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EQUIVALENCE CONSTANTS FOR THE l_p-NORMS AND THE l_q-SYMMETRIC MULTILINEAR OPERATOR NORMS OF VECTORS IN $\mathbb{C}^n$

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Abstract. We investigate the best equivalence constants for the l_p-norms and the l_q-symmetric multilinear operator norms of vectors in $\mathbb{C}^n$ which are induced by symmetric n-linear forms. In this paper, we provides estimates which are either best possible or close to best possible.

1. Introduction

In the following all scalars are complex. Let $m, n \geq 2$ and $1 \leq p, q \leq \infty$. We write l_p^n for the Banach space of all n-tuples $(z_1, \ldots, z_n)$ equipped with the l_p-norm. For $m \times n$ matrix $A = (a_{ij})$, we denote the l_p-coefficient norms by

$$|A|_p := (\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |a_{ij}|^p)^{\frac{1}{p}}, \quad (1 \leq p < \infty)$$

and

$$|A|_\infty := \max_{1 \leq i \leq m,\ 1 \leq j \leq n} |a_{ij}|, \quad (p = \infty).$$

We denote the l_p-operator norms by

$$\|A\|_p := \sup\{\|A(z_1, \ldots, z_n)^t\|_p : \|(z_1, \ldots, z_n)\|_p \leq 1\}.$$

The problem of determining the equivalence constants for the l_p-coefficient norms and the l_p-operator norms was raised by Goldberg [4] who showed that for $1 \leq p, q \leq \infty$,

$$\|A\|_p \leq m^{\max(\frac{1}{p} - \frac{1}{q}, 0)} n^{\max(\frac{1}{q'} - \frac{1}{p}, 0)} |A|_q,$$

where q' denotes the index conjugate to q. He also proved that these inequalities are sharp.

Goldberg left an open question: Determine the best constant $c = c(m, n, p, q)$ such that $c|A|_q \leq \|A\|_p$ for all $m \times n$ matrices A.

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Feng [2, 3] and Tonge [5] investigated the equivalence constants for the l_p-coefficient norms and l_q-operator norms of complex $m \times n$ matrices, and provided estimates which are either best possible or close to best possible.

We denote $\mathcal{L}_s({}^n l_p^2)$ by the Banach space of symmetric n-linear forms on l_p^2 equipped with the norm

$$\|T\|_{\mathcal{L}_s({}^n l_p^2)} = \sup_{\|(x_i,y_i)\|_p=1,\ 1\le i\le n} |T((x_1,y_1),\ldots,(x_n,y_n))|. \quad \text{(see [1])}$$

For $j = 0, \ldots, n$, we let

$$F_j((x_1,y_1),\ldots,(x_n,y_n)) := \sum_{\{l_1,\ldots,l_j,k_1,\ldots,k_{n-j}\}=\{1,\ldots,n\}} x_{l_1}\cdots x_{l_j} y_{k_1}\cdots y_{k_{n-j}},$$

where $(x_i, y_i) \in l_p^2$ for $1 \le i \le n$. Then, $\{F_0, \ldots, F_n\}$ is a basis for $\mathcal{L}_s({}^n l_p^2)$. Hence, $\dim(\mathcal{L}_s({}^n l_p^2)) = n + 1$. If $T \in \mathcal{L}_s({}^n l_p^2)$, then

$$T = \sum_{j=0}^{n} a_j F_j$$

for some $a_0, \ldots, a_n \in \mathbb{C}$. By simplicity, we denote $T = (a_0, \ldots, a_n)^t$.

We define the *l_p-symmetric multilinear operator norm* on $\mathbb{C}^{n+1}$ by

$$\|(z_0,\ldots,z_n)\|_{\mathcal{L}_s({}^n l_p^2)} := \|T\|_{\mathcal{L}_s({}^n l_p^2)},$$

where $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ and $T = (z_0, \ldots, z_n)^t \in \mathcal{L}_s({}^n l_p^2)$.

