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REMARKS ON A THEOREM OF CUPIT-FOUTOU AND ZAFFRAN

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Abstract. There is a well-known class of compact, complex, non-Kählerian manifolds constructed by Bosio, called the LVMB manifolds, which properly includes the Hopf manifold, the Calabi-Eckmann manifold, and the LVM manifolds. As in the case of LVM manifolds, these LVMB manifolds can admit a regular holomorphic foliation $\mathcal{F}$. Moreover, later Meersseman showed that if an LVMB manifold is actually an LVM manifold, then the regular holomorphic foliation $\mathcal{F}$ is actually transverse Kähler. The aim of this paper is to deal with a converse question and to give a simple and new proof of a well-known result of Cupit-Foutou and Zaffran. That is, we show that, when the holomorphic foliation $\mathcal{F}$ on an LVMB manifold N is transverse Kähler with respect to a basic and transverse Kähler form and the leaf space $N/\mathcal{F}$ is an orbifold, $N/\mathcal{F}$ is projective, and thus N is actually an LVM manifold.

1. Introduction and main results

One well-known example of a compact, complex, non-Kählerian manifold is the Hopf manifold, diffeomorphic to the product $S^{2n-1} \times S^1$ of spheres, which can be obtained by taking the quotient of $\mathbb{C}^n\backslash\{0\}$ by a holomorphic totally discontinuous action of $\mathbb{Z}$ ([14]). Another example is the Calabi-Eckmann manifold which is given by the existence of complex structures on $S^{2k-1}\times S^{2l-1}$ ([8]). To achieve it, Calabi and Eckmann consider the smooth fibration

$$S^{2k-1} \times S^{2l-1} \to \mathbb{CP}^{k-1} \times \mathbb{CP}^{l-1},$$

equipped with the torus fiber of the bundle with a structure of an elliptic curve. In the paper [18], López de Medrano and Verjovsky constructed a family of compact, complex, non-symplectic manifolds which can be obtained by taking the quotient of a open dense subset of $\mathbb{CP}^n$ by the holomorphic action of $\mathbb{C}$. This construction was extended to the case of a holomorphic action of $\mathbb{C}^m$ by Meersseman in [19]. These non-Kählerian manifolds are called *LVM manifolds.* Meersseman also constructed a holomorphic foliation $\mathcal{F}$ on each LVM manifold,

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and showed that $\mathcal{F}$ is transverse Kähler with respect to the Euler class of a certain S^1-bundle (refer to [19, Theorem 7]).

Finally, in his paper [5] Bosio showed that Meersseman's construction can be generalized to more general holomorphic actions of $\mathbb{C}^m$, so that he obtained the so-called *LVMB manifolds* $N = N(\mathcal{L}, \mathcal{E}_{m,n})$ (see Section 2 for a precise definition). The class of these manifolds properly includes the family of LVM manifolds. So there exists an LVMB manifold which is not biholomorphic to any LVM manifold (see, e.g., [9, Example 1.2]). It turns out that many interesting properties of LVM manifolds continue to hold for LVMB manifolds. In addition, as in the case of LVM manifolds there exists a holomorphic foliation $\mathcal{F}$ on each LVMB manifold.

We say that an LVMB manifold $N(\mathcal{L}, \mathcal{E}_{m,n})$ satisfies *condition* (K) if there exists a real affine automorphism of the dual space $(\mathbb{C}^m)^*$ of $\mathbb{C}^m$ as a real vector space $\mathbb{R}^{2m}$ sending each component of an admissible configuration $\mathcal{L}$ to a vector with integer coefficients. In the paper [9], Cupit-Foutou and Zaffran showed that if the holomorphic foliation $\mathcal{F}$ on an LVMB manifold $N = N(\mathcal{L}, \mathcal{E}_{m,n})$ is transverse Kähler and N satisfies the condition (K), then N is actually an LVM manifold. So it is natural to ask if, when the holomorphic foliation $\mathcal{F}$ on an LVMB manifold N is simply transverse Kähler, N is actually an LVM manifold.

In [15], recently Ishida gave a proof asserting that any LVMB manifold equipped with a transverse Kähler foliation is indeed LVM. This present paper can be regarded as a consequence of our attempt to give a more complex-geometric proof of [15, Corollary 5.8]. The complex-geometric proof in its own seems to be interesting, because by this way we can see how much LVMB manifolds are different from usual compact Kähler manifolds.

