Commun. Korean Math. Soc. **35** (2020), No. 2, pp. 571–589
https://doi.org/10.4134/CKMS.c190255
pISSN: 1225-1763 / eISSN: 2234-3024

RELATION BETWEEN KNEADING MATRICES OF A MAP AND ITS ITERATES

Chaitanya Gopalakrishna and Murugan Veerapazham

Abstract. It is known that the kneading matrix associated with a continuous piecewise monotone self-map of an interval contains crucial combinatorial information of the map and all its iterates, however for every iterate of such a map we can associate its kneading matrix. In this paper, we describe the relation between kneading matrices of maps and their iterates for a family of chaotic maps. We also give a new definition for the kneading matrix and describe the relationship between the corresponding determinant and the usual kneading determinant of such maps.

1. Introduction

Continuous piecewise monotone self-maps of a compact interval in the real line provide interesting examples of discrete dynamical systems [3, 4, 9, 10], however their behaviour can be very complicated. As defined in [7], an element $f \in \mathcal{C}(I)$, where $I = [a, b]$ is a compact interval in $\mathbb{R}$ and $\mathcal{C}(I)$ denotes the set of all continuous self-maps of I, is said to be *piecewise monotone* if there exists a partition $a = c_0 < c_1 < \cdots < c_m < c_{m+1} = b$ of I such that the restriction of f to subintervals $I_j = [c_{j-1}, c_j]$ is strictly monotone for $1 \leq j \leq m + 1$. Let $f \in \mathcal{M}(I)$, the set of all piecewise monotone mappings in $\mathcal{C}(I)$, and suppose that the minimal choice for the c_i's is made so that f is not monotone in any neighbourhood of c_i for $1 \leq i \leq m$. Then the points $c_1, c_2, \ldots, c_m$ are called the *turning points* of f and the subintervals $I_j, j = 1, 2, \ldots, m + 1$, the *laps* of f. An $f \in \mathcal{M}(I)$ with exactly one turning point is called a *unimodal map*. For $f \in \mathcal{M}(I)$, let $T(f)$ denote the set of turning points of f, $|T(f)|$ the number of turning points of f and $L(f)$ the set of laps of f.

The set $\mathcal{M}(I)$ is closed with respect to composition of maps. In fact,

$$T(f \circ g) = \big(T(g) \cup g^{-1}\big(T(f)\big)\big) \cap (a, b). \tag{1.1}$$

Received July 24, 2019; Revised September 25, 2019; Accepted November 19, 2019.
2010 *Mathematics Subject Classification.* Primary 37E05; Secondary 15A24.
Key words and phrases. Dynamical system, piecewise monotone map, kneading matrix, kneading determinant.
The second author was supported by the Science and Engineering Research Board (SERB), DST, Government of India, through the project *ECR*/2017/000765.

So, in particular, if $f \in \mathcal{M}(I)$, then $f^k \in \mathcal{M}(I)$ such that

$$T(f^k) = \{x \in (a,b) : f^l(x) \in T(f) \text{ for some } 0 \le l \le k-1\} \tag{1.2}$$

for each $k \in \mathbb{N}$, where for each $k \ge 0$, f^k denotes the k-th order iterate of f defined recursively by

$$f^0 := \mathrm{id}_I \text{ and } f^k := f \circ f^{k-1},$$

id_I being the identity map on I. On the other hand, if $f, g \in \mathcal{C}(I)$ such that $f \circ g \in \mathcal{M}(I)$, then $g \in \mathcal{M}(I)$. In particular, if $f \in \mathcal{C}(I)$ such that $f^k \in \mathcal{M}(I)$ for some $k \in \mathbb{N}$, then $f \in \mathcal{M}(I)$.

Milnor and Thurston, in their kneading theory [6, 7] to study the iterates of mappings in $\mathcal{M}(I)$, have associated with each element of $\mathcal{M}(I)$ a matrix and a determinant called the kneading matrix and kneading determinant, respectively. In some sense, this matrix contains most of the crucial combinatorial information of the map and all its iterates [2, 11]. Moreover, it is proved in [7] that these matrix and determinant are invariant under orientation-preserving conjugacy. Being an important area of research in symbolic dynamics, kneading theory has been developed in various aspects, see for example, kneading theory for piecewise monotone maps with discontinuities [11], tree maps [1], triangular maps [5] and circle maps [8].

In this paper, we investigate some dynamical behaviours of mappings in $\mathcal{M}_0(I)$, a specific yet very important subclass of $\mathcal{M}(I)$ consisting of all chaotic maps whose restrictions to each of their laps are onto. The kneading matrix of an $f \in \mathcal{M}(I)$ with m turning points is an $m \times (m+1)$ matrix with entries from the ring of formal power series with integer coefficients. Moreover, the iterates of f satisfy the ascending relation

$$|T(f)| \le |T(f^2)| \le |T(f^3)| \le \cdots .$$

Therefore the process of finding the kneading matrices of higher-order iterates of f involves tedious computations. In the next section, with a view to introduce some notations and recall some definitions, we give a brief account of Milnor-Thurston's kneading theory for mappings in $\mathcal{M}(I)$. For arbitrary $f, g \in \mathcal{M}_0(I)$, in Section 3 we prove that the composite maps satisfy either of the matrix identities

$$N(f \circ g; t) = N(g \circ f; t) \text{ or } N(f \circ g; t) = -S_k N(g \circ f; t) S_{k+1}$$

for some $k \in \mathbb{N}$, where S_k denotes the $k \times k$ matrix $[k_{ij}]$ defined by

$$k_{ij} = \begin{cases} 1 & \text{if } i + j = k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we prove the identities

$$M(f; t) = \mathcal{I}_m M(h; t) R_{3\times(m+1)} \tag{1.3}$$

and

$$M(f;t) = \begin{bmatrix} \mathcal{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} M(\tilde{h};t) R_{4\times(m+1)} \tag{1.4}$$

for mappings in $\mathcal{M}_0(I)$. Here h and $\tilde{h}$ denote respectively the bimodal and trimodal uniformly piecewise linear maps in $\mathcal{M}_0(I)$, $\mathbb{O}_{k\times l}$ the zero matrix of order $k\times l$, $\mathbb{I}_k$ the identity matrix of order k, $\mathcal{I}_k$ the transpose of $[\mathbb{I}_2\ \mathbb{I}_2\ \cdots \mathbb{I}_2]_{2\times k}$ for even k, $R_{k\times l}$ the $k\times l$ matrix $[r_{ij}]$ defined by

$$r_{ij} = \begin{cases} 1 & \text{if } i=j=1 \text{ or } i=k \text{ and } j=l,\ 1\le i\le k, 1\le j\le l, \\ 0 & \text{otherwise,} \end{cases}$$

and $M(f;t) = N(f;t) - N_0(f;t)$, where $N_0(f;t)$ is the $m\times(m+1)$ matrix $[N^0_{ij}(f;t)]$ given by

$$N^0_{ij}(f;t) = \begin{cases} -1 & \text{if } j=i,\ 1\le i\le m, 1\le j\le m+1, \\ 1 & \text{if } j=i+1,\ 1\le i\le m, 1\le j\le m+1, \\ 0 & \text{otherwise.} \end{cases}$$

The identities (1.3) and (1.4) describe the relation between kneading matrices of mappings in $\mathcal{M}_0(I)$ with that of uniformly piecewise linear maps whose dynamical behaviours are relatively easy to investigate. We also prove similar identities which relate kneading matrices of mappings in $\mathcal{M}_0(I)$ with that of their iterates. Finally, in Section 4, we define the modified kneading matrix for such maps and exhibit a relation between the corresponding determinant and the usual kneading determinant.

2. Preliminaries

In this section, through a brief introduction to Milnor-Thurston's kneading theory, we introduce the notations and definitions that are used in our further discussions. For the entirety of this section, unless otherwise stated, let $f \in \mathcal{M}(I)$ with

$$T(f) = \{c_1, c_2, \ldots, c_m\} \text{ and } L(f) = \{I_1, I_2, \ldots, I_{m+1}\},$$

where $I_j = [c_{j-1}, c_j]$ for $1\le j\le m+1$. We recall several formal power series associated with the map f, which serves as raw ingredients to develop this kneading theory.