Since the norms $\|\cdot\|_q$ and $\|\cdot\|_{\mathcal{L}_s({}^n l_p^2)}$ are equivalent on $\mathbb{C}^{n+1}$, there are $A > 0$ and $B > 0$ such that for all $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$,

$$A\|(z_0,\ldots,z_n)\|_q \le \|(z_0,\ldots,z_n)\|_{\mathcal{L}_s({}^n l_p^2)} \le B\|(z_0,\ldots,z_n)\|_q.$$

Let

$$\begin{aligned} A_{(p,q:n)} := \sup\{A > 0 : A\|(z_0,\ldots,z_n)\|_q &\le \|(z_0,\ldots,z_n)\|_{\mathcal{L}_s({}^n l_p^2)} \\ &\text{for all } (z_0,\ldots,z_n) \in \mathbb{C}^{n+1}\} \end{aligned}$$

and

$$\begin{aligned} B_{(p,q:n)} := \inf\{B > 0 : \|(z_0,\ldots,z_n)\|_{\mathcal{L}_s({}^n l_p^2)} &\le B\|(z_0,\ldots,z_n)\|_q \\ &\text{for all } (z_0,\ldots,z_n) \in \mathbb{C}^{n+1}\}. \end{aligned}$$

We say that $A_{(p,q:n)}$ and $B_{(p,q:n)}$ are *the best equivalence constants* for the norms $\|\cdot\|_q$ and $\|\cdot\|_{\mathcal{L}_s({}^n l_p^2)}$ on $\mathbb{C}^{n+1}$.

In this paper, we show that for $n \ge 2$,

(a) $B_{(p,\infty:n)} = 2^{n(1-\frac{1}{p})}$ for every $1 \le p \le \infty$;
(b) $(n+1)^{-\frac{1}{q}} \le A_{(p,q:n)} \le 1$ for $1 \le p < \infty, 1 \le q \le \infty$;

(c) $2^{n(1-\frac{1}{p})}(n+1)^{-\frac{1}{q}} \leq B_{(p,q:n)}$

$$\leq \Big(2^{(1-\frac{1}{p})} + 2^{p-1} \sum_{1\leq j\leq n-1} ([\frac{{}_nC_j + 1}{2}])^{\frac{q}{q-1}} \Big)^{1-\frac{1}{q}}$$

for $1 \leq p, q < \infty$.

2. Results

Theorem 2.1. *Let $1 \leq p \leq \infty$, $n \geq 2$. Then*

$$B_{(p,\infty:n)} = 2^{n(1-\frac{1}{p})}.$$

Proof. Let $T \in \mathcal{L}_s({}^n l_p^2)$ with $T = (a_0, \ldots, a_n)^t$ for some $a_0, \ldots, a_n \in \mathbb{C}$. Let $(x_i, y_i) \in l_p^2$ with $\|(x_i, y_i)\|_p = 1$ for $1 \leq i \leq n$. It follows that

$$\begin{aligned}
&|T((x_1,y_1),\ldots,(x_n,y_n))| \\
\leq &\sum_{0\leq j\leq n} |a_j|\ |F_j((x_1,y_1),\ldots,(x_n,y_n))| \\
\leq &\|(a_0,\ldots,a_n)\|_\infty \sum_{0\leq j\leq n} |F_j((x_1,y_1),\ldots,(x_n,y_n))| \\
\leq &\|(a_0,\ldots,a_n)\|_\infty \sum_{0\leq j\leq n} \Big(\sum_{\{l_1,\ldots,l_j,k_1,\ldots,k_{n-j}\}=\{1,\ldots,n\}} |x_{l_1}|\cdots|x_{l_j}|\ |y_{k_1}|\cdots|y_{k_{n-j}}| \Big) \\
= &\|(a_0,\ldots,a_n)\|_\infty\ (|x_1|+|y_1|)\times\cdots\times(|x_n|+|y_n|) \\
\leq &\|(a_0,\ldots,a_n)\|_\infty\ 2^{(1-\frac{1}{p})}(|x_1|^p+|y_1|^p)^{\frac{1}{p}} \times\cdots\times 2^{(1-\frac{1}{p})}(|x_n|^p+|y_n|^p)^{\frac{1}{p}} \\
&\text{(by Hölder's inequality)} \\
= &\|(a_0,\ldots,a_n)\|_\infty\ 2^{n(1-\frac{1}{p})},
\end{aligned}$$

which shows that

$$\|T\|_{\mathcal{L}_s({}^n l_p^2)} = \|(a_0,\ldots,a_n)\|_{\mathcal{L}_s({}^n l_p^2)} \leq 2^{n(1-\frac{1}{p})}\ \|(a_0,\ldots,a_n)\|_\infty$$

for all $a_j \in \mathbb{C}$. Hence, $B_{(p,\infty:n)} \leq 2^{n(1-\frac{1}{p})}$.

Claim: $(*)$ $\quad \|(1,\ldots,1)\|_{\mathcal{L}_s({}^n l_p^2)} \geq 2^{n(1-\frac{1}{p})}$.

It follows that

$$\begin{aligned}
\|(1,\ldots,1)\|_{\mathcal{L}_s({}^n l_p^2)} &= \sup_{\|(x_i,y_i)\|_p=1,\ 1\leq i\leq n} \Big| \sum_{0\leq j\leq n} F_j((x_1,y_1),\ldots,(x_n,y_n)) \Big| \\
&\geq \Big| \sum_{0\leq j\leq n} F_j((2^{-\frac{1}{p}},2^{-\frac{1}{p}}),\ldots,(2^{-\frac{1}{p}},2^{-\frac{1}{p}})) \Big| \\
&= \Big(\sum_{0\leq j\leq n} {}_nC_j \Big)\ 2^{-\frac{n}{p}} \\
&= 2^{n(1-\frac{1}{p})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}2^{n(1-\frac{1}{p})} &\le \|(1,\dots,1)\|_{\mathcal{L}_s({}^n l_p^2)}\\ &\le B_{(p,\infty:n)}\|(1,\dots,1)\|_\infty\\ &= B_{(p,\infty:n)}\\ &\le 2^{n(1-\frac{1}{p})}.\end{aligned}$$

□

Theorem 2.2. *Let $1 \le p < \infty$, $1 \le q \le \infty, n \ge 2$. Then*

$$(n+1)^{-\frac{1}{q}} \le A_{(p,q:n)} \le 1.$$

Proof. Note that

$$A_{(p,q:n)} = A_{(p,q:n)}\|(1,0,\dots,0)\|_q \le \|(1,0,\dots,0)\|_{\mathcal{L}_s({}^n l_p^2)} = 1.$$

Hence, $A_{(p,q:n)} \le 1$. Let $T \in \mathcal{L}_s({}^n l_p^2)$ with $T = (a_0,\dots,a_n)^t$ for some $a_0,\dots,a_n \in \mathbb{C}$.

It follows that

$$\begin{aligned}\|(a_0,\dots,a_n)\|_{\mathcal{L}_s({}^n l_p^2)} &= \sup_{\|(x_i,y_i)\|_p=1,\ 1\le i\le n} |T((x_1,y_1),\dots,(x_n,y_n))|\\ &\ge \max\{|T(e_2,\dots,e_2,e_1,\dots,e_1)| : 0 \le i \le n\\ &\quad \text{and } e_2 \text{ appears } i \text{ times}\}\\ &= \|(a_0,\dots,a_n)\|_\infty\\ &\ge (n+1)^{-\frac{1}{q}}\|(a_0,\dots,a_n)\|_q,\end{aligned}$$

which shows that $(n+1)^{-\frac{1}{q}} \le A_{(p,q:n)}$. □

Since $A_{(p,\infty:n)} = 1$, the estimates of Theorem 2.2 are best possible.

Lemma 2.3. *Let $1 \le p < \infty$, $1 \le q < \infty$, $n \ge 2$. Suppose that $(x_i,y_i) \in l_p^2$ with $\|(x_i,y_i)\|_p = 1$ for $1 \le i \le n$. Then, for $1 \le j \le n-1$,*

$$|F_j((x_1,y_1),\dots,(x_n,y_n))| \le 2^{1-\frac{1}{p}}\ [\frac{{}_nC_j+1}{2}],$$

where $[\frac{{}_nC_j+1}{2}]$ is the largest integer less than or equal to $\frac{{}_nC_j+1}{2}$.

Proof. It follows that

$$\begin{aligned}&|F_j((x_1,y_1),\dots,(x_n,y_n))|\\ \le& \sum_{\{l_1,\dots,l_j,k_1,\dots,k_{n-j}\}=\{1,\dots,n\}} |x_{l_1}|\cdots|x_{l_j}|\ |y_{k_1}|\cdots|y_{k_{n-j}}|\\ \le& \sum_{1\le s\le[\frac{{}_nC_j+1}{2}],\ 1\le k_s\le n} (|x_{k_s}|+|y_{k_s}|)\\ \le& \sum_{1\le s\le[\frac{{}_nC_j+1}{2}],\ 1\le k_s\le n} 2^{1-\frac{1}{p}}\ (|x_{k_s}|^p+|y_{k_s}|^p)^{\frac{1}{p}} \text{ (by Hölder's inequality)}\end{aligned}$$

$$= \sum_{1\le s\le [\frac{{}_nC_j+1}{2}]} 2^{1-\frac{1}{p}}$$

$$= 2^{1-\frac{1}{p}}\ [\frac{{}_nC_j+1}{2}].$$

□

Note that $\|F_0\|_{\mathcal{L}_s({}^n l_p^2)} = 1 = \|F_n\|_{\mathcal{L}_s({}^n l_p^2)}$. By Lemma 2.3, $\|F_j\|_{\mathcal{L}_s({}^n l_p^2)} \le 2^{1-\frac{1}{p}}\ [\frac{{}_nC_j+1}{2}]$ for $1 \le j \le n-1$.

Theorem 2.4. *Let* $1 \le p < \infty$, $1 \le q < \infty$, $n \ge 2$. *Then*

$$2^{n(1-\frac{1}{p})}(n+1)^{-\frac{1}{q}} \le B_{(p,q:n)} \le (2^{(1-\frac{1}{p})} + 2^{p-1} \sum_{1\le j\le n-1} ([\frac{{}_nC_j+1}{2}])^{\frac{q}{q-1}})^{1-\frac{1}{q}}.$$

Proof. Let $T \in \mathcal{L}_s({}^n l_p^2)$ with $T = (a_0, \dots, a_n)^t$ for some $a_0, \dots, a_n \in \mathbb{C}$. Let $(x_i, y_i) \in l_p^2$ with $\|(x_i, y_i)\|_p = 1$ for $1 \le i \le n$. It follows that

$$\begin{aligned}
&|T((x_1,y_1),\dots,(x_n,y_n))|\\
&\le \sum_{0\le j\le n} |a_j||F_j((x_1,y_1),\dots,(x_n,y_n))|\\
&\le \|(a_0,\dots,a_n)\|_q(|F_0((x_1,y_1),\dots,(x_n,y_n))|^{\frac{q}{q-1}}\\
&\quad + |F_n((x_1,y_1),\dots,(x_n,y_n))|^{\frac{q}{q-1}}\\
&\quad + \sum_{1\le j\le n-1} |F_j((x_1,y_1),\dots,(x_n,y_n))|^{\frac{q}{q-1}})^{1-\frac{1}{q}} \text{ (by Hölder's inequality)}\\
&\le \|(a_0,\dots,a_n)\|_q((|x_1| + |y_1|)\\
&\quad + \sum_{1\le j\le n-1} (2^{(1-\frac{1}{p})}\ [\frac{{}_nC_j+1}{2}])^{\frac{q}{q-1}})^{1-\frac{1}{q}} \text{ (by Lemma 2.3)}\\
&\le \|(a_0,\dots,a_n)\|_q(2^{(1-\frac{1}{p})}\\
&\quad + \sum_{1\le j\le n-1} (2^{(1-\frac{1}{p})}\ [\frac{{}_nC_j+1}{2}])^{\frac{q}{q-1}})^{1-\frac{1}{q}} \text{ (by Hölder's inequality)}\\
&= \|(a_0,\dots,a_n)\|_q(2^{(1-\frac{1}{p})} + 2^{p-1} \sum_{1\le j\le n-1} ([\frac{{}_nC_j+1}{2}])^{\frac{q}{q-1}})^{1-\frac{1}{q}},
\end{aligned}$$

which shows that for every $(a_0, \dots, a_n) \in \mathbb{C}^{n+1}$,

$$\begin{aligned}
\|T\|_{\mathcal{L}_s({}^n l_p^2)} &= \|(a_0,\dots,a_n)\|_{\mathcal{L}_s({}^n l_p^2)}\\
&\le (2^{(1-\frac{1}{p})} + 2^{p-1} \sum_{1\le j\le n-1} ([\frac{{}_nC_j+1}{2}])^{\frac{q}{q-1}})^{1-\frac{1}{q}} \|(a_0,\dots,a_n)\|_q.
\end{aligned}$$

Hence, $B_{(p,q:n)} \le (2^{(1-\frac{1}{p})} + 2^{p-1} \sum_{1\le j\le n-1} ([\frac{{}_nC_j+1}{2}])^{\frac{q}{q-1}})^{1-\frac{1}{q}}$ for every $1 \le p < \infty$ and $1 \le q < \infty$.

By $(*)$ in the proof of Theorem 2.1, we have

$$2^{n(1-\frac{1}{p})} \leq \|(1,\ldots,1)\|_{\mathcal{L}_s({}^n l_p^2)} \leq B_{(p,q:n)}\|(1,\ldots,1)\|_q = B_{(p,q:n)}(n+1)^{\frac{1}{q}},$$

which implies that $\frac{2^{n(1-\frac{1}{p})}}{(n+1)^{\frac{1}{q}}} \leq B_{(p,q:n)}$ for every $1 \leq p < \infty$ and $1 \leq q < \infty$. $\square$

We do not know whether the estimates of Theorem 2.4 are best possible.

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