To be more precise, our primary aim of this paper is to give a complex-geometric proof of the following theorem which immediately implies the main result of Cupit-Foutou and Zaffran in [9] (see Section 2 for precise definitions of some terminologies).

Theorem 1.1. *Let N be an LVMB manifold, and let $\mathcal{F}$ be the holomorphic foliation on N such that $\mathcal{F}$ is transverse Kähler with respect to a basic and transverse Kähler form. If the leaf space $N/\mathcal{F}$ is an orbifold, then $N/\mathcal{F}$ is projective, and thus N is actually an LVM manifold.*

As for the transverse Kähler foliation $\mathcal{F}$ as in Theorem 1.1, it turns out that the leaf space $N/\mathcal{F}$ is a quasifold in the sense of [21, Section 1]. If a quasifold version of the Kodaira embedding theorem that seems to be currently out of reach happens to be true, the proof of Theorem 1.1 given in Section 3 would imply that, without the additional assumption that $N/\mathcal{F}$ is an orbifold, N is an LVM manifold. Note that the orbifold condition of the leaf space $N/\mathcal{F}$ is equivalent to the condition (K).

As an immediate consequence of Theorem 1.1, we have the following corollary which recovers [9, Theorem 3.7].

Corollary 1.2. *Let N be an LVMB manifold satisfying condition* (K)*, and let $\mathcal{F}$ be the holomorphic foliation on N such that $\mathcal{F}$ is transverse Kähler with respect to a transverse, but not necessarily basic, Kähler form. Then the leaf space $N/\mathcal{F}$ is a projective orbifold and N is an LVM manifold.*

Proof. Since by assumption the LVMB manifold N satisfies the condition (K), the leaves of the foliation $\mathcal{F}$ of N transverse to a transverse Kähler form ω are compact complex tori of the same complex dimension, and the leaf space $N/\mathcal{F}$ is an orbifold (see [20]). But then one can obtain a basic and transverse Kähler form from ω by taking the usual averaging procedure. Hence it is immediate to see from Theorem 1.1 that the LVMB manifold N is actually LVM, as desired. $\square$

We organize this paper, as follows. In Section 2, we collect basic definitions and well-known properties of LVMB manifolds. Section 3 is devoted to giving a proof of Theorem 1.1. Our proof is elementary, and uses an orbifold version of the well-known Kodaira embedding theorem saying that every compact Kähler orbifold with an integral Kähler form is always projective (see, e.g., [1] and [12]).

2. LVMB manifolds and transverse Kähler foliations

The aim of this section is to briefly recall the construction of the so-called LVMB manifold and to set up some basic notations and definitions necessary for the proof of Theorem 1.1 given in Section 3 (see [9] for more details).

To do so, let m and $n > 2m$ be positive integers. Let $\mathcal{L} = (l_1, \dots, l_n)$ be n linear forms of $\mathbb{C}^m$ such that any subcollection of $2m+1$ elements of $\mathcal{L}$ is an $\mathbb{R}$-affine basis of $(\mathbb{C}^m)^*$, where $(\mathbb{C}^m)^*$ denotes the dual space of $\mathbb{C}^m$. We remark that our n corresponds to $n-1$ in [9, Section 1].

Let $\mathcal{E}_{m,n} = \{\mathcal{E}_\alpha\}_\alpha$ denote a family of subsets of $[n] := \{1, 2, \dots, n\}$, each of these subsets having the cardinality $2m+1$. Let us denote by $(\mathcal{L}, \mathcal{E}_{m,n})$ these data. To a given data $(\mathcal{L}, \mathcal{E}_{m,n})$, we can also give the following definitions:

- A subset A of $[n]$ is *acceptable* if it contains an element of $\mathcal{E}_{m,n}$, and the set of all acceptable subsets of $[n]$ will be denoted by $\mathcal{A}$.
- An element of $[n]$ is *indispensable* if it belongs to every element of $\mathcal{E}_{m,n}$.
- For each $z \in \mathbb{C}^n$, let I_z denote the set of all $i \in [n]$ such that $i \in I_z$ if and only if $z_i \neq 0$. One can then define two open sets $\mathcal{S}$ in $\mathbb{C}^n$ and $\mathcal{V}$ in $\mathbb{CP}^{n-1}$, as follows.