Let V be the $(m+1)$-dimensional vector space over $\mathbb{Q}$ with an ordered basis the set of formal symbols $I_1, I_2, \ldots, I_{m+1}$ and $V[[t]]$ be the $\mathbb{Q}[[t]]$-module consisting of all formal power series with coefficients in V. For $x\in I$ and $k\ge 0$, let

$$A_k(x,f) := \begin{cases} I_j & \text{if } f^k(x)\in I_j,\ 1\le j\le m+1 \text{ and } f^k(x)\notin T(f), \\ C_i & \text{if } f^k(x) = c_i,\ 1\le i\le m, \end{cases}$$

where $C_i := \frac{1}{2}(I_i + I_{i+1})$ for $1\le i\le m$. The symbol $A_0(x,f)$ is called the *address* of x.

For each subinterval I' of I, we write $f \nearrow I'$ (resp. $f \searrow I'$) to mean f is strictly increasing (resp. strictly decreasing) on I'. For each symbol I_j, define the *sign* by

$$\epsilon(I_j) = \begin{cases} +1 & \text{if} \quad f \nearrow I_j, \\ -1 & \text{if} \quad f \searrow I_j, \end{cases}$$

and for each of the vector C_j corresponding to the turning point c_j, let $\epsilon(C_j) := 0$. For each $x \in I$, let $\epsilon_k(x, f) := \epsilon(A_k(x, f))$ for $k \geq 0$, and

$$\theta_0(x, f) := A_0(x, f) \text{ and } \theta_k(x, f) := \left(\prod_{l=0}^{k-1} \epsilon_l(x, f)\right) A_k(x, f) \text{ for } k \geq 1.$$

The corresponding formal power series is defined by

$$\theta(x, f; t) = \sum_{k \geq 0} \theta_k(x, f) t^k.$$

Consider $V[[t]]$ in the formal power series topology in which the submodules $t^k V[[t]]$ form a basis for the neighbourhoods of zero. For each $x \in [a, b)$ and $k \geq 0$, let

$$x+ := \mathrm{id}_I(x+), \; A(f^k(x+)) := \lim_{y \downarrow x} A_k(y, f), \; \epsilon_k(x+, f) := \lim_{y \downarrow x} \epsilon_k(y, f)$$

and $\theta_k(x+, f) := \lim_{y \downarrow x} \theta_k(y, f)$. The corresponding left-hand limits are defined similarly. Then it follows that

$$\epsilon_k(x+, f) = \epsilon(A_k(x+, f)) \text{ for } x \in [a, b), \; k \geq 0,$$

and

$$\epsilon_k(x-, f) = \epsilon(A_k(x-, f)) \text{ for } x \in (a, b], \; k \geq 0,$$

where $A_k(x+, f)$ and $A_k(x-, f)$ denote $A(f^k(x+))$ and $A(f^k(x-))$, respectively. Moreover,

$$A_k(c_i+, f) = A_k(c_i-, f)$$

for $1 \leq i \leq m$ and $k \in \mathbb{N}$. For each $x \in [a, b)$, let $\theta(x+, f) := \lim_{y \downarrow x} \theta(y, f)$ and for each $x \in (a, b]$, let $\theta(x-, f) := \lim_{y \uparrow x} \theta(y, f)$. Then $\theta(x+, f; t) = \sum_{k \geq 0} \theta_k(x+, f) t^k$ for $x \in [a, b)$ and $\theta(x-, f; t) = \sum_{k \geq 0} \theta_k(x-, f) t^k$ for $x \in (a, b]$.

As defined in [7], the formal power series $\theta(c_i+, f; t) - \theta(c_i-, f; t)$ is called the i^{th} *kneading increment* $\nu(c_i, f; t)$ of f for $1 \leq i \leq m$. The matrix $N(f; t) = [N_{ij}(f; t)]$ of order $m \times (m + 1)$, with entries in $\mathbb{Z}[[t]]$, obtained by setting

$$\nu(c_i, f; t) = N_{i1}(f; t) I_1 + N_{i2}(f; t) I_2 + \cdots + N_{i,m+1}(f; t) I_{m+1}, \text{ for } 1 \leq i \leq m,$$

is called the *kneading matrix* of f. We can write the matrix $N(f; t)$ as a power series $\sum_{k \geq 0} [N_{ij}^k(f; t)] t^k$ where the coefficients $[N_{ij}^0(f; t)]$, $[N_{ij}^1(f; t)]$, ... are matrices with integer entries. For $k = 0$, the matrix $[N^0_{ij}(f;t)]$ is given by

$$[N^0_{ij}(f;t)] = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{m\times(m+1)}$$

and in fact, it is independent of the mapping f. Let $N_k(f;t)$ denote the matrix $[N^k_{ij}(f;t)]$ for $k \geq 0$, and $M(f;t) := \sum_{k\geq 1} N_k(f;t)t^k$. For $1 \leq j \leq m+1$, let $N^{(j)}(f;t)$ denote the $m \times m$ matrix obtained by deleting the j^{th} column of $N(f;t)$. Then the power series $(-1)^{j+1}\big(1-\epsilon(I_j)t\big)^{-1}\det\big(N^{(j)}(f;t)\big)$ is indeed independent of the choice of j for $1 \leq j \leq m+1$ and this common expression, denoted by $D(f;t)$, is called the *kneading determinant* of f ([7]).

3. Kneading matrices of iterates of f

For each $f \in \mathcal{C}(I)$, let $\mathcal{I}_f := \{f^k \mid k \geq 0\}$, the set of iterates of f. As noted in the introduction, the kneading matrix $N(f;t)$ of any $f \in \mathcal{M}(I)$ contains some important combinatorial information concerning all the elements of $\mathcal{I}_f$ and hence that of $\mathcal{I}_{f^k}$ for any $k \in \mathbb{N}$, because $\mathcal{I}_{f^k} \subseteq \mathcal{I}_f$. Motivated by this observation, we expect that $N(f^k;t)$ and $N(f;t)$ are related for every $k \in \mathbb{N}$. But the problem of finding a matrix equation that relates these two matrices is not so trivial, as the order of these matrices are different and moreover the problem of computing the kneading matrix of a map is very hard. In this section, we derive matrix equations that relate the kneading matrices of function and its iterates for a particular family of chaotic piecewise monotone maps, namely

$$\mathcal{M}_0(I) = \big\{f \in \mathcal{M}(I) : f\big(T(f) \cup \{a,b\}\big) \subseteq \{a,b\}\big\},$$

the set of all continuous piecewise monotone self-maps of I which are onto on each of their laps.

For each $k \in \mathbb{N}$ and $n_1, n_2, \ldots, n_k \in \mathbb{N} \cup \{0\}$, let

$$S(n_1, n_2, \ldots, n_k) := \sum_{j=1}^{k} S_j(n_1, n_2, \ldots, n_k),$$

where for $1 \leq j \leq k$, let

$$S_j(n_1, n_2, \ldots, n_k) := \sum_{1\leq i_1<i_2<\cdots<i_j\leq k} n_{i_1} n_{i_2} \cdots n_{i_j}.$$

Proposition 3.1. (1) *If $f_1, f_2, \ldots, f_k \in \mathcal{M}_0(I)$, then*

$$|T(f_1 \circ f_2 \circ \cdots \circ f_k)| = S\big(|T(f_1)|, |T(f_2)|, \ldots, |T(f_k)|\big).$$

(2) *If $f \in \mathcal{M}_0(I)$ such that $|T(f)| = m$, then $|T(f^k)| = (m+1)^k - 1$, $\forall k \in \mathbb{N}$.*

(3) $|T(f^k)| \equiv |T(f)| (\text{mod } 2)$, $\forall f \in \mathcal{M}_0(I)$ *and* $\forall k \in \mathbb{N}$.

Proof. We prove the first result by mathematical induction on k. For any $f_1 \in \mathcal{M}_0(I)$, we have $S(|T(f_1)|) = S_1(|T(f_1)|) = |T(f_1)|$, and therefore the result is true for $k = 1$.

