) and (16), we get

$$ds(X) = 2(s-t)B(X). \tag{17}$$

Putting $X = B^\sharp$ in (15), we have

$$\begin{aligned} & B(B^\sharp)Ric(Y,Z) - B(Z)Ric(B^\sharp, Y) \\ = & \frac{s}{(n-1)}[B(B^\sharp)g(Y,Z) - B(Z)g(B^\sharp, Y)] \\ & - \frac{1}{(n-1)}[Ric(B^\sharp, B^\sharp)g(Y,Z) - Ric(Z,B^\sharp)g(B^\sharp, Y)]. \end{aligned}$$

Taking account of (16), the last relation yields

$$||B||^2 Ric(Y,Z) - tB(Y)B(Z) = \frac{s}{(n-1)}[||B||^2 g(Y,Z) - B(Y)B(Z)]$$

$$-\frac{1}{(n-1)}[t||B||^2 g(Y,Z) - tB(Y)B(Z)],$$

which leads to

$$Ric(Y,Z) = ag(Y,Z) + bU(Y)U(Z), \tag{18}$$

where $a = \frac{s-t}{n-1}, b = \frac{nt-s}{n-1}$ and $U = \frac{B}{||B||}$.

Hence the manifold is quasi-Einstein. This completes the proof. □

As a consequence, we obtain:

**Corollary 2.5.** *Let $(M^n, g)(n > 3)$ be a conformally flat and almost quasi Ricci symmetric manifold. Then the vector field $B^\sharp$ corresponding to the 1-form $B$ is an eigenvector of the Ricci tensor corresponding to the eigenvalue $(a+b)$.*

*Proof.* Putting $Z = B^\sharp$ in (18) we get

$$Ric(Y, B^\sharp) = (a+b)g(Y, B^\sharp).$$

This completes the proof. □

A Riemannian manifold $(M^n, g)$ is said to be of quasi-constant curvature if its curvature tensor $R$ satisfies

$$\begin{aligned} R(X,Y,Z,W) = {} & \alpha[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + \beta[g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z) \\ & + g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W)], \end{aligned}$$

where $\alpha$ and $\beta$ are smooth functions and $T$ is a 1-form. If $\beta = 0$, then the manifold reduces to a space of constant curvature. Now we can state the following:

**Theorem 2.6.** *Let $(M^n, g)(n > 3)$ be a conformally flat and almost quasi Ricci symmetric manifold. Then the manifold is of quasi-constant curvature.*

*Proof.* By virtue of (13) we have

$$\begin{aligned} R(X,Y,Z,W) = {} & \frac{1}{n-2}(Ric(Y,Z)g(X,W) - Ric(Y,W)g(X,Z) \\ & + g(Y,Z)Ric(X,W) - g(Y,W)Ric(X,Z)) \\ & - \frac{s}{(n-1)(n-2)}(g(Y,Z)g(X,W) - g(Y,W)g(X,Z)). \end{aligned}$$

Taking account of (18) and the last relation, we obtain

$$\begin{aligned} R(X,Y,Z,W) = {} & (\frac{2a}{n-2} - \frac{s}{(n-1)(n-2)})[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)] \\ & + \frac{b}{(n-2)}[g(X,W)U(Z)U(Y) - g(Y,W)U(X)U(Z) \\ & + g(Y,Z)U(X)U(W) - g(X,Z)U(Y)U(W)], \end{aligned}$$

showing that the manifold is of quasi-constant curvature. This completes the proof. □

A vector field $V$ on a Riemannian manifold $(M^n, g)$ is said to be torse-forming if it satisfies

$$\nabla_X V = fX + \mu(X)V, \tag{19}$$

where $f$ and $\mu$ are a smooth function and a 1-form, respectively. Now we can state the following:

**Theorem 2.7.** *Let $(M^n, g)(n > 3)$ be a conformally flat and almost quasi Ricci symmetric manifold. If $b$ in (18) never vanishes, then the associated vector field $B^\sharp$ of 1-form $B$ in (2) is a torse-forming vector field.*

*Proof.* Putting $Z = B^\sharp$ in (2), we get

$$\begin{aligned}&(\nabla_X Ric)(Y, B^\sharp)\\ &= [A(X) + B(X)]Ric(Y, B^\sharp) - A(Y)Ric(X, B^\sharp) - A(B^\sharp)Ric(X, Y).\end{aligned}$$