$$\mathcal{S} = \{z \in \mathbb{C}^n \mid I_z \in \mathcal{A}\}, \quad \mathcal{V} = \{[z] \in \mathbb{CP}^{n-1} \mid I_z \in \mathcal{A}\}.$$

  Since $[n] \in \mathcal{A}$, it is easy to see that $\mathcal{S}$ contains $(\mathbb{C}^*)^n$ as an open subset.
- For each $\mathcal{E}_\alpha \in \mathcal{E}_{m,n}$, we denote by C_α the convex hull in $\mathbb{R}^{2m}$ of the l_i's for $i \in \mathcal{E}_\alpha$. Here l_i is regarded as an element of $\mathbb{R}^{2m}$ by using the natural identification $l_i \mapsto (\mathrm{Re}(l_i), \mathrm{Im}(l_i)) \in \mathbb{R}^m \times \mathbb{R}^m \cong \mathbb{R}^{2m}$.

It turns out that the open subset $\mathcal{S}$ of $\mathbb{C}^n$ is given by the complement of an arrangement of coordinate subspaces in $\mathbb{C}^n$. Indeed, let $\mathcal{P}$ denote the simplicial complex such that $I_z^c \in \mathcal{P}$ for $I_z \in \mathcal{A}$. Then it follows from [23, Proposition 1.1 or Proposition 1.2] that

$$\mathcal{S} = \mathbb{C}^n - E,$$

where

$$E = \bigcup_{(i_1,\dots,i_k)\notin\mathcal{P}} \{z \in \mathbb{C}^n \mid z_{i_1} = \cdots = z_{i_k} = 0\}.$$

From now on, let d denote the complex codimension of E in $\mathbb{C}^n$.

Note that there is a $\mathbb{C}^* \times \mathbb{C}^m$-action on $\mathcal{S}$ defined by

$$\begin{aligned} &(\mathbb{C}^* \times \mathbb{C}^m) \times \mathcal{S} \to \mathcal{S} \\ &((\alpha, Z), (z_1, z_2, \dots, z_n)) \mapsto (\alpha e^{l_1(Z)} z_1, \alpha e^{l_2(Z)} z_2, \dots, \alpha e^{l_n(Z)} z_n). \end{aligned} \tag{2.1}$$

Then it has been shown in [5, 1.4] that the action of $\mathbb{C}^* \times \mathbb{C}^m$ on $\mathcal{S}$ yields a compact complex quotient manifold $\mathcal{S}/(\mathbb{C}^* \times \mathbb{C}^m)$ if and only if the following two conditions hold:

(1) (Imbrication Condition) For any $\mathcal{E}_\alpha, \mathcal{E}_\beta \in \mathcal{E}_{m,n}$, we have $C_\alpha^\circ \cap C_\beta^\circ \neq \emptyset$. Here C_α° (resp. C_β°) means the relative interior of C_α (resp. C_β).

(2) (Substitute Existence Principle) For all $\mathcal{E}_\alpha \in \mathcal{E}_{m,n}$ and for all $i \in [n]$, there is an element $j \in \mathcal{E}_\alpha$ such that

$$(\mathcal{E}_\alpha \backslash \{j\}) \cup \{i\} \in \mathcal{E}_{m,n}.$$

We say that the pair $(\mathcal{L}, \mathcal{E}_{m,n})$ is an *LVMB datum* if it satisfies the above two conditions (1) and (2), and denote by $N = N(\mathcal{L}, \mathcal{E}_{m,n})$ the compact complex manifold $\mathcal{S}/(\mathbb{C}^* \times \mathbb{C}^m)$ of dimension $n - m - 1$, called an *LVMB manifold.* Moreover, if, in addition, we have

$$\cap_{\mathcal{E}_\alpha \in \mathcal{E}_{m,n}} C_\alpha^\circ \neq \emptyset,$$

then the $\mathbb{C}^* \times \mathbb{C}^m$-action on $\mathcal{S}$ is called an *LVM-action.* When the $\mathbb{C}^* \times \mathbb{C}^m$-action is LVM, the compact complex manifold $N = N(\mathcal{L}, \mathcal{E}_{m,n})$ is called an *LVM manifold* that is exactly the manifold constructed by Meersseman in [19].