To prove the result for $k = 2$, consider any $f_1, f_2 \in \mathcal{M}_0(I)$ such that $|T(f_1)| = m_1$ and $|T(f_2)| = m_2$. If both m_1 and m_2 are zero, then f_1, f_2 and hence $f_1 \circ f_2$ is strictly monotone on I, implying that $|T(f_1 \circ f_2)| = 0 = S(0, 0) = S(m_1, m_2)$. If $m_1 = 0$ and $m_2 \neq 0$, then by (1.1), $T(f_1 \circ f_2) = T(f_2)$, and hence

$$|T(f_1 \circ f_2)| = m_2 = S(0, m_2) = S(m_1, m_2).$$

If $m_1 \neq 0$ and $m_2 = 0$, then again by (1.1), $T(f_1 \circ f_2) = f_2^{-1}(T(f_1)) \cap (a, b)$. Since f_2 is strictly monotone on I, it follows that $|f_2^{-1}(T(f_1)) \cap (a, b)| = |T(f_1)|$, and therefore

$$|T(f_1 \circ f_2)| = |T(f_1)| = m_1 = S(m_1, 0) = S(m_1, m_2).$$

Now, let both m_1 and m_2 be non-zero. Let

$$T(f_1) = \{c_1, c_2, \ldots, c_{m_1}\},\ T(f_2) = \{d_1, d_2, \ldots, d_{m_2}\},$$
$$L(f_1) = \{I_1, I_2, \ldots, I_{m_1+1}\} \text{ and } L(f_2) = \{J_1, J_2, \ldots, J_{m_2+1}\},$$

where $a = c_0 < c_1 < \cdots < c_{m_1} < c_{m_1+1} = b$, $I_j = [c_{j-1}, c_j]$ for $1 \leq j \leq m_1 + 1$, $a = d_0 < d_1 < d_2 < \cdots < d_{m_2} < d_{m_2+1} = b$ and $J_i = [d_{i-1}, d_i]$ for $1 \leq i \leq m_2 + 1$. Since $f_2(T(f_2)) \subseteq \{a, b\}$, by using (1.1), we have

$$T(f_1 \circ f_2) = T(f_2) \bigsqcup \left(\bigsqcup_{j=0}^{m_2} (f_2^{-1}(T(f_1)) \cap (d_j, d_{j+1})) \right), \tag{3.1}$$

where $\sqcup$ indicates that the union is disjoint. Now for $0 \leq j \leq m_2$ and $1 \leq i \leq m_1$, since f_2 is strictly monotone on (d_j, d_{j+1}), there exists unique $p_i \in (d_j, d_{j+1})$ such that $f_2(p_i) = c_i$. That is, $f_2^{-1}(c_i) \cap (d_j, d_{j+1})$ is a singleton set for $1 \leq i \leq m_1$ and $0 \leq j \leq m_2$. Hence from (3.1), we have

$$\begin{aligned}
|T(f_1 \circ f_2)| &= |T(f_2)| + \sum_{j=0}^{m_2} |f_2^{-1}(T(f_1)) \cap (d_j, d_{j+1})| \\
&= m_2 + \sum_{j=0}^{m_2} \sum_{i=1}^{m_1} |f_2^{-1}(c_i) \cap (d_j, d_{j+1})| \\
&= m_2 + \sum_{j=0}^{m_2} \sum_{i=1}^{m_1} 1 \\
&= m_2 + m_1(m_2 + 1) \\
&= m_1 + m_2 + m_1 m_2 = S(m_1, m_2).
\end{aligned}$$

Therefore the result is true for $k = 2$. Now suppose that the result is true for certain $k \geq 2$. In order to prove the result for $k+1$, consider any $f_1, f_2, \ldots, f_{k+1}$

in $\mathcal{M}_0(I)$ such that $|T(f_j)| = m_j$ for $1 \leq j \leq k+1$. Let $g = f_1 \circ f_2 \circ \cdots \circ f_k$. Then by using the result for the case $k = 2$,

$$\begin{aligned} |T(g \circ f_{k+1})| &= S(|T(g)|, m_{k+1}) = S_1(|T(g)|, m_{k+1}) + S_2(|T(g)|, m_{k+1}) \\ &= |T(g)| + m_{k+1} + |T(g)| \cdot m_{k+1}. \end{aligned} \tag{3.2}$$

By induction hypothesis,

$$|T(g)| = S(m_1, m_2, \ldots, m_k).$$

Therefore by (3.2), we have

$$|T(g \circ f_{k+1})| = S(m_1, m_2, \ldots, m_k) + m_{k+1} + S(m_1, m_2, \ldots, m_k)m_{k+1}. \tag{3.3}$$

Now

$$S_1(m_1, m_2, \ldots, m_{k+1}) = S_1(m_1, m_2, \ldots, m_k) + m_{k+1}, \tag{3.4}$$

$$S_{k+1}(m_1, m_2, \ldots, m_{k+1}) = S_k(m_1, m_2, \ldots, m_k)m_{k+1} \tag{3.5}$$

and

$$\begin{aligned} S_j(m_1, m_2, \ldots, m_{k+1}) = {} & S_j(m_1, m_2, \ldots, m_k) \\ & + S_{j-1}(m_1, m_2, \ldots, m_k)m_{k+1} \end{aligned} \tag{3.6}$$

for $2 \leq j \leq k$. Therefore by adding (3.4), (3.5) and (3.6), on simplification, we obtain

$$\begin{aligned} S(m_1, m_2, \ldots, m_{k+1}) &= S(m_1, m_2, \ldots, m_k) + m_{k+1} \\ &\quad + S(m_1, m_2, \ldots, m_k)m_{k+1} \\ &= |T(g \circ f_{k+1})| \text{ (by (3.3))} \\ &= |T(f_1 \circ f_2 \circ \cdots \circ f_{k+1})|. \end{aligned}$$

Thus the result is true for $k+1$ and therefore by mathematical induction it is true for every $k \in \mathbb{N}$. This proves result (1).

In order to prove the second result, consider any $f \in \mathcal{M}_0(I)$ such that $|T(f)| = m$ and let $k \in \mathbb{N}$. Put $m_j = m$ for $1 \leq j \leq k$. Then

$$S_j(m_1, m_2, \ldots, m_k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} m^j = \binom{k}{j} m^j$$

for $1 \leq j \leq k$, and therefore

$$S(m_1, m_2, \ldots, m_k) = \sum_{j=1}^{k} \binom{k}{j} m^j = (m+1)^k - 1.$$

Hence by result (1), we have $|T(f^k)| = S(m_1, m_2, \ldots, m_k) = (m+1)^k - 1$. Result (3) follows from result (2) by noting that $(m+1)^k - 1 \equiv m (\text{mod } 2)$. $\square$

Now we introduce some particular subsets of $\mathcal{M}_0(I)$. Let

$$\begin{aligned}
\mathcal{M}_{\nearrow}(I) &:= \{f \in \mathcal{M}_0(I) : T(f) = \emptyset, f(a) = a \text{ and } f(b) = b\}\,,\\
\mathcal{M}_{\searrow}(I) &:= \{f \in \mathcal{M}_0(I) : T(f) = \emptyset, f(a) = b \text{ and } f(b) = a\}\,,\\
\mathcal{M}_{\wedge}(I) &:= \{f \in \mathcal{M}_0(I) : f \text{ is unimodal and } f(a) = f(b) = a\}\,,\\
\mathcal{M}_{\vee}(I) &:= \{f \in \mathcal{M}_0(I) : f \text{ is unimodal and } f(a) = f(b) = b\}\,,\\
\mathcal{M}_{\mathrm{N}}(I) &:= \{f \in \mathcal{M}_0(I) : T(f) \neq \emptyset, f(a) = a \text{ and } f(b) = b\}\,,\\
\mathcal{M}_{\text{И}}(I) &:= \{f \in \mathcal{M}_0(I) : T(f) \neq \emptyset, f(a) = b \text{ and } f(b) = a\}\,,\\
\mathcal{M}_{\mathrm{M}}(I) &:= \{f \in \mathcal{M}_0(I) : T(f) \neq \emptyset \text{ and } f(a) = f(b) = a\}\,,\\
\mathcal{M}_{\mathrm{W}}(I) &:= \{f \in \mathcal{M}_0(I) : T(f) \neq \emptyset \text{ and } f(a) = f(b) = b\}\,.
\end{aligned}$$

Then $\mathcal{M}_0(I)$ is indeed the disjoint union of $\mathcal{M}_{\nearrow}(I)$, $\mathcal{M}_{\searrow}(I)$, $\mathcal{M}_{\mathrm{N}}(I)$, $\mathcal{M}_{\text{И}}(I)$, $\mathcal{M}_{\mathrm{M}}(I)$ and $\mathcal{M}_{\mathrm{W}}(I)$.

