

RIGHT AND LEFT QUOTIENT OF TWO BOUNDED OPERATORS ON HILBERT SPACES

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Dedicated to the memory of R. G. Douglas (1938-2018)

ABSTRACT. We define a left quotient as well as a right quotient of two bounded operators between Hilbert spaces, and we parametrize these two concepts using the Moore-Penrose inverse. In particular, we show that the adjoint of a left quotient is a right quotient and conversely. An explicit formulae for computing left (resp. right) quotient which correspond to adjoint, sum, and product of given left (resp. right) quotient of two bounded operators are also shown.

1. Introduction

Let A and B be bounded linear operators on a Hilbert space \mathcal{H} with *the kernel condition*

$$\mathcal{N}(B) \subset \mathcal{N}(A).$$

The quotient operator $[A/B]$ of A and B (not necessary bounded) is defined as the mapping $Bx \mapsto Ax$, $x \in \mathcal{H}$. If we write $\mathcal{G}(B, A)$ for the set $\{(Bx, Ax) : x \in \mathcal{H}\}$ in the product Hilbert space $\mathcal{H} \times \mathcal{H}$, then $\mathcal{G}(B, A)$ is a graph and we can define $[A/B]$ as the operator corresponding to this graph, see [18]. A quotient (of bounded operators) so defined appeared for the first time in the work of Dixmier [5] by the name “opérateur J uniforme” and investigated by Kaufman under the name “semi closed operators” in [17–19], where several characterizations are given. It is worth noting that this extension was anticipated well before by Douglas in [4], where he used in the proof of his famous lemma. In [17] Kaufman showed that a linear operator T on \mathcal{H} is closed if and only if T is represented as a quotient $[A/B]$ using A and B such that $\mathcal{R}(A^*) + \mathcal{R}(B^*) = \{A^*x + B^*y : x, y \in \mathcal{H}\}$ is closed in \mathcal{H} , so that every closed operator is included in the class of quotients ($\mathcal{R}(A)$ denotes the range of

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an operator A). Later, many authors studied this class of operators. We can mention, for example, the study of algebraic operations, adjoint, weak adjoint and selfadjoint extension of quotient operators are included in [8, 9, 12, 14, 15], various topologies on the sets of unbounded linear Hilbert space operators by means of quotient operators are investigated in [2, 10, 11, 20, 21], and for other topics we can cite [1, 7, 13]. Recently, Koliha in [22] extend Kaufman's results to operators between two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 as follows: if A is a bounded operator from \mathcal{H}_1 to \mathcal{H}_2 and B is bounded operator on \mathcal{H}_1 , we can define the quotient $[A/B]$ as the set $\mathcal{G}(B, A) = \{(Bx, Ax) : x \in \mathcal{H}_1\} \subset \mathcal{H}_1 \times \mathcal{H}_2$. He showed that $[A/B]$ is a linear operator from \mathcal{H}_1 to \mathcal{H}_2 , with domain $\mathcal{R}(B)$ and range $\mathcal{R}(A)$ if and only if $\mathcal{N}(B) \subset \mathcal{N}(A)$. For more details see [22].

This paper is a continuation and refinement of the research treatment of the class of quotient operators. From now, the concept of the left quotient of bounded operators between two Hilbert spaces (shortly, a left quotient) means the Douglas solution of a given range operators inclusion. A right quotient means quotient of bounded operators as defined above. In this work we extend this two definitions to bounded operators acting between two possibly different Hilbert spaces. We also collect; unify and generalize earlier work and derive new or strengthened results, including a complete description of the this two class of operators in terms of the Moore-Penrose pseudoinverse. We next prove that the adjoint of a left quotient is always exist and coincide with the right quotient of the adjoints of this given operators. We present reasonable left quotient which coincide with the adjoint of a given right quotient. We investigate a conditions under which the sum and product of two left (resp. right) quotient operators are also left (resp. right) quotient operators and we show how to compute the sum and product of given quotients. We also give some other applications which show the advantage of fractional representations of operators.