Let M be a complex manifold equipped with a regular holomorphic foliation $\mathcal{F}$, and let ω be a closed real 2-form on M. The foliation $\mathcal{F}$ is called *transverse Kähler* with respect to ω if the following conditions are satisfied:

- The form ω is J-invariant, where J denotes the almost complex structure on the tangent bundle of M.
- For all $z \in M$, the kernel of $\omega(z)$ is the tangent to the foliation $\mathcal{F}$.
- The quadratic form $h(u_1, u_2) = \omega(Ju_1, u_2) + \sqrt{-1}\omega(u_1, u_2)$ is positive-definite on the normal bundle $N\mathcal{F}$ of the foliation $\mathcal{F}$.

Such a closed real 2-form ω is said to be a *transverse Kähler form.*

Next we briefly recall the notion of a basic differential form that we need in this paper. We refer the reader to [11, Section 4] for more details. Let M be a complex manifold, as before. A p-form α on M is called *basic* with respect to a vector field ξ if we have

$$\iota_\xi \alpha = 0, \quad \mathcal{L}_\xi \alpha = 0,$$

where $\mathcal{L}$ denotes the Lie derivative. Then we can consider the basic de Rham complexes as well as basic Dolbeault complexes on M whose cohomology groups are called the *basic cohomology groups*. Similarly, we can also consider the *basic harmonic forms*. Furthermore, results of El Kacimi-Alaoui in [10] show that we have the isomorphisms between basic cohomology groups and the space of harmonic forms, as expected.

When a complex manifold M admits a regular transverse Kähler foliation $\mathcal{F}$ with respect to a basic and transverse Kähler form ω, it follows from the third item of the definition of a transverse Kähler foliation that the leaf space X always admits the structure of a Kähler quasifold by using the normal bundle $N\mathcal{F}$ of the foliation $\mathcal{F}$ and the transverse Kähler form ω in the natural way. More precisely, we have the following proposition which seems to be known to some experts. For the sake of reader's convenience, we give a sketch of the proof.

Proposition 2.1. *Let M be a complex n-dimensional manifold equipped with a transverse Kähler foliation $\mathcal{F}$ generated by m-dimensional leaves with respect to a basic and closed real 2-form ω. Then the leaf space X admits the structure of a Kähler quasifold of complex dimension $n-m$ in the natural way.*

Proof. To begin with the proof, let us first denote by $\pi : M \to X$ the projection map with the quotient topology on X. Let $(U_\alpha, \varphi_\alpha)$ be a local coordinate neighborhood of M with the coordinates

$$(w_1, w_2, \dots, w_m, z_1, z_2, \dots, z_{n-m}) \in \mathbb{C}^n$$

such that $(w_1, w_2, \dots, w_m) \in \mathbb{C}^m$ are the coordinates along the leaves of the foliation $\mathcal{F}$. This kind of a local coordinate neighborhood for a transverse foliation is usually called a *foliation chart*.

Let $\pi_\alpha := \pi|_{U_\alpha}$, and let $V_\alpha := \pi_\alpha(U_\alpha)$. Also let $p_\alpha : \varphi_\alpha(U_\alpha) \to \mathbb{C}^{n-m}$ be the natural projection onto the last $(n-m)$ components of $\varphi_\alpha(U_\alpha)$. Then, by using the standard representation theory we may assume that π_α restricted to the normal slice $N\mathcal{F}$ of $\mathcal{F}$ in U_α is a quotient map of $N\mathcal{F}$ of $\mathcal{F}$ by a discrete subgroup Γ_α of $(\mathbb{C}^*)^{n-m}$. Here Γ_α is the maximal subgroup of $(\mathbb{C}^*)^{n-m}$ under which ω is invariant. The fact that Γ_α is discrete can be also seen in other way, since the intersection of a leaf of the foliation and the foliated chart should be countable (see, e.g., the proof of a theorem in [22, Theorem 6]). If the discrete subgroup Γ_α of $(\mathbb{C}^*)^{n-m}$ happens to be finite, then the quotient space admits an orbifold structure, while otherwise it admits just a quasifold structure as in [21, Section 1] and [2].

Note also that if Γ_α is closed, then we can show the finiteness of Γ_α by using the well-known fact that any closed discrete subgroup of $\mathbb{C}^*$ is always finite (see, e.g., [7, Example 2.1.8]).