Proposition 3.2. (1) *If $f, g \in \mathcal{M}_0(I)$, then $f \circ g \in \mathcal{M}_0(I)$. This is also true when $\mathcal{M}_0(I)$ is replaced by $\mathcal{M}_{\nearrow}(I), \mathcal{M}_N(I), \mathcal{M}_M(I)$ and $\mathcal{M}_W(I)$.*

(2) *If $f, g \in \mathcal{C}(I)$ such that $f \circ g \in \mathcal{M}_0(I)$ and $f^{-1}(\{a, b\}) \subseteq \{a, b\}$, then $g \in \mathcal{M}_0(I)$.*

(3) *If $f^k \in \mathcal{M}_0(I)$ for some $k \in \mathbb{N}$, then $f \in \mathcal{M}_0(I)$. This is also true when $\mathcal{M}_0(I)$ is replaced by $\mathcal{M}_M(I)$ and $\mathcal{M}_W(I)$.*

Proof. Let $f, g \in \mathcal{M}_0(I)$. Since $f, g \in \mathcal{M}(I)$, clearly $f \circ g \in \mathcal{M}(I)$. Also, since $f(\{a, b\}) \subseteq \{a, b\}$ and $g(\{a, b\}) \subseteq \{a, b\}$, we have $(f \circ g)(\{a, b\}) \subseteq \{a, b\}$. Now, consider any $c \in T(f \circ g)$. Then by (1.1), either $c \in T(g)$ or $c \in g^{-1}(T(f)) \cap (a, b)$. If $c \in T(g)$, then $g(c) \in \{a, b\}$, implying that $(f \circ g)(c) \in \{a, b\}$. If $c \in g^{-1}(T(f)) \cap (a, b)$, then $g(c) \in T(f)$, and hence $(f \circ g)(c) \in \{a, b\}$. Thus

$$(f \circ g)(T(f \circ g) \cup \{a, b\}) \subseteq \{a, b\},$$

and therefore $f \circ g \in \mathcal{M}_0(I)$. This proves the first part of result (1). Now consider any $f, g \in \mathcal{M}_N(I)$. Then by using result (1) for $\mathcal{M}_0(I)$, we have $f \circ g \in \mathcal{M}_0(I)$. Also, $f(a) = g(a) = a$ and $f(b) = g(b) = b$, implying that $(f \circ g)(a) = a$ and $(f \circ g)(b) = b$. Hence $f \circ g \in \mathcal{M}_N(I)$, proving result (1) for $\mathcal{M}_N(I)$. The proofs for $\mathcal{M}_{\nearrow}(I), \mathcal{M}_M(I)$ and $\mathcal{M}_W(I)$ are similar.

In order to prove the second result, consider any $f, g \in \mathcal{C}(I)$ such that $f \circ g \in \mathcal{M}_0(I)$ and $f^{-1}(\{a, b\}) \subseteq \{a, b\}$. Since $f \circ g \in \mathcal{M}(I)$, we have $g \in \mathcal{M}(I)$. Since $(f \circ g)(\{a, b\}) \subseteq \{a, b\}$, we have $g(\{a, b\}) \subseteq \{a, b\}$. Now it remains to prove that $g(T(g)) \subseteq \{a, b\}$. So, let $c \in T(g)$. Since $T(g) \subseteq T(f \circ g)$, we get that $(f \circ g)(c) \in \{a, b\}$. Therefore $g(c) \in f^{-1}(\{a, b\})$, implying that $g(c) \in \{a, b\}$, because by assumption $f^{-1}(\{a, b\}) \subseteq \{a, b\}$.

To prove result (3), consider any $f \in \mathcal{C}(I)$ such that $f^k \in \mathcal{M}_0(I)$ for some $k \in \mathbb{N}$. If $k = 1$, then there is nothing to prove. So, let $k > 1$. Since $f^k \in \mathcal{M}(I)$, we have $f \in \mathcal{M}(I)$.

Case (a): Suppose that $T(f^k) = \emptyset$. Then $T(f) = \emptyset$, implying that f is strictly monotone on I. Also, since f^k is onto on I, so is f. Therefore $f(\{a, b\}) \subseteq \{a, b\}$, and hence $f \in \mathcal{M}_0(I)$.
Case (b): Suppose that $T(f^k) \neq \emptyset$. Then $T(f) \neq \emptyset$. If $a \in f^{-(k-1)}(a)$, then $f^{k-1}(a) = a$, implying that $f(a) = f(f^{k-1}(a)) = f^k(a) \in \{a, b\}$. If $b \in f^{-(k-1)}(a)$, then $f^{k-1}(b) = a$, and therefore $f(a) = f(f^{k-1}(b)) = f^k(b) \in \{a, b\}$. If $a, b \notin f^{-(k-1)}(a)$, then as f^{k-1} is onto, there exists $c \in (a, b)$ such that $c \in f^{-(k-1)}(a)$. This implies that $c \in T(f^{k-1})$, and hence $c \in T(f^k)$, since $T(f^{k-1}) \subseteq T(f^k)$. Therefore $f^k(c) \in \{a, b\}$ so that $f(a) = f(f^{k-1}(c)) = f^k(c) \in \{a, b\}$. This proves that $f(a) \in \{a, b\}$. By a similar argument, it follows that $f(b) \in \{a, b\}$. Now, it remains to prove that $f(T(f)) \subseteq \{a, b\}$. So, let $c \in T(f)$. Since f^{k-1} is onto, there exists $d \in I$ such that $f^{k-1}(d) = c$, implying that $d \in f^{-(k-1)}(c)$. Then $d \in T(f^k)$, since by (1.2) we have $f^{-(k-1)}(T(f)) \subseteq T(f^k)$. So $f^k(d) \in \{a, b\}$, and therefore $f(c) = f(f^{k-1}(d)) = f^k(d) \in \{a, b\}$. ☐

For each $m \in \mathbb{N} \cup \{0\}$, let $\mathcal{M}_{M,m}(I) := \{f \in \mathcal{M}_M(I) \ : \ |T(f)| = m\}$ and $\mathcal{M}_{W,m}(I)$, $\mathcal{M}_{N,m}(I)$, $\mathcal{M}_{V\!\Lambda,m}(I)$ be defined similarly.

Lemma 3.3. *For each $m \in \mathbb{N}$, the kneading matrix $N(f; t)$ is independent of the choice of f in $\mathcal{M}_{M,m}(I)$. This is also true when $\mathcal{M}_{M,m}(I)$ is replaced by $\mathcal{M}_{W,m}(I)$, $\mathcal{M}_{N,m}(I)$ and $\mathcal{M}_{V\!\Lambda,m}(I)$.*

Proof. Let $m \in \mathbb{N}$ and $f \in \mathcal{M}_{M,m}(I)$. Then

$$f(c_i) = \begin{cases} b & \text{if } i \in \{1, 3, \dots, m\}, \\ a & \text{if } i \in \{2, 4, \dots, m-1\}, \end{cases}$$

and

$$\epsilon(I_j) = \begin{cases} +1 & \text{for} \quad j \in \{1, 3, 5, \dots, m\}, \\ -1 & \text{for} \quad j \in \{2, 4, 6, \dots, m+1\}. \end{cases} \tag{3.7}$$

Since $f(a) = a$ and $f(b) = a$, we have

$$f^k(c_i) = \begin{cases} a & \text{if } i \in \{1, 3, \dots, m\} \text{ and } k \geq 2, \\ a & \text{if } i \in \{2, 4, \dots, m-1\} \text{ and } k \geq 1. \end{cases} \tag{3.8}$$

Let $i \in \{2, 4, 6, \dots, m-1\}$. Note that $A_0(c_i+, f) = I_{i+1}$ and from (3.8), $A_k(c_i+, f) = I_1$ for $k \geq 1$. Therefore by (3.7), $\epsilon_k(c_i+, f) = 1$ for $k \geq 0$. Hence $\theta_0(c_i+, f) = A_0(c_i+, f) = I_{i+1}$, and

$$\begin{aligned} \theta_k(c_i+, f) &= \left(\prod_{l=0}^{k-1} \epsilon_l(c_i+, f) \right) A_k(c_i+, f) \\ &= (1 \cdot 1 \cdots (k \text{ times}) \cdots 1) \cdot I_1 = I_1 \end{aligned}$$

for $k \geq 1$. This implies that

$$\theta(c_i+, f; t) = \sum_{k \geq 0} \theta_k(c_i+, f) t^k = I_{i+1} + I_1 t + I_1 t^2 + I_1 t^3 + \cdots$$

$$= (t + t^2 + t^3 + \cdots)I_1 + I_{i+1}.$$

Also, $A_0(c_i-, f) = I_i$ and since $A_k(c_i-, f) = A_k(c_i+, f)$, we have $A_k(c_i-, f) = I_1$ for $k \geq 1$. Hence by (3.7), $\epsilon_0(c_i-, f) = -1$ and $\epsilon_k(c_i-, f) = 1$ for $k \geq 1$. Therefore $\theta_0(c_i-, f) = I_i$ and

$$\theta_k(c_i-, f) = (-1) \cdot (1 \cdot 1 \cdots (k-1) \text{ times} \cdots 1) \cdot I_1 = -I_1 \text{ for } k \geq 1.$$