Taking account of (18) and the last relation, we obtain

$$\begin{aligned}(\nabla_X Ric)(Y, B^\sharp) = {}& [A(X) + B(X)]((a+b)B(Y)) - A(Y)((a+b)B(X))\\ & -A(B^\sharp)(ag(X, Y) + bU(X)U(Y)),\end{aligned}$$

which leads to

$$\begin{aligned}(\nabla_X Ric)(Y, B^\sharp) = {}& -aA(B^\sharp)g(X, Y) + (a+b)(A(X)B(Y) - A(Y)B(X))\\ & + ((a+b) - b\frac{A(B^\sharp)}{||B||^2})B(X)B(Y).\end{aligned} \tag{20}$$

On the other hand, we know

$$(\nabla_X Ric)(Y, B^\sharp) = \nabla_X(Ric(Y, B^\sharp)) - Ric(\nabla_X Y, B^\sharp) - Ric(Y, \nabla_X B^\sharp),$$

which yields from (18)

$$\begin{aligned}(\nabla_X Ric)(Y, B^\sharp) = {}& \nabla_X((a+b)B(Y)) - (a+b)B(\nabla_X Y) - ag(Y, \nabla_X B^\sharp)\\ & - bU(Y)U(\nabla_X B^\sharp)\\ = {}& (\nabla_X(a+b))B(Y) + b(\nabla_X B)(Y) - bB(Y)\frac{\nabla_X(||B||^2)}{2||B||^2}.\end{aligned} \tag{21}$$

Since $Ric^\sharp X = aX + \frac{b}{||B||^2}B(X)B^\sharp$, where $g(Ric^\sharp X, Y) = Ric(X, Y)$, we have

$$A(Ric^\sharp X) = aA(X) + \frac{b}{||B||^2}B(X)A(B^\sharp). \tag{22}$$

By virtue of (7), (17) and (22), we get

$$2(s-t)B(X) = [A(X) + B(X)]s - 2aA(X) - \frac{2b}{||B||^2}B(X)A(B^\sharp),$$

which leads to

$$A(X) = \lambda B(X), \tag{23}$$

where $\lambda = \frac{1}{(s-2a)}[s - 2t + \frac{2b}{||B||^2} A(B^\sharp)]$. From (20), (21) and (23) it follows that

$$\nabla_X B^\sharp = fX + \mu(X) B^\sharp, \tag{24}$$

where $f = -\frac{a}{b} A(B^\sharp)$ and $\mu(X) = \frac{1}{b}[((a+b) - \frac{bA(B^\sharp)}{2||B||^2}) B(X) - \nabla_X(a+b) + b\frac{\nabla_X(||B||^2)}{2||B||^2}]$. This completes the proof. □

As a consequence, we have:

**Theorem 2.8.** *Let $(M^n, g)(n > 3)$ be a conformally flat and almost quasi Ricci symmetric manifold. If $b$ in* (18) *never vanishes, then the manifold is a product manifold.*

*Proof.* Let $(B^\sharp)^\perp$ be the $(n-1)$-dimensional distribution in $(M^n, g)$ orthogonal to $B^\sharp$. If $X$ and $Y$ belong to $(B^\sharp)^\perp$, then

$$g(X, B^\sharp) = g(Y, B^\sharp) = 0. \tag{25}$$

Since $(\nabla_X g)(Y, B^\sharp) = (\nabla_Y g)(X, B^\sharp) = 0$, we have

$$0 = \nabla_X(g(Y, B^\sharp)) - g(\nabla_X Y, B^\sharp) - g(Y, \nabla_X B^\sharp),$$

which yields from (25)

$$g(\nabla_X Y, B^\sharp) = -g(Y, \nabla_X B^\sharp). \tag{26}$$

From (24), (25) and (26) it follows that

$$g(\nabla_X Y, B^\sharp) = -fg(X, Y),$$

which implies

$$g(\nabla_X Y, B^\sharp) = g(\nabla_Y X, B^\sharp). \tag{27}$$

Since $[X, Y] = \nabla_X Y - \nabla_Y X$, we have from (27)

$$g([X, Y], B^\sharp) = g(\nabla_X Y - \nabla_Y X, B^\sharp) = g(\nabla_X Y, B^\sharp) - g(\nabla_Y X, B^\sharp) = 0.$$

Therefore $[X, Y]$ belongs to $(B^\sharp)^\perp$, showing that the $(n-1)$-dimensional distribution $(B^\sharp)^\perp$ is involutive. From Frobenius' theorem it follows that $(B^\sharp)^\perp$ is integrable. And this implies that the manifold is a product manifold. This completes the proof. □

Now we show that there exists an almost quasi Ricci symmetric manifold.