Now the rest of the proof follows from the classical result of Holmann (see, e.g., [20, Theorem 2.7]). To be a bit more precise, by using the same notation as above let

$$\psi_\alpha : V_\alpha \to p_\alpha(\varphi_\alpha(U_\alpha)) \subset \mathbb{C}^{n-m}, \quad \pi_\alpha(x) \mapsto p_\alpha \circ \varphi_\alpha(x).$$

Here, clearly V_α and $p_\alpha(\varphi_\alpha(U_\alpha))$ are open subsets of X and $\mathbb{C}^{n-m}$, respectively, and ψ_α is well-defined. When $V_\alpha \cap V_\beta \neq \emptyset$, note also that

$$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(V_\alpha \cap V_\beta) \to \psi_\alpha(V_\alpha \cap V_\beta)$$

is biholomorphic. Therefore, the leaf space X admits the complex orbifold structure of complex dimension $n-m$ given by the collection of coordinate neighborhoods (V_α, ψ_α)'s as above.

In addition, it is now easy to see that on each V_α we can give a Kähler structure ω_α, induced from the closed real 2-form ω, which may be singular along the singular locus of V_α (see, e.g., [11, Section 3] for certain similar cases). Note also that the induced Kähler form ω_α is the same as the restriction of the globally defined closed real 2-form ω to the normal slice of the foliation $\mathcal{F}$ on each U_α which is invariant under the finite subgroup Γ_α of $(\mathbb{C}^*)^{n-m}$. This implies that there is a well-defined global singular Kähler form on the leaf space X. Therefore, the leaf space X is actually a Kähler orbifold of complex dimension $n-m$, as desired. □

In view of Proposition 2.1, from now on we shall assume that the leaf space X of a transverse Kähler foliation $\mathcal{F}$ with respect to a basic and closed differential 2-form is a Kähler orbifold, without further mentioning.

Given an LVMB datum $(\mathcal{L}, \mathcal{E}_{m,n})$, we next consider the vector fields η_j $(1 \leq j \leq m)$ of $\mathbb{C}^n$ given by

$$\eta_j = \sum_{i=1}^{n} \mathrm{Re}(l_i^j) z_i \frac{\partial}{\partial z_i},$$

where $l_i = (l_i^1, \ldots, l_i^j, \ldots, l_i^m)$. Let ξ_j $(1 \leq j \leq m)$ be m holomorphic commuting vector fields on $\mathbb{C}^n$ given by

$$\xi_j(z) = \sum_{i=1}^{n} l_i^j z_i \frac{\partial}{\partial z_i},$$

associated to the holomorphic action of $\mathbb{C}^m$ on $\mathbb{C}^n$ defined in (2.1), and let R be the holomorphic vector field on $\mathbb{C}^n \backslash \{0\}$ defined by

$$R(z) = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}.$$

Then the holomorphic vector fields $R, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_m$ on $\mathbb{C}^n$ all commute to each other, and are linearly independent at each point of $\mathbb{C}^n$. So their projections $\tilde{\eta}_1, \ldots, \tilde{\eta}_m$ are linearly independent on each point of an LVMB manifold N, and generate a regular holomorphic foliation $\mathcal{F}$ of dimension m on N.

Generalizing the result of Loeb and Nicolau in [17], Meersseman shows in [19, Theorem 7] that the regular holomorphic foliation $\mathcal{F}$ of dimension m on an LVM manifold N is actually transverse Kähler with respect to the Euler class of the S^1-bundle $\pi_1 : M_1 \to N$ and a Kähler form ω that is the projection of the standard Kähler form on $\mathbb{C}^n$ onto N. Here $\pi_1 : M_1 \to N$ is the pullback of the bundle $S^{2n-1} \to \mathbb{CP}^{n-1}$ by the smooth embedding of N into the projective space.

As another remarkable result related to the transverse Kähler foliation, we can mention that if an LVM manifold M admitting a regular transverse Kähler foliation also satisfies the condition (K), then it has been shown in [20] that M admits the structure of a holomorphic principal Seifert bundle with compact complex tori as fibers over the projective toric orbifold. As in the remark given in [20, Theorem A] the condition (K) is optimal in the sense that the transverse Kähler foliation $\mathcal{F}$ without the condition (K) does not necessarily have compact leaves.

Remark 2.2. As mentioned above, it follows from the construction that there is a fibration on an LVMB manifold N induced from the foliation $\mathcal{F}$ such that

$$\mathbb{C}^m \hookrightarrow N \to X.$$

Assume that the foliation $\mathcal{F}$ is transverse Kähler with a closed 2-form ω. Since $\mathbb{C}^m$ is not compact, it is not always possible to make ω to be basic by taking the usual averaging procedure.