This implies that

$$\theta(c_i-, f; t) = I_i - I_1 t - I_1 t^2 - I_1 t^3 - \cdots = (-t - t^2 - t^3 - \cdots)I_1 + I_i,$$

and therefore

$$\begin{aligned} \nu(c_i, f; t) &= \theta(c_i+, f; t) - \theta(c_i-, f; t) \\ &= (I_{i+1} + I_1 t + I_1 t^2 + \cdots) - (I_i - I_1 t - I_1 t^2 - \cdots) \\ &= (I_{i+1} - I_i) + 2I_1 t + 2I_1 t^2 + \cdots \\ &= (2t + 2t^2 + \cdots)I_1 - I_i + I_{i+1}. \end{aligned}$$

By a similar argument as above, we obtain

$$\nu(c_i, f; t) = (2t^2 + 2t^3 + \cdots)I_1 - I_i + I_{i+1} - 2tI_{m+1}$$

for each $i \in \{1, 3, 5, \ldots, m\}$. Hence the kneading matrix of f is given by

$$N(f;t) = \begin{bmatrix} -1 + 2t^2 + 2t^3 + \cdots & & -2t \\ 2t + 2t^2 + \cdots & & 0 \\ 2t^2 + 2t^3 + \cdots & & -2t \\ 2t + 2t^2 + \cdots & & 0 \\ \vdots & M_m & \vdots \\ 2t + 2t^2 + \cdots & & 0 \\ 2t^2 + 2t^3 + \cdots & & 1 - 2t \end{bmatrix}_{m \times (m+1)}, \tag{3.9}$$

where

$$M_m = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}_{m \times (m-1)}.$$

Since $f \in \mathcal{M}_{M,m}(I)$ was arbitrary, (3.9) is true for every $f \in \mathcal{M}_{M,m}(I)$. Therefore $N(f;t)$ is independent of choice of f in $\mathcal{M}_{M,m}(I)$. A proof for the cases where $\mathcal{M}_{M,m}(I)$ is replaced by $\mathcal{M}_{W,m}(I)$, $\mathcal{M}_{N,m}(I)$ and $\mathcal{M}_{\text{И},m}(I)$ is exactly similar. In fact, it follows that, if $f \in \mathcal{M}_{N,m}(I)$, then

$$N(f;t) = \begin{bmatrix} -1 & & -2t-2t^2-2t^3-\cdots \\ 2t+2t^2+\cdots & & 0 \\ 0 & & -2t-2t^2-2t^3-\cdots \\ 2t+2t^2+\cdots & & 0 \\ \vdots & M_m & \vdots \\ 0 & & -2t-2t^2-2t^3-\cdots \\ 2t+2t^2+\cdots & & 1 \end{bmatrix}_{m\times(m+1)}, \tag{3.10}$$

if $f \in \mathcal{M}_{W,m}(I)$, then

$$N(f;t) = \begin{bmatrix} -1+2t & & -2t^2-2t^3-\cdots \\ 0 & & -2t-2t^2-\cdots \\ 2t & & -2t^2-2t^3-\cdots \\ 0 & & -2t-2t^2-\cdots \\ \vdots & M_m & \vdots \\ 0 & & -2t-2t^2-\cdots \\ 2t & & 1-2t^2-2t^3-\cdots \end{bmatrix}_{m\times(m+1)}, \tag{3.11}$$

and if $f \in \mathcal{M}_{\text{И},m}(I)$, then

$$N(f;t) = \begin{bmatrix} -1+2t+2t^3+\cdots & & -2t^2-2t^4-\cdots \\ 2t^2+2t^4+\cdots & & -2t-2t^3-\cdots \\ 2t+2t^3+\cdots & & -2t^2-2t^4-\cdots \\ 2t^2+2t^4+\cdots & & -2t-2t^3-\cdots \\ \vdots & M_m & \vdots \\ 2t+2t^3+\cdots & & -2t^2-2t^4-\cdots \\ 2t^2+2t^4+\cdots & & 1-2t-2t^3-\cdots \end{bmatrix}_{m\times(m+1)}. \tag{3.12}$$

□

For each $m \in \mathbb{N}$, let $N_{M,m}(t) := N(f;t)$ for some $f \in \mathcal{M}_{M,m}(I)$. The matrices $N_{W,m}(t)$, $N_{N,m}(t)$ and $N_{\text{И},m}(t)$ are defined similarly. For $k \geq 1$, let S_k be as defined in the introduction. Although any two elements f and g of $\mathcal{M}_0(I)$ do not commute in general, the kneading matrices $N(f\circ g)$ and $N(g\circ f)$ are related as specified in the following theorem.

Theorem 3.4. *If $f, g \in \mathcal{M}_0(I)$, then either $N(f\circ g;t) = N(g\circ f;t)$ or $N(f\circ g;t) = -S_m N(g\circ f;t) S_{m+1}$ for some $m \in \mathbb{N}$.*

Proof. Consider any $f, g \in \mathcal{M}_0(I)$. Without loss of generality, we assume that either $|T(f)| \neq \emptyset$ or $|T(g)| \neq \emptyset$. Let $|T(f)| = m_1$ and $|T(g)| = m_2$ such that $m_1, m_2 \geq 0$, but not both zero. Since $S(m_1, m_2) = S(m_2, m_1)$, we have $|T(f\circ g)| = |T(g\circ f)|$. Let this common number be m.

Now, suppose that both m_1 and m_2 are odd. Then it suffices to consider the following cases.

TABLE 1. Comparison of $N(f\circ g;t)$ and $N(g\circ f;t)$

Parity	$f\in$	$g\in$	$f\circ g\in$	$g\circ f\in$	$N(f\circ g;t)$	$N(g\circ f;t)$	Conclusion
	$\mathcal{M}_{M,m_1}(I)$	$\mathcal{M}_{N,m_2}(I)$	$\mathcal{M}_{M,m}(I)$	$\mathcal{M}_{M,m}(I)$	$N_{M,m}(t)$	$N_{M,m}(t)$	$(*)$
m_1 odd	$\mathcal{M}_{M,m_1}(I)$	$\mathcal{M}_{\text{И},m_2}(I)$	$\mathcal{M}_{M,m}(I)$	$\mathcal{M}_{W,m}(I)$	$N_{M,m}(t)$	$N_{W,m}(t)$	$(**)$
m_2 even	$\mathcal{M}_{W,m_1}(I)$	$\mathcal{M}_{N,m_2}(I)$	$\mathcal{M}_{W,m}(I)$	$\mathcal{M}_{W,m}(I)$	$N_{W,m}(t)$	$N_{W,m}(t)$	$(*)$
	$\mathcal{M}_{W,m_1}(I)$	$\mathcal{M}_{\text{И},m_2}(I)$	$\mathcal{M}_{W,m}(I)$	$\mathcal{M}_{M,m}(I)$	$N_{W,m}(t)$	$N_{M,m}(t)$	$(**)$
	$\mathcal{M}_{N,m_1}(I)$	$\mathcal{M}_{N,m_2}(I)$	$\mathcal{M}_{N,m}(I)$	$\mathcal{M}_{N,m}(I)$	$N_{N,m}(t)$	$N_{N,m}(t)$	$(*)$
m_1 even	$\mathcal{M}_{N,m_1}(I)$	$\mathcal{M}_{\text{И},m_2}(I)$	$\mathcal{M}_{\text{И},m}(I)$	$\mathcal{M}_{\text{И},m}(I)$	$N_{\text{И},m}(t)$	$N_{\text{И},m}(t)$	$(*)$
m_2 even	$\mathcal{M}_{\text{И},m_1}(I)$	$\mathcal{M}_{\text{И},m_2}(I)$	$\mathcal{M}_{N,m}(I)$	$\mathcal{M}_{N,m}(I)$	$N_{N,m}(t)$	$N_{N,m}(t)$	$(*)$