2. The left quotient of bounded operators

In what follows \mathcal{H} , \mathcal{H}_1 , \mathcal{H}_2 , etc. denote Hilbert spaces endowed with the appropriate scalar product and the associated norm. The inner product in $\mathcal{H}_1 \times \mathcal{H}_2$ is defined by $\langle (x, y), (x', y') \rangle = \langle x, x' \rangle + \langle y, y' \rangle$. For T linear operator from \mathcal{H}_1 to \mathcal{H}_2 , the symbols $\mathcal{D}(T) \subset \mathcal{H}_1$, $\mathcal{N}(T) \subset \mathcal{H}_1$ and $\mathcal{R}(T) \subset \mathcal{H}_2$ will denote the domain, null space and the range space of T , respectively. The set $\mathcal{G}(T) = \{(x, Tx) : x \in \mathcal{D}(T)\} \subset \mathcal{H}_1 \times \mathcal{H}_2$ is called the graph of T . The operator T is closed if and only if $\mathcal{G}(T)$ is a closed subset of $\mathcal{H}_1 \times \mathcal{H}_2$, and is densely defined if $\overline{\mathcal{D}(T)} = \mathcal{H}_1$, where $\overline{\mathcal{D}(T)}$ denote the closure of $\mathcal{D}(T)$ in \mathcal{H}_1 . Denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of all bounded linear operators from \mathcal{H}_1 to \mathcal{H}_2 . If $\mathcal{H}_1 = \mathcal{H}_2$, write $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1)$.

Before to introduce the left quotient, we need the following theorem, which is referred to as the Douglas lemma [4]. This lemma is a basic tool for our discussions.

Theorem 2.1 ([4, Theorem 1]). *For $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$, the following statements are equivalent:*

- (1) (range inclusion) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;
- (2) (majorization) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and
- (3) (factorization) *there exists a bounded operator C from \mathcal{H}_1 to \mathcal{H}_2 such that $A = BC$.*

Moreover, if one of this conditions ((1), (2) or (3)) holds, then there exists a unique operator C so that

- (a) $\|C\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\}$;
- (b) $\mathcal{N}(C) = \mathcal{N}(A)$; and
- (c) $\mathcal{R}(C) \subseteq \mathcal{R}(B^*)$.

We shall call this uniquely determined C the Douglas solution of the equation $BX = A$.

As remarked by Fillmore and Williams [6, Remark, p. 259], that the condition (c) is crucial to determine uniquely the operator C , but, in general, the operator C may be not unique and (a) or (b) is verified, as shown in the following examples:

Example 2.2. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

We have $\mathcal{R}(A) = \mathcal{R}(B) = \text{span}\{e_1\}$, $AA^* = \frac{1}{2}BB^*$ and $A = BC$ for

$$C = \begin{bmatrix} \alpha & \beta \\ -\alpha & 1 - \beta \end{bmatrix} \quad \text{for } \alpha, \beta \in \mathbb{R}.$$

Now if we assume that the matrix C such that $\mathcal{N}(C) = \mathcal{N}(A)$ (the condition (b)), we get $\alpha = 0$ and $A = BC$ is verified for all C , with

$$C = \begin{bmatrix} 0 & \beta \\ 0 & 1 - \beta \end{bmatrix} \quad \text{for all } \beta \in \mathbb{R}.$$

Further, if we added the condition (c), we obtain $\beta = \frac{1}{2}$; so

$$C = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Example 2.3. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By a simple calculus, we get $\mathcal{R}(A) = \text{span}\{e_1\} \subset \mathcal{R}(B) = \text{span}\{e_1, e_2\}$, $AA^* \leq BB^*$ and $A = BC$ for

$$C = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for all } \alpha, \beta, \gamma \in \mathbb{R}.$$

Now if we assume that the matrix C such that $\mathcal{N}(C) = \mathcal{N}(A) = \text{span}\{e_1, e_2\}$ (the condition (b)), we get $\alpha = \beta = 0$ and $A = BC$ is verified for all C , with

$$C = \begin{bmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for all } \gamma \in \mathbb{R}.$$

Furthermore; $\|C\| = 1$; at this stage C is not unique. Hence C is uniquely determined by using the condition (c), which gives

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In view of the observation above, a definition of a left quotient of two operators can be given as follows:

Definition (Left quotient of bounded operators). Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ such that the *range inclusion*: $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ holds. The unique operator

$$C : \mathcal{H}_1 \longrightarrow \mathcal{N}(B)^\perp$$

as a Douglas solution of $A = BX$ is called *left quotient* of A by B , and will be noted by $[B \setminus A]$.