Example 2.3. Note that the standard Hopf surface obtained by taking the quotient of $\mathbb{C}^2\backslash\{(0,0)\}$ by the group generated by a diagonal matrix, for example,

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha > 1$$

can possibly provide an example showing that the leaf space $N/\mathcal{F}$ in Proposition 2.1 is a quasifold, but not an orbifold. Indeed, the Hopf manifold is an LVM manifold which admits a transverse Kähler foliation given by the flow of a well-chosen linear vector field on $\mathbb{C}^2$. Such a foliation contains at least two elliptic leaves corresponding to two axes of $\mathbb{C}^2$. Now it is possible to perturb the foliation above in such a way that the Hopf surface contains only two elliptic leaves and its holonomy group, say Γ_α appeared in the proof of Proposition 2.1, is infinite. See also [16, Example 1.3] for a similar example.

As a much simpler example that we have recently learned, let Γ be a lattice generated by two vectors (a, b) and (c, d) in $\mathbb{C}^2$ with coordinates z_1 and z_2 such that a, b, c, and d are all non-zero constants, and let $M = \mathbb{C}^2/\Gamma$. Let $\mathcal{F}$ be the holomorphic foliation on M generated by $\frac{\partial}{\partial z_1}$. Then $\mathcal{F}$ is transverse Kähler with respect to the closed 2-form $-\frac{1}{2\sqrt{-1}} dz_2 \wedge d\bar{z}_2$. With a suitable choice of a, b, c, and d, each leaf of $\mathcal{F}$ is not closed, and so the leaf space $M/\mathcal{F}$ is not an orbifold, but a quasifold.

Remark 2.4. It is important to note that by Proposition 2.1 there is no difficulty in applying an orbifold version of the Kodaira embedding theorem as well as other well-known theorems in complex geometry to the leaf space X, when X admits an integral Hodge class. This fact will play an important role in the proof of Theorem 1.1 given in Section 3.

3. Proof of Theorem 1.1

The aim of this section is to give a proof of our main Theorem 1.1, as follows.

Proof of Theorem 1.1. In order to prove it, assume that N admits the transverse Kähler foliation $\mathcal{F}$, and let X denote the leaf space of the foliation $\mathcal{F}$ on N. For the rest of the paper, the de Rham cohomology and Dolbeault cohomology of the leaf space X will always mean those of the complexes of basic forms, unless stated otherwise.

With these understood, it also follows from a work [10] of El Kacimi-Alaoui that, roughly speaking, the cohomology of the complex of basic forms on $\mathcal{S}$ is the same as the de Rham cohomology of the leaf space X obtained from the complexes of basic differential forms on N. In particular, we have

$$H^2_B(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \cong H^2_{\bar\partial}(X, \mathcal{O}_X),$$

where $\mathcal{O}_{\mathcal{S}}$ (resp. $\mathcal{O}_X$) denotes the sheaf of germs of holomorphic functions on $\mathcal{S}$ (resp. X).

Indeed, consider the following short exact sequence on $\mathcal{S}$

$$0 \longrightarrow \mathcal{O}^{inv}_{\mathcal{S}} \longrightarrow \mathcal{O}_{\mathcal{S}} \xrightarrow{\mathcal{L}} \mathcal{O}_{etr} \longrightarrow 0. \tag{3.1}$$

Here, $\mathcal{O}_{etr}$ is the image of $\mathcal{O}_{\mathcal{S}}$ in $\mathcal{O}^{\oplus 2m+1}_{\mathcal{S}}$, and $\mathcal{O}^{inv}_{\mathcal{S}}$ denotes the sheaf of germs of holomorphic functions on $\mathcal{S}$ which are invariant along the linear foliation generated by R, $\xi_1, \ldots, \xi_m$, $\eta_1, \ldots, \eta_m$. Moreover, let $\mathcal{L}_R$, $\mathcal{L}_{\xi_i}$, and $\mathcal{L}_{\eta_i}$ denote the Lie derivatives with respect to R, ξ_i, and η_i, respectively. Then $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{L}_R \oplus \mathcal{L}_{\xi_1} \oplus \cdots \oplus \mathcal{L}_{\xi_m} \oplus \mathcal{L}_{\eta_1} \oplus \cdots \oplus \mathcal{L}_{\eta_m}.$$