Case (a): If $f\in\mathcal{M}_{M,m_1}(I)$ and $g\in\mathcal{M}_{M,m_2}(I)$, then $f\circ g,\ g\circ f\in\mathcal{M}_{M,m}(I)$, and hence by Lemma 3.3, $N(f\circ g;t)=N_{M,m}(t)=N(g\circ f;t)$.
Case (b): If $f\in\mathcal{M}_{M,m_1}(I)$ and $g\in\mathcal{M}_{W,m_2}(I)$, then $f\circ g\in\mathcal{M}_{M,m}(I)$ and $g\circ f\in\mathcal{M}_{W,m}(I)$. So, by Lemma 3.3, $N(f\circ g;t)=N_{M,m}(t)$ and $N(g\circ f;t)=N_{W,m}(t)$. This implies

$$N(f\circ g;t)=N_{M,m}(t)=-S_mN_{W,m}(t)S_{m+1}=-S_mN(g\circ f;t)S_{m+1}.$$

Case (c): If $f\in\mathcal{M}_{W,m_1}(I)$ and $g\in\mathcal{M}_{W,m_2}(I)$, then $f\circ g,\ g\circ f\in\mathcal{M}_{W,m}(I)$. So, again by Lemma 3.3, $N(f\circ g;t)=N_{W,m}(t)=N(g\circ f;t)$.

Remaining instances for the parity of m_1, m_2 and the corresponding cases can be discussed similarly. A summary of premises and the corresponding conclusions is given in Table 1, where $(*)$ and $(**)$ denote the equations $N(f\circ g;t)=N(g\circ f;t)$ and $N(f\circ g;t)=-S_mN(g\circ f;t)S_{m+1}$, respectively. □

Lemma 3.5. *Let $f,g\in\mathcal{M}(I)$ such that $N(g;t)=-S_mN(f;t)S_{m+1}$ for some $m\in\mathbb{N}$. Then $D(g;t)=D(f;t)$.*

Proof. By hypothesis, there exists $m\in\mathbb{N}$ such that $N(g;t)=-S_mN(f;t)S_{m+1}$. So, we have $|T(f)|=|T(g)|=m$. Let

$$T(f)=\{c_1,c_2,\dots,c_m\},\ T(g)=\{d_1,d_2,\dots,d_m\},$$
$$L(f)=\{I_1,I_2,\dots,I_{m+1}\}\text{ and }L(g)=\{J_1,J_2,\dots,J_{m+1}\},$$

where $a=c_0<c_1<\cdots<c_m<c_{m+1}=b$, $I_j=[c_{j-1},c_j]$ for $1\le j\le m+1$, $a=d_0<d_1<d_2<\cdots<d_m<d_{m+1}=b$ and $J_i=[d_{i-1},d_i]$ for $1\le i\le m+1$. Without loss of generality, assume that f is strictly increasing on I_1. We have

$$\begin{aligned}D(f;t)&=(-1)^{1+1}\big(1-\epsilon(I_1)t\big)^{-1}\det\big(N^{(1)}(f;t)\big)\\&=(1-t)^{-1}\det\big(N^{(1)}(f;t)\big),\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}D(g;t)&=(-1)^{(m+1)+1}\big(1-\epsilon(J_{m+1})t\big)^{-1}\det\big(N^{(m+1)}(g;t)\big)\\&=(-1)^{m+2}\big(1-\epsilon(J_{m+1})t\big)^{-1}\det\big(N^{(m+1)}(g;t)\big).\end{aligned}\tag{3.14}$$

Since $N(g;t)=-S_mN(f;t)S_{m+1}$, we get that $N^{(m+1)}(g;t)=-S_mN^{(1)}(f;t)S_m$, and therefore

$$\begin{aligned}\det\big(N^{(m+1)}(g;t)\big) &= (-1)^m(\det S_m)^2\det\big(N^{(1)}(f;t)\big)\\ &= (-1)^m\det\big(N^{(1)}(f;t)\big),\end{aligned}$$

where the last equality is true, because $\det S_m=(-1)^{\lfloor\frac{m}{2}\rfloor}$. Hence from (3.13) and (3.14), we obtain

$$D(g;t)=\big(1-\epsilon(J_{m+1})t\big)^{-1}(1-t)D(f;t). \tag{3.15}$$

Moreover, $\epsilon(J_{m+1})=\epsilon(I_1)$, and so $\epsilon(J_{m+1})=1$, because $\epsilon(I_1)=1$. Therefore (3.15) implies that $D(g;t)=(1-t)^{-1}(1-t)D(f;t)=D(f;t)$. □

Corollary 3.6. *$D(f\circ g;t)=D(g\circ f;t)$ for every $f,g\in\mathcal{M}_0(I)$.*

Proof. Since $f,g\in\mathcal{M}_0(I)$, by Theorems 3.4, it follows that either $N(f\circ g;t)=N(g\circ f;t)$ or $N(f\circ g;t)=-S_mN(g\circ f;t)S_{m+1}$ for some $m\in\mathbb{N}$. In the first case, the equality $D(f\circ g;t)=D(g\circ f;t)$ follows from the definition of kneading determinant, while in the second, this equality follows from Lemma 3.5. □

3.1. Relation between $N(f^k;t)$ and $N(f;t)$

Although we aim to describe a relation between $N(f^k;t)$ and $N(f;t)$, in view of the relation $N(f;t)=N_0(f;t)+M(f;t)$, where $N_0(f;t)$ is independent of choice of f, it suffices to describe a relation between $M(f^k;t)$ and $M(f;t)$. So in what follows, we prove results for $M(f;t)$ instead of $N(f;t)$.

For $k,l\geq 1$, let e_k denote the matrix $[0,0,\dots,0,1]_{1\times k}$ and $\mathbb{I}_k$, $\mathbb{O}_{k\times l}$, $R_{k\times l}$ be as defined in the introduction. As defined in [10], $f\in\mathcal{M}(I)$ is said to be *uniformly piecewise linear* if it is linear on each of its laps with slope $\pm\alpha$ for some positive real α. For $k\geq 1$, let $f_{N,k}$, $f_{M,k}$, $f_{W,k}$ and $f_{V\!\!\Lambda,k}$ be the uniformly piecewise linear maps in $\mathcal{M}_{N,k}(I)$, $\mathcal{M}_{M,k}(I)$, $\mathcal{M}_{W,k}(I)$ and $\mathcal{M}_{V\!\!\Lambda,k}(I)$, respectively. The following theorem describe the relation between kneading matrices of elements of $\mathcal{M}_0(I)$ with that of bimodal/trimodal uniformly piecewise linear maps, whose dynamical properties are relatively easy to investigate.

Theorem 3.7. (1) *If $f\in\mathcal{M}_{N,m}(I)$, then*

$$M(f;t)=\mathcal{I}_mM(f_{N,2};t)R_{3\times(m+1)}. \tag{3.16}$$

(2) *If $f\in\mathcal{M}_{M,m}(I)$, then*

$$M(f;t)=\begin{bmatrix}\mathcal{I}_{m-1} & \mathbb{O}_{(m-1)\times 1}\\ \mathbb{O}_{1\times 2} & 1\end{bmatrix}M(f_{M,3};t)R_{4\times(m+1)}.$$

This is also true when $\mathcal{M}_{M,m}(I)$ is replaced by $\mathcal{M}_{W,m}(I)$.