**Example 2.9.** Let $(R_+^{n+4}, g)$ be a Riemannian manifold given by

$$R_+^{n+4} = \{(x_1, x_2, x_3, x_4, \ldots, x_{n+4}) \,|\, x_4 > 0\}$$

and

$$g = (x_4)^{\frac{4}{3}}[(dx_1)^2 + (dx_2)^2 + (dx_3)^2] + (dx_4)^2 + (dx_5)^2 + \cdots + (dx_{n+4})^2.$$

This kind of metric was appeared in [3]. In the metric described as above, the only nonvanishing components for the Christoffel symbols $\Gamma^k_{ij}$, the curvature tensors $R_{ijkl}$ and their covariant derivatives $R_{ijkl;p}$ are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x_4},$$
$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{-2}{3}(x_4)^{\frac{1}{3}},$$
$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x_4)^{\frac{2}{3}},$$
$$R_{1441} = R_{2442} = R_{3443} = \frac{-2}{9(x_4)^{\frac{2}{3}}}$$

and the components obtained by the symmetry properties. The nonvanishing components of the Ricci tensor $Ric_{jk}$ and their covariant derivatives $Ric_{jk;l}$ are

$$\begin{aligned} &Ric_{11} = Ric_{22} = Ric_{33} = \frac{2}{3(x_4)^{\frac{2}{3}}}, Ric_{44} = \frac{-2}{3(x_4)^2}, \\ &Ric_{11;4} = Ric_{22;4} = Ric_{33;4} = \frac{4}{9(x_4)^{\frac{5}{3}}}, Ric_{44;4} = \frac{4}{3(x_4)^3}. \end{aligned} \tag{28}$$

Let us define the associated 1-forms $A, B$ of (2) on $(R^{n+4}_+, g)$ as follows:

$$A_i = 0 \text{ for } i = 1, 2, \ldots, n+4; \;\; B_i = \frac{-2}{(x_4)} \text{ for } i = 4 \text{ and } 0 \text{ otherwise.}$$

In the manifold $(R^{n+4}_+, g)$, the relation (2) reduces to the following equations:

$$Ric_{11;4} = [A_4 + B_4]Ric_{11} - A_1 Ric_{41} - A_1 Ric_{14}, \tag{29}$$

$$Ric_{22;4} = [A_4 + B_4]Ric_{22} - A_2 Ric_{42} - A_2 Ric_{24}, \tag{30}$$

$$Ric_{33;4} = [A_4 + B_4]Ric_{33} - A_3 Ric_{43} - A_3 Ric_{34}, \tag{31}$$

$$Ric_{44;4} = [A_4 + B_4]Ric_{44} - A_4 Ric_{44} - A_4 Ric_{44}, \tag{32}$$

because for the other cases, the components of each term of (2) vanishes identically. From (28) and the definition of $A, B$ it follows that the last equations hold. For instance, in case of (29),

$$Ric_{11;4} = \frac{4}{9(x_4)^{\frac{5}{3}}}$$

and

$$[A_4 + B_4]Ric_{11} - A_1 Ric_{41} - A_1 Ric_{14} = B_4 Ric_{11} = \frac{4}{9(x_4)^{\frac{5}{3}}},$$

showing that (29) holds. By similar arguments, it can be shown that (30) and (31) hold. In case of (32),

$$Ric_{44;4} = \frac{4}{3(x_4)^3}$$

and

$$[A_4 + B_4]Ric_{44} - A_4 Ric_{44} - A_4 Ric_{44} = B_4 Ric_{44} = \frac{4}{3(x_4)^3},$$

showing that (32) holds too. Hence the Riemannian manifold $(R_+^{n+4}, g)$ with $A, B$ mentioned in the above is an almost quasi Ricci symmetric manifold.

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