Thus it follows from the short exact sequence (3.1) that we have

$$H^2_{\bar\partial}(X, \mathcal{O}_X) \cong H^2_{\bar\partial}(\mathcal{S}, \mathcal{O}^{inv}_{\mathcal{S}}) =: H^2_B(\mathcal{S}, \mathcal{O}_{\mathcal{S}}).$$

We next claim that we have

$$H^2_{\bar\partial}(X, \mathcal{O}_X) = 0.$$

To prove the claim, as before let d denote the complex codimension of E in $\mathbb{C}^n$. Then we divide the proof of the claim into two cases, and besides we will complete the proof of Theorem 1.1 for each case.

So we first assume that d is greater than or equal to 2. Then, in particular, $\mathcal{S}$ is a 2-connected open subset in $\mathbb{C}^n$. We next consider the homotopy sequence of the fibration

$$\mathbb{C}^* \times \mathbb{C}^m \hookrightarrow \mathcal{S} \to N.$$

Since $\mathcal{S}$ is 2-connected, it is easy to see that

$$\pi_2(N) \cong \pi_1(\mathbb{C}^* \times \mathbb{C}^m) \cong \mathbb{Z},\ \pi_1(N) \cong \pi_1(\mathcal{S}) = 0.$$

Thus, by the Hurewicz isomorphism theorem ([6], p. 225) we have

$$H^2(N;\mathbb{Z}) \cong \pi_2(N) \cong \mathbb{Z}.$$

Moreover, by considering the fibration induced from the foliation $\mathcal{F}$ on N

$$\mathbb{C}^m \hookrightarrow N \to X$$

we also have $\pi_i(N) \cong \pi_i(X)$ for all non-negative integers i. Thus, by applying the Whitehead's theorem ([13], p. 346) we have $H^i(N;\mathbb{R}) \cong H^i(X;\mathbb{R})$ for all non-negative integers i. In particular, we have $H^2(X;\mathbb{R}) \cong \mathbb{R}$. We note that if $d > 1$, then this fits well with the computation given in [9, Proposition 3.4] or [5, Proposition 2.1] for the cases dealt with in their papers.

Since N is assumed to admit a transverse Kähler foliation $\mathcal{F}$, as mentioned above and in Section 2, it is well known (see, e.g., [11]) that, by using a work [10] of El Kacimi-Alaoui and the notion of basic differential forms, the Hodge decomposition theorem on the leaf space X holds, as follows.

$$H^2_{DR}(X,\mathbb{C}) \cong H^{0,2}_{\bar\partial}(X) \oplus H^{1,1}_{\bar\partial}(X) \oplus H^{2,0}_{\bar\partial}(X). \tag{3.2}$$

On the other hand, since $H^2(X,\mathbb{R}) \cong \mathbb{R}$ and $H^{1,1}_{\bar\partial}(X) \neq 0$, it follows from (3.2) that we have

$$H^2_{\bar\partial}(X,\mathcal{O}_X) = 0,$$

as claimed. This implies that we have

$$\mathbb{R} \cong H^2(X,\mathbb{R}) \cong H^{1,1}_{\bar\partial}(X).$$

Since $H^2(X,\mathbb{Q}) \subset H^2(X,\mathbb{R}) \cong H^{1,1}_{\bar\partial}(X)$, we can conclude that there should be an integral Kähler form on X whose lift to N is a basic and transverse Kähler form. Hence it follows from the assumption of Theorem 1.1 that our orbit space X is, in fact, a Kähler orbifold with an integral Kähler form, so X should be projective by an orbifold version of the Kodaira embedding theorem ([1] and [12, p. 191]).

Further, observe that our leaf space X is a toric orbifold. To see it, note first that by Proposition 2.1 X is an orbifold of complex dimension $n-(2m+1)$. Moreover, since $\mathbb{C}^n$ admits a holomorphic action of $(\mathbb{C}^*)^n$ under the multiplication and $\mathcal{S}$ is an open subset of $\mathbb{C}^n$ invariant under the holomorphic action of $(\mathbb{C}^*)^n$, it follows from the construction that our LVMB manifold N admits a holomorphic action of $(\mathbb{C}^*)^{n-m-1}$. Thus the leaf space X now admits a holomorphic action of $(\mathbb{C}^*)^{n-(2m+1)}$, as desired.