(3) *If $f\in\mathcal{M}_{V\!\!\Lambda,m}(I)$, then*

$$M(f;t)=\mathcal{I}_mM(f_{V\!\!\Lambda,2};t)R_{3\times(m+1)}. \tag{3.17}$$

Proof. Let $f \in \mathcal{M}_{N,m}(I)$. Since $f_{N,2} \in \mathcal{M}_{N,2}(I)$, from (3.10) we have

$$M(f_{N,2};t) = \begin{bmatrix} 0 & 0 & -2t-2t^2-2t^3-\cdots \\ 2t+2t^2+\cdots & 0 & 0 \end{bmatrix}_{2\times 3}.$$

Put

$$A = \begin{bmatrix} 0 \\ 2t+2t^2+\cdots \end{bmatrix}_{2\times 1} \text{ and } B = \begin{bmatrix} -2t-2t^2-2t^3-\cdots \\ 0 \end{bmatrix}_{2\times 1}.$$

Then $M(f_{N,2};t) = \begin{bmatrix} A & \mathbb{O}_{2\times 1} & B \end{bmatrix}_{2\times 3}$, and by (3.10), we have

$$\begin{aligned} M(f;t) &= \begin{bmatrix} A & & B \\ A & & B \\ \vdots & \mathbb{O}_{m\times(m-1)} & \vdots \\ A & & B \end{bmatrix}_{m\times(m+1)} \\ &= \begin{bmatrix} \mathbb{I}_2 \\ \mathbb{I}_2 \\ \vdots \\ \mathbb{I}_2 \end{bmatrix}_{m\times 2} \begin{bmatrix} A & \mathbb{O}_{2\times 1} & B \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{3\times(m+1)} \\ &= \mathcal{I}_m M(f_{N,2};t) R_{3\times(m+1)}. \end{aligned}$$

This proves result (1). Now let $f \in \mathcal{M}_{M,m}(I)$. Since $f_{M,3} \in \mathcal{M}_{M,3}(I)$, from (3.9) we have

$$M(f_{M,3};t) = \begin{bmatrix} 2t^2+2t^3+\cdots & 0 & 0 & -2t \\ 2t+2t^2+\cdots & 0 & 0 & 0 \\ 2t^2+2t^3+\cdots & 0 & 0 & -2t \end{bmatrix}_{3\times 4}.$$

Put

$$A = \begin{bmatrix} 2t^2+2t^3+\cdots \\ 2t+2t^2+\cdots \end{bmatrix}_{2\times 1} \text{ and } B = \begin{bmatrix} -2t \\ 0 \end{bmatrix}_{2\times 1}.$$

Then

$$M(f_{M,3};t) = \begin{bmatrix} A & \mathbb{O}_{2\times 1} & \mathbb{O}_{2\times 1} & B \\ 2t^2+2t^3+\cdots & 0 & 0 & -2t \end{bmatrix}_{3\times 4},$$

and by (3.9), we have

$$
\begin{aligned}
M(f;t) &= \begin{bmatrix} A & & B \\ A & & B \\ \vdots & \mathbb{O}_{m\times(m-1)} & \vdots \\ A & & B \\ 2t^2+2t^3+\cdots & & -2t \end{bmatrix}_{m\times(m+1)} \\
&= \begin{bmatrix} \mathbb{I}_2 & \\ \mathbb{I}_2 & \\ \vdots & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{I}_2 & \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix}_{m\times 3} \begin{bmatrix} A & \mathbb{O}_{2\times 1} & \mathbb{O}_{2\times 1} & B \\ 2t^2+2t^3+\cdots & 0 & 0 & -2t \end{bmatrix} R_{4\times(m+1)} \\
&= \begin{bmatrix} \mathcal{I}_{m-1} & \mathbb{O}_{(m-1)\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} M(f_{M,3};t) R_{4\times(m+1)}.
\end{aligned}
$$

The proofs of result (2) for $\mathcal{M}_{W,m}(I)$ and that of result (3) are similar. □

Theorem 3.8. (1) *If $f \in \mathcal{M}_{N,m}(I)$, then*

$$M(f^k;t) = [\mathcal{I}_l \quad \mathbb{O}_{l\times(m-2)}] M(f;t) R_{(m+1)\times(l+1)}, \ \forall k \geq 1,$$

where $l = (m+1)^k - 1$. This is also true when $\mathcal{M}_{N,m}(I)$ is replaced by $\mathcal{M}_{V\!\!\!\Lambda,m}(I)$ and k is a positive odd integer.

(2) *If $f \in \mathcal{M}_{M,m}(I)$, then*

$$M(f^k;t) = \begin{bmatrix} \mathcal{I}_{l-1} & \mathbb{O}_{(l-1)\times(m-2)} \\ \mathbb{O}_{1\times 2} & e_{m-2} \end{bmatrix} M(f;t) R_{(m+1)\times(l+1)}, \ \forall k \geq 1,$$

where $l = (m+1)^k - 2$. This is also true when $\mathcal{M}_{M,m}(I)$ is replaced by $\mathcal{M}_{W,m}(I)$.

Proof. Let $f \in \mathcal{M}_{N,m}(I)$ and $k \in \mathbb{N}$. Then by result (1) of Proposition 3.2, $f^k \in \mathcal{M}_{N,m}(I)$ and from result (2) of Proposition 3.1, $|T(f^k)| = (m+1)^k - 1$. Thus $f \in \mathcal{M}_{N,(m+1)^k-1}(I)$, and therefore by Lemma 3.3,

$$M(f;t) = \begin{bmatrix} A & & B \\ A & & B \\ \vdots & \mathbb{O}_{l\times(l-1)} & \vdots \\ A & & B \end{bmatrix}_{l\times(l+1)},$$

where $l = (m+1)^k - 1$,

$$A = \begin{bmatrix} 0 \\ 2t+2t^2+\cdots \end{bmatrix}_{2\times 1} \text{ and } B = \begin{bmatrix} -2t-2t^2-2t^3-\cdots \\ 0 \end{bmatrix}_{2\times 1}.$$

This implies that

$$\begin{aligned} M(f;t) &= \begin{bmatrix} \mathbb{I}_2 & \\ \mathbb{I}_2 & \\ \vdots & \mathbb{O}_{l\times(m-2)} \\ \mathbb{I}_2 & \end{bmatrix}_{l\times m} \begin{bmatrix} A & & B \\ A & & B \\ \vdots & \mathbb{O}_{m\times(m-1)} & \vdots \\ A & & B \end{bmatrix}_{m\times(m+1)} R_{(m+1)\times(l+1)} \\ &= [\mathcal{I}_l \quad \mathbb{O}_{l\times(m-2)}] M(f;t) R_{(m+1)\times(l+1)}, \end{aligned}$$

proving first part of result (1). The proofs of second part of result (1) and result (2) are similar. ☐

3.2. Relation between $D(f^k;t)$ and $D(f;t)$

Lemma 3.9. *Let $m \in \mathbb{N}$. Then*

$$\det(N^{(1)}_{M,m}(t)) = \det(N^{(1)}_{W,m}(t)) = 1-(m+1)t \tag{3.18}$$

and

$$\det(N^{(m+1)}_{N,m}(t)) = \det(N^{(m+1)}_{\mathsf{V\!\!\backslash},m}(t)) = \frac{1-(m+1)t}{1-t}.$$

Proof. Follows by mathematical induction, using (3.9), (3.11), (3.10) and (3.12). ☐

Theorem 3.10. *If $f \in \mathcal{M}_{M,m}(I) \cup \mathcal{M}_{W,m}(I) \cup \mathcal{M}_{N,m}(I)$, then*

$$D(f^k;t) = \frac{1-(m+1)^k t}{1-(m+1)t} D(f;t) \text{ for } k \in \mathbb{N}. \tag{3.19}$$

This is also true when $\mathcal{M}_{N,m}(I)$ is replaced by $\mathcal{M}_{\mathsf{V\!\!\backslash},m}(I)$ and k is any positive odd integer.

Proof. First, consider the case that $f \in \mathcal{M}_{M,m}(I)$, where $m \in \mathbb{N}$, and let $k \in \mathbb{N}$ be fixed. By definition,

$$D(f;t) = (1-\epsilon(I_1))^{-1} \det(N^{(1)}(f;t)). \tag{3.20}$$

Since $f \in \mathcal{M}_{M,m}(I)$, we have $\epsilon(I_1) = 1$ and $N(f;t) = N_{\mathcal{M},m}(t)$. This implies that $N^{(1)}(f;t) = N^{(1)}_{M,m}(t)$ and therefore by (3.18), $\det(N^{(1)}(f;t)) = 1-(m+1)t$. Hence by (3.20),

$$D(f;t) = (1-t)^{-1}(1-(m+1)t). \tag{3.21}$$

Since $f \in \mathcal{M}_{M,m}(I)$, by Propositions 3.2 and 3.1, we have $f^k \in \mathcal{M}_{M,(m+1)^k-1}(I)$. Therefore $N(f^k;t) = N_{M,(m+1)^k-1}(t)$ and $\epsilon(I'_1) = 1$, where I'_1 is the first lap of f^k. This implies by (3.18) that, $\det(N^{(1)}(f^k;t)) = 1-(m+1)^k t$ and therefore

$$D(f^k;t) = (1-\epsilon(I'_1)t)^{-1} \det(N^{(1)}(f^k;t)) = (1-t)^{-1}(1-(m+1)^k t). \tag{3.22}$$

So, (3.19) follows from (3.21) and (3.22). The proofs for other cases are similar. ☐

4. Modified kneading matrix

As observed in Section 2, the kneading matrix of an $f \in \mathcal{M}(I)$ is defined using only the kneading increments corresponding to the turning points of f. In what follows, we use the 'kneading data' associated with endpoints a and b of I, with suitable one-sided limits, to define a new kneading matrix for f.