A direct consequence of the Douglas lemma, we have:

- (i) $[B \setminus A]$ is bounded operator from \mathcal{H}_1 to \mathcal{H}_2 , with

$$\|[B \setminus A]\| = \sqrt{\inf\{\mu : AA^* \leq \mu BB^*\}}.$$

- (ii) $\mathcal{N}([B \setminus A]) = \mathcal{N}(A)$ and $[B \setminus A]$ from $\mathcal{N}(A)^\perp$ to $\mathcal{N}(B)^\perp$ is an injective operator.

- (iii) $B[B \setminus A]x = Ax$ for all $x \in \mathcal{H}_1$ and $\mathcal{R}(C) = \mathcal{R}([B \setminus A]) \subseteq \overline{\mathcal{R}(B^*)}$. So, the left quotient $[B \setminus A]$ is the unique solution of the equation $A = BX$.

Next, in order to give a complete description of the left quotient using the Moore-Penrose inverse, let recall the definition of this generalized inverse.

Definition ([3]). Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then there exists a unique closed densely defined operator T^\dagger , with domain $\mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ and has the following properties:

- (1) $T^\dagger T x = P_{\mathcal{N}(T)^\perp} x$ for all $x \in \mathcal{H}_1$.
- (2) $T T^\dagger y = P_{\overline{\mathcal{R}(T)}} y$ for all $y \in \mathcal{D}(T^\dagger)$.
- (3) $\mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp$.

Where $P_{\mathcal{M}}$ denotes the orthogonal projection onto a closed subspace \mathcal{M} .

This unique operator T^\dagger is called the Moore-Penrose inverse of T (or the Maximal Tseng generalized Inverse in the terminology of [3]). In this case, T is written in a matrix form with respect to mutually orthogonal subspaces decompositions as follows

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{R}(T)^\perp \end{bmatrix}$$

with T_1 from $\mathcal{N}(T)^\perp$ to $\mathcal{R}(T)$ is injective with dense range in $\overline{\mathcal{R}(T)}$. Hence, its Moore-Penrose inverse is given by

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{R}(T)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{bmatrix}$$

as a closed operator with domain $\mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ dense in \mathcal{H}_2 . As a consequence of the closed graph theorem, T^\dagger is bounded if and only if $\mathcal{R}(T)$ is closed in \mathcal{H}_2 , see [3].

In the next result, we redefine the left quotient via the Moore-Penrose inverse.

Lemma 2.4. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ such that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. Then*

$$[B \setminus A] = B^\dagger A : \mathcal{H}_1 \longrightarrow \mathcal{N}(B)^\perp.$$

In particular, if B is invertible, then $[B \setminus A] = B^{-1}A$.

Proof. Since $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, $B^\dagger A$ is a bounded operator from \mathcal{H}_1 to \mathcal{H}_2 as a composition of closed operator and bounded one (see [16, Problem 5.22, p. 167]). Further, A and B has the following form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix},$$

with B_1 is an invertible operator from $\mathcal{N}(B)^\perp$ to $\mathcal{R}(B)$. Moreover,

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix}.$$

Then

$$B^\dagger A = \begin{bmatrix} B_1^{-1} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix}.$$

This implies that

$$(1) \quad BB^\dagger A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = A : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix}.$$

On the other hand,

$$(2) \quad BB^\dagger A = P_{\overline{\mathcal{R}(B)}} Ax = Ax \quad \text{for all } x \in \mathcal{H}_1.$$

By (1) and (2), it follows that $BB^\dagger A = A$ with

$$B^\dagger A : \mathcal{H}_1 \longrightarrow \mathcal{N}(B)^\perp.$$

Now by the uniqueness of the Douglas solution we have the desired result. \square

Remark 2.5. For $A \in \mathcal{B}(\mathcal{H})$, we have

- $[I \setminus A] = A$.
- If A is surjective, then $[A \setminus I]$ is the right inverse of A and given by $[A \setminus I] = A^\dagger = A^*(AA^*)^{-1}$.

Corollary 2.6. *Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{n \times p}$ such that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. Then $[B \setminus A] = B^\dagger A \in \mathbb{C}^{m \times p}$.*

Example 2.7. For the matrices A and B given in Example 2.2 and Example 2.3, the left quotient of A by B is given by

$$[B \setminus A] = B^\dagger A = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

and

$$[B \setminus A] = B^\dagger A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

respectively, where B^\dagger is the inverse of Moore-Penrose of the matrix B .

If $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ such that $\mathcal{R}(A) = \mathcal{R}(B)$, then we can define $[B \setminus A]$ and $[A \setminus B]$ as follows

$$[B \setminus A] : \mathcal{H}_1 \longrightarrow \mathcal{N}(B)^\perp$$

and

$$[A \setminus B] : \mathcal{H}_2 \longrightarrow \mathcal{N}(A)^\perp$$

with

$$[B \setminus A][A \setminus B] \text{ is the identity operator on } \mathcal{N}(B)^\perp$$

and

$$[A \setminus B][B \setminus A] \text{ is the identity operator on } \mathcal{N}(A)^\perp.$$

Therefore, $[B \setminus A]$ is an invertible operator from $\mathcal{N}(A)^\perp$ to $\mathcal{N}(B)^\perp$ with

$$[B \setminus A]^{-1} = [A \setminus B].$$

Corollary 2.8. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ have the same range. Then $[B \setminus A]$ is an invertible operator from $\mathcal{N}(A)^\perp$ to $\mathcal{N}(B)^\perp$ with*

$$[B \setminus A]^{-1} = [A \setminus B] : \mathcal{N}(B)^\perp \longrightarrow \mathcal{N}(A)^\perp$$

and bounded.

3. The right quotient of bounded operators

We now present the definition of the second kind of quotient of two bounded operators in the following generalized form.

Definition (Right quotient of bounded operators). Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ such that the *kernel inclusion*: $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ holds. The *right quotient operator* $[A/B]$ of A by B is defined as the mapping

$$Bx \longmapsto Ax; \quad x \in \mathcal{H}_1.$$

A direct consequence of this definition, we have:

- (i) If we write $\mathcal{G}(B, A)$ for the set $\{(Bx, Ax) : x \in \mathcal{H}_1\}$ in the product Hilbert space $\mathcal{H}_3 \times \mathcal{H}_2$, then $\mathcal{G}(B, A)$ is a graph and we can define $[A/B]$ as the operator from \mathcal{H}_3 to \mathcal{H}_2 corresponding to this graph.
- (ii) $[A/B]Bx = Ax$ for all $x \in \mathcal{H}_1$. So, the right quotient $[A/B]$ is the unique solution of the equation $A = XB$.
- (iii) $\mathcal{D}([A/B]) = \mathcal{R}(B)$, $\mathcal{R}([A/B]) = \mathcal{R}(A)$ and $\mathcal{N}([A/B]) = B(\mathcal{N}(A))$.
- (iv) By Douglas lemma, we have $\mathcal{R}(A^*) \subset \mathcal{R}(B^*)$ if and only if $\|Ax\| \leq \mu \|Bx\|$ for some $\mu > 0$ and all $x \in \mathcal{H}_1$. Consequently $[A/B]$ is bounded from $\mathcal{R}(B)$ to $\mathcal{R}(A)$ if and only if $\mathcal{R}(A^*) \subset \mathcal{R}(B^*)$.
- (v) If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$, this definition coincides with those given by Kaufman, [18].
- (vi) If $\mathcal{H}_1 = \mathcal{H}_3$, this definition coincides with those given by Koliha, [22].

As remarked above a right quotient of bounded operators is not necessary bounded neither closed. The following lemma expresses a right quotient in terms of the Moore-Penrose inverse.

Lemma 3.1. Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ such that $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. Then

$$[A/B] = [AP_{\overline{\mathcal{R}(B^*)}}/B] = AB^\dagger : \mathcal{R}(B) \longrightarrow \mathcal{R}(A).$$

If $\mathcal{R}(B)$ is closed, then $[A/B]$ is bounded. In particular, if B is invertible, then $[A/B] = AB^{-1}$.

Proof. As in the proof of Lemma 2.4 the Moore-Penrose generalized inverse of B is given by

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix}.$$

Since $\mathcal{N}(B) \subseteq \mathcal{N}(A)$, then

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix}.$$

So

$$AB^\dagger = \begin{bmatrix} A_1 B_1^{-1} & 0 \\ A_2 B_1^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}.$$

This implies that

$$(3) \quad AB^\dagger B = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} = A : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}.$$

On the other hand, for all $x \in \mathcal{H}_1 = \mathcal{N}(B)^\perp \oplus \mathcal{N}(B)$, we have

$$(4) \quad AB^\dagger Bx = AP_{\mathcal{N}(B)^\perp}x = AP_{\mathcal{N}(B)^\perp}x + AP_{\mathcal{N}(B)}x = Ax.$$

By (3) and (4), it follows that $[A/B] = [AP_{\mathcal{N}(B)^\perp}/B] = AB^\dagger$ with

$$AB^\dagger : \mathcal{R}(B) \longrightarrow \mathcal{R}(A).$$

Since $D([A/B]) = \mathcal{R}(B)$, it follows that $[A/B]$ is densely defined in $\overline{\mathcal{R}(B)}$ and is bounded when $\mathcal{R}(B)$ is closed. \square

Remark 3.2. For $A \in \mathcal{B}(\mathcal{H})$, we have

- $[A/I] = A$.
- If A is injective with closed range, then $[I/A]$ is the left inverse of A and given by $[I/A] = A^\dagger = (A^*A)^{-1}A^*$.

Corollary 3.3. *Let $A, \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times n}$ such that $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. Then $[A/B] = AB^\dagger \in \mathbb{C}^{p \times m}$.*

If $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $\mathcal{N}(A) = \mathcal{N}(B)$, then we can define $[B/A]$ and $[A/B]$ with

$$[B/A][A/B] \text{ is the identity operator on } \mathcal{R}(B)$$

and

$$[A/B][B/A] \text{ is the identity operator on } \mathcal{R}(A).$$

Therefore, $[A/B]$ is an invertible operator from $\mathcal{R}(B)$ to $\mathcal{R}(A)$ with

$$[A/B]^{-1} = [B/A].$$

Corollary 3.4. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ have the same kernel. Then $[A/B]$ is an invertible operator from $\mathcal{R}(B)$ to $\mathcal{R}(A)$ with*

$$[A/B]^{-1} = [B/A].$$

If $\mathcal{R}(A)$ is closed, then $[A/B]^{-1}$ is bounded.

The graph of a right quotient operator $[A/B]$ (with the condition $\mathcal{N}(B) \subseteq \mathcal{N}(A)$) is the set

$$\mathcal{G}(B, A) = \{(Bx, Ax) : x \in \mathcal{H}_1\} \subset \mathcal{H}_3 \times \mathcal{H}_2.$$

Note that $\mathcal{G}(B, A)$ is the range of the operator T defined by:

$$T = \begin{bmatrix} B \\ A \end{bmatrix} : \mathcal{H}_1 \longrightarrow \mathcal{H}_3 \times \mathcal{H}_2.$$

So its adjoint is given by

$$T^* = \begin{bmatrix} B^* & A^* \end{bmatrix} : \mathcal{H}_3 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_1.$$

We have $\mathcal{R}(T) = \mathcal{G}(B, A)$ and $\mathcal{R}(T^*) = \mathcal{R}(A^*) + \mathcal{R}(B^*)$. Hence $[A/B]$ is closed if and only if $\mathcal{R}(T)$ is closed in $\mathcal{H}_3 \times \mathcal{H}_2$; equivalently to $\mathcal{R}(T^*) = \mathcal{R}(A^*) + \mathcal{R}(B^*)$ is closed in \mathcal{H}_1 . This proves the following result.

Lemma 3.5. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ such that $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. $[A/B]$ is closed if and only if $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is closed in \mathcal{H}_1 .*

For $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$, let the following selfadjoint operator

$$R_{A^*, B^*} = (A^*A + B^*B)^{1/2} : \mathcal{H}_1 \longrightarrow \mathcal{H}_1;$$

It well known that $\mathcal{R}(R_{A^*, B^*}) = \mathcal{R}(A^*) + \mathcal{R}(B^*)$ [6, Theorem 2.2]. Thus $\mathcal{N}(R_{A^*, B^*}) = (\mathcal{R}(A^*) + \mathcal{R}(B^*))^\perp = \mathcal{N}(A) \cap \mathcal{N}(B)$. If, further, $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is closed in \mathcal{H}_1 , then R_{A^*, B^*}^\dagger exists and bounded, with

$$R_{A^*, B^*}^\dagger = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) + \mathcal{R}(B^*) \\ \mathcal{N}(A) \cap \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A^*) + \mathcal{R}(B^*) \\ \mathcal{N}(A) \cap \mathcal{N}(B) \end{bmatrix},$$

where R is an isomorphism on $\mathcal{R}(A^*) + \mathcal{R}(B^*)$. As in [17], we define an operator $J : \mathcal{R}(A^*) + \mathcal{R}(B^*) \longrightarrow \mathcal{H}_3 \times \mathcal{H}_2$ by

$$Jx = (BR_{A^*, B^*}^\dagger x, AR_{A^*, B^*}^\dagger x).$$

Then for each $x \in \mathcal{R}(A^*) + \mathcal{R}(B^*)$,

$$\begin{aligned} \|Jx\|^2 &= \|BR_{A^*, B^*}^\dagger x\|^2 + \|AR_{A^*, B^*}^\dagger x\|^2 \\ &= \left\langle x, R_{A^*, B^*}^\dagger R_{A^*, B^*}^2 R_{A^*, B^*}^\dagger x \right\rangle \\ &= \|x\|^2. \end{aligned}$$

Hence, if $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is closed, then J is a linear isometry from $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ to $\mathcal{H}_3 \times \mathcal{H}_2$. If we assume further, $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. We have also $\overline{\mathcal{R}(A^*)} \subset \overline{\mathcal{R}(B^*)}$, this implies that $\mathcal{R}(A^*) + \mathcal{R}(B^*) \subset \overline{\mathcal{R}(B^*)}$. Since $\mathcal{N}(R_{A^*, B^*}) = \mathcal{N}(B)$; then $\mathcal{R}(A^*) + \mathcal{R}(B^*) = \overline{\mathcal{R}(B^*)}$. In this case, since $\mathcal{H}_1 = \overline{\mathcal{R}(B^*)} \oplus \mathcal{N}(B)$, the graph of the linear operator $[A/B]$ is given

$$\mathcal{G}(B, A) = \{(Bx, Ax) : x \in \overline{\mathcal{R}(B^*)} = \mathcal{R}(A^*) + \mathcal{R}(B^*)\},$$

and hence the linear isometry J maps the closed set $\overline{\mathcal{R}(B^*)}$ onto $\mathcal{G}(B, A)$. This gives another proof of the preceding Lemma.

4. Dual properties

In this section we prove some dual relationships between the left quotient and the right quotient of two bounded operators.

Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$. Since $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ then $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$, so we can define $[A^*/B^*]$ the right quotient of A^* by B^* from $\mathcal{R}(B^*)$ to $\mathcal{R}(A^*)$ such that

$$(5) \quad [A^*/B^*]B^*x = A^*x \quad \text{for all } x \in \mathcal{H}_3.$$

Since $[B \setminus A]$ is bounded, then its adjoint exists.

Theorem 4.1. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ such that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. We have, if $\mathcal{R}(B^*)$ is closed, then*

$$[B \setminus A]^* = [A^* / B^*] = A^*(B^*)^\dagger.$$

Proof. For all $x, y \in \mathcal{H}_1$,

$$\begin{aligned} \langle Ax, y \rangle &= \langle B[B \setminus A]x, y \rangle \\ &= \langle x, [B \setminus A]^* B^* y \rangle \\ &= \langle x, A^* y \rangle. \end{aligned}$$

Then

$$[B \setminus A]^* B^* y = A^* y \quad \text{for all } y \in \mathcal{H}_3.$$

Compare this with (5), we obtain $[B \setminus A]^* = [A^* / B^*]$. \square

Remark 4.2. Without the closure of the range of B^* , as seen in the proof of the Douglas lemma, the operator $[B \setminus A]^*$ is the natural extension of $[A^* / B^*]$ to the closure of the range of B^* ; in the following sense.

Since $[A^* / B^*]$ is bounded from $\mathcal{R}(B^*)$ to $\mathcal{R}(A^*)$; that is $\sup\{\|A^* x\| / \|B^* x\| : x \in \mathcal{H}_3, B^* x \neq 0\} < \infty$, then we can define its natural extension $[A^* / B^*]$ to $\overline{\mathcal{R}(B^*)}$ by

$$[\widetilde{A^* / B^*}]x = \begin{cases} \lim_{n \rightarrow \infty} [A^* / B^*]x_n & \text{for } x \in \overline{\mathcal{R}(B^*)}, x_n \rightarrow x, \\ 0 & \text{for } x \in \mathcal{R}(B^*)^\perp. \end{cases}$$

It well known that if T is densely defined from \mathcal{H}_1 to \mathcal{H}_2 , the adjoint T^* of T exists, is unique and T^* is also defined from \mathcal{H}_2 to \mathcal{H}_1 . Now, let $[A/B]$ be a right quotient with the domain $\mathcal{R}(B)$, which is dense in $\overline{\mathcal{R}(B)}$, and let $\mathcal{G}(B, A)$ be its graph. Then the adjoint $\mathcal{G}(B, A)^*$ is naturally defined as the set of elements $(x, y) \in \mathcal{H}_2 \times \mathcal{H}_3$ such that $\langle Az, x \rangle = \langle Bz, y \rangle$ for all $z \in \mathcal{H}_1$. We can see that

$$\mathcal{G}(B, A)^* = \{(x, y) : A^* x = B^* y\} \subset \mathcal{H}_2 \times \mathcal{H}_3,$$

and it is a graph again. The corresponding operator is just the adjoint of $[B/A]$, that is $[B/A]^*$. Furthermore, we have

$$\mathcal{G}(B, A)^* = \mathcal{V}(\mathcal{N}(T^*));$$

with

$$T^* = \begin{bmatrix} B^* & A^* \end{bmatrix} : \mathcal{H}_3 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_1$$

and \mathcal{V} is the isomorphisms from $\mathcal{H}_3 \times \mathcal{H}_2$ to $\mathcal{H}_2 \times \mathcal{H}_3$ defined by

$$\mathcal{V}(x, y) = (-y, x).$$

Theorem 4.3. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ such that $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. For a right quotient $[A/B]$,*

- its adjoint $[A/B]^*$ exist and closed as an operator from $\overline{R(A)}$ to $\overline{R(B)}$ and

$$[A/B]^* = (B^\dagger)^* A^*;$$

with uniquely extended to an operator from \mathcal{H}_2 to $\overline{R(B)}$.

- If $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is closed in \mathcal{H}_1 , then $[A/B]^*$ is closed densely defined from \mathcal{H}_2 to $\overline{R(B)}$.
- If $\mathcal{R}(B^*)$ is closed in \mathcal{H}_1 , then

$$[A/B]^* = [B^* \setminus A^*] = (B^*)^\dagger A^*,$$

and

$$[A/B]^{**} = [A/B].$$

Proof. Let $[A/B]$ be a right quotient with domain $\mathcal{R}(B)$, which is dense in $\overline{R(B)}$, then its adjoint $[A/B]^*$ exist and closed as an operator from $\overline{R(A)}$ to $\overline{R(B)}$ with

$$\mathcal{D}([A/B]^*) = \{x \in \overline{R(A)} : \exists y \in \overline{R(B)}, \langle [A/B]Az, x \rangle = \langle Bz, y \rangle; \forall z \in \mathcal{H}_1\}$$

and

$$[A/B]^*x = y.$$

By Lemma 3.1 $[A/B] = AB^\dagger$, this implies that $[A/B]^* = (B^\dagger)^* A^*$. Further, for all $x \in \mathcal{D}([A/B]^*)$,

$$B^*[A/B]^*x = B^*(B^\dagger)^* A^*x = (B^\dagger B)^* A^*x = P_{\overline{R(B^*)}} A^*x = A^*x;$$

this holds because $\mathcal{R}(A^*) \subset \overline{\mathcal{R}(B^*)}$. The last equality is also valid if $x \in \mathcal{N}(A^*)$, so it holds for all $x \in \mathcal{D}([A/B]^*) \oplus \mathcal{N}(A^*) \subseteq \mathcal{H}_2$; this implies that we can extend $[A/B]^*$ to a closed operator from \mathcal{H}_2 to $\overline{R(B)}$. Further; if $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is closed in \mathcal{H}_1 , then $[A/B]$ is closed densely defined from $\overline{R(B)}$ to $\overline{R(A)}$. So, $[A/B]^*$ is closed densely defined from \mathcal{H}_2 to $\overline{R(B)}$, with $\mathcal{D}([A/B]^*)$ is dense in $\overline{R(A)}$.

Now if $\mathcal{R}(B^*)$ is closed in \mathcal{H}_1 , in that case $\mathcal{R}(A^*) \subset \mathcal{R}(B^*)$, so $[A/B]$ is bounded and its adjoint $[A/B]^*$ exist and bounded. Again by Lemma 3.1,

$$[A/B]^* = (B^\dagger)^* A^* = (B^*)^\dagger A^* = [B^* \setminus A^*],$$

and

$$B^*[B^* \setminus A^*]x = B^*(B^*)^\dagger A^*x = P_{\mathcal{R}(B^*)} A^*x = A^*x \quad \text{for all } x \in \mathcal{H}_2.$$

Then

$$B^*y = A^*x \quad \text{for all } x \in \mathcal{H}_2 \quad \text{and} \quad y = [