Recall that the associated polytope of a projective toric manifold is a Delzant (or moment) polytope which is, in particular, polytopal. Clearly this fact applies to our toric projective manifold (or orbifold) X, and so the associated polytope of X is polytopal. Therefore, N is actually an LVM manifold by [4, Theorem 3.10][1].

We next assume that d is equal to 1. This implies that there is an indispensable element $i \in [n]$, say $i = 1$. So we may write

$$\mathcal{S} = \mathbb{C}^* \times \mathcal{S}',$$

where $\mathcal{S}' = \mathbb{C}^{n-1} - E'$ and E' is an arrangement of coordinate subspaces in $\mathbb{C}^{n-1}$ whose complex codimension d' is at least two. Note that we have

$$\pi_1(\mathcal{S}) \cong \mathbb{Z}, \quad \pi_i(\mathcal{S}) \cong \pi_i(\mathcal{S}'), \; i \geq 2.$$

As in the above arguments, it is also easy to see that

$$\pi_2(\mathcal{S}) \cong \pi_2(\mathcal{S}') \cong \mathbb{Z}.$$

Now, as before we consider the homotopy sequence of the fibration

$$\mathbb{C}^* \times \mathbb{C}^m \hookrightarrow \mathcal{S} \to N.$$

Then we can obtain

$$\begin{aligned} &\longrightarrow \pi_2(\mathbb{C}^* \times \mathbb{C}^m) = 0 \longrightarrow \pi_2(\mathcal{S}) = \mathbb{Z} \longrightarrow \pi_2(N) \\ &\longrightarrow \pi_1(\mathbb{C}^* \times \mathbb{C}^m) = \mathbb{Z} \longrightarrow \pi_1(\mathcal{S}) = \mathbb{Z} \longrightarrow \pi_1(N) \longrightarrow 0. \end{aligned} \tag{3.3}$$

By the standard diagram-chasing of (3.3), it is not difficult to obtain the following two possibilities:

$$\pi_2(N) \cong \mathbb{Z} \text{ and } \pi_1(N) = 0, \text{ or } \pi_2(N) = \mathbb{Z} \rtimes \mathbb{Z} \text{ and } \pi_1(N) = \mathbb{Z}.$$

Once again, by considering the fibration induced from the foliation $\mathcal{F}$ on N

$$\mathbb{C}^m \hookrightarrow N \to X,$$

we have $\pi_i(N) \cong \pi_i(X)$ for all non-negative integers i. Thus, by applying the Whitehead's theorem ([13], p. 346) we have $H^i(N;\mathbb{R}) \cong H^i(X;\mathbb{R})$ for all non-negative integers i. In particular, we have $H^2(X;\mathbb{R}) \cong \mathbb{R}$ or $\mathbb{R} \oplus \mathbb{R}$. Once again, this fits well with the computation given in [9, Proposition 3.4] or [5, Proposition 2.1] for the case of $d = 1$.

Next, we apply the Hodge decomposition on the leaf space X, as follows.

$$H^2_{DR}(X, \mathbb{C}) \cong H^{0,2}_{\bar\partial}(X) \oplus H^{1,1}_{\bar\partial}(X) \oplus H^{2,0}_{\bar\partial}(X).$$

Since $H^2_{DR}(X;\mathbb{C}) \cong \mathbb{C}$ or $\mathbb{C} \oplus \mathbb{C}$, $H^{0,2}_{\bar\partial}(X) \cong H^{2,0}_{\bar\partial}(X)$, and $H^{1,1}_{\bar\partial}(X)$ is nontrivial, we should have $H^{0,2}_{\bar\partial}(X) = 0$, as claimed.

Finally, the rest of the proof goes exactly as in the case of $d \geq 2$. That is, since $H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R}) \cong H^{1,1}_{\bar\partial}(X)$, there should be an integral Kähler form on X whose lift to N is a basic and transverse Kähler form. Hence once again our orbit space X is a Kähler orbifold with an integral Kähler form, so

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[1] Theorem 3.10 in [4] is stated in a slightly different way, but, as is well-known, it says that if the associated underlying simplicial complex of X is polytopal as in our case, then N is an LVM manifold. For instance, see [3]

that X is projective by an orbifold version of the Kodaira embedding theorem. Therefore, N is actually an LVM manifold by [4, Theorem 3.10].

This completes the proof of Theorem 1.1. □

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