Let $\nu(c_0, f; t) := \theta(c_0+, f; t)$ and $\nu(c_{m+1}, f; t) := -\theta(c_{m+1}-, f; t)$. Then the *modified kneading matrix* of f, denoted by $N'(f; t)$, is defined by

$$N'(f;t) = \begin{bmatrix} N'_{01}(f;t) & N'_{02}(f;t) & \cdots & N'_{0,m+1}(f;t) \\ & & N(f;t) & \\ N'_{m+1,1}(f;t) & N'_{m+1,2}(f;t) & \cdots & N'_{m+1,m+1}(f;t) \end{bmatrix}_{(m+2)\times(m+1)},$$

where the entries $N'_{ij}(f;t)$, $i = 0, m+1$, $j = 1, 2, \dots, m+1$ are obtained by setting

$$\nu(c_0, f; t) = N'_{01}(f;t)I_1 + N'_{02}(f;t)I_2 + \cdots + N'_{0,m+1}(f;t)I_{m+1}$$

and

$$\nu(c_{m+1}, f; t) = N'_{m+1,1}(f;t)I_1 + N'_{m+1,2}(f;t)I_2 + \cdots + N'_{m+1,m+1}(f;t)I_{m+1}.$$

For $1 \le i \le m+2$, let $N'_{(i)}(f;t)$ denote the $(m+1)\times(m+1)$ matrix obtained by deleting the i^{th} row of $N'(f;t)$.

Theorem 4.1. (1) *If $f \in \mathcal{M}_{M,m}(I) \cup \mathcal{M}_{W,m}(I)$, then*

$$D(f;t) = \det N'_{(i)}(f;t), \; i = 1, m+2. \tag{4.1}$$

(2) *If $f \in \mathcal{M}_{N,m}(I) \cup \mathcal{M}_{\text{VI},m}(I)$, then*

$$D(f;t) = (-1)^i \det N'_{(i)}(f;t), \; i = 1, m+2. \tag{4.2}$$

Proof. Let $f \in \mathcal{M}_{M,m}(I)$, where $m \in \mathbb{N}$. Since $f(c_0) = f(c_{m+1}) = a$, we have $f^k(c_0) = f^k(c_{m+1}) = a$ for each $k \in \mathbb{N}$. Also, $A_k(c_0+, f) = I_1$, and therefore $\epsilon_k(c_i+, f) = 1$ for $k \ge 0$. Hence $\theta_k(c_0+, f) = I_1$ for $k \ge 0$. This implies that

$$\begin{aligned}\nu(c_0, f; t) = \theta(c_0+, f; t) &= I_1 + I_1 t + I_1 t^2 + I_1 t^3 + \cdots \\ &= (1 + t + t^2 + t^3 + \cdots)I_1.\end{aligned}$$

Also, $A_0(c_{m+1}-, f) = I_{m+1}$ and $A_k(c_{m+1}-, f) = I_1$ for $k \ge 1$. Therefore $\epsilon_0(c_{m+1}-, f) = -1$ and $\epsilon_k(c_{m+1}-, f) = 1$ for $k \ge 1$. Hence $\theta_0(c_{m+1}-, f) = I_{m+1}$ and $\theta_k(c_{m+1}-, f) = -I_1$ for $k \ge 1$. This implies that

$$\begin{aligned}\nu(c_{m+1}, f; t) = -\theta(c_{m+1}-, f; t) &= -I_{m+1} + I_1 t + I_1 t^2 + I_1 t^3 + \cdots \\ &= (t + t^2 + t^3 + \cdots)I_1 - I_{m+1}.\end{aligned}$$

Moreover, since $f \in \mathcal{M}_{M,m}(I)$, we have $N(f;t) = N_{M,m}(t)$. Thus

$$N'(f;t) = \begin{bmatrix} 1 + t + t^2 + \cdots & 0 & 0 & \cdots & 0 & 0 \\ & & & N_{M,m}(f;t) & & \\ t + t^2 + t^3 + \cdots & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}_{(m+2)\times(m+1)},$$

and hence

$$\begin{aligned}\det N'_{(m+2)}(f;t) &= (1+t+t^2+\cdots)\det N^{(1)}(f;t)\\ &= (1-t)^{-1}\det N^{(1)}(f;t)\\ &= (-1)^{1+1}(1-\epsilon(I_1)t)^{-1}\det N^{(1)}(f;t)\\ &= D(f;t).\end{aligned}$$

Also, sine m is odd, we have

$$\det N^{(1)}(f;t) = -(1-t)(1+t)^{-1}\det N^{(m+1)}(f;t).$$

Therefore

$$\begin{aligned}\det N'_{(1)}(f;t) &= (-1)^{(m+1)+1}(t+t^2+\cdots)\det N^{(1)}(f;t)\\ &\quad + (-1)^{(m+1)+(m+1)}(-1)\det N^{(m+1)}(f;t)\\ &= (t+t^2+\cdots)(1-t)(1+t)^{-1}\det N^{(m+1)}(f;t)\\ &\quad - \det N^{(m+1)}(f;t)\\ &= -(1+t)^{-1}\det N^{(m+1)}(f;t)\\ &= (-1)^{(m+1)+1}(1-\epsilon(I_{m+1})t)^{-1}\det N^{(m+1)}(f;t)\\ &= D(f;t).\end{aligned}$$

This proves (4.1) for $f \in \mathcal{M}_{M,m}(I)$. The proofs of (4.1) for $f \in \mathcal{M}_{W,m}(I)$ and that of result (2) are similar. □

References

[1] J. F. Alves and J. Sousa Ramos, *Kneading theory for tree maps*, Ergodic Theory Dynam. Systems **24** (2004), no. 4, 957–985. https://doi.org/10.1017/S014338570400015X

[2] V. Baladi, *Infinite kneading matrices and weighted zeta functions of interval maps*, J. Funct. Anal. **128** (1995), no. 1, 226–244. https://doi.org/10.1006/jfan.1995.1029

[3] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, reprint of the second (1989) edition, Studies in Nonlinearity, Westview Press, Boulder, CO, 2003.

[4] R. A. Holmgren, *A First Course in Discrete Dynamical Systems*, Universitext, Springer-Verlag, New York, 1994. https://doi.org/10.1007/978-1-4684-0222-3

[5] D. A. Mendes and J. S. Ramos, *Kneading theory for triangular maps*, Int. J. Pure Appl. Math. **10** (2004), no. 4, 421–450.

[6] J. Milnor and W. Thurston, *On iterated maps of the interval. I. the kneading matrix, and II. periodic points*, Preprint, Princeton University, 1977.

[7] ———, *On iterated maps of the interval*, in Dynamical systems (College Park, MD, 1986–87), 465–563, Lecture Notes in Math., **1342**, Springer, Berlin, 1988. https://doi.org/10.1007/BFb0082847

[8] E. Piña, *Kneading theory of the circle map*, Phys. Rev. A (3) **34** (1986), no. 1, 574–581. https://doi.org/10.1103/PhysRevA.34.574

[9] C. Preston, *Iterates of maps on an interval*, Lecture Notes in Mathematics, **999**, Springer-Verlag, Berlin, 1983. https://doi.org/10.1007/BFb0061749

[10] ———, *Iterates of piecewise monotone mappings on an interval*, Lecture Notes in Mathematics, **1347**, Springer-Verlag, Berlin, 1988. https://doi.org/10.1007/BFb0079769

[11] ———, *What you need to know to knead*, Adv. Math. **78** (1989), no. 2, 192–252. https://doi.org/10.1016/0001-8708(89)90033-9

Chaitanya Gopalakrishna
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka Suarthkal
Mangalore- 575 025, India
Email address: cberbalaje@gmail.com

Murugan Veerapazham
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka Suarthkal
Mangalore- 575 025, India
Email address: murugan@nitk.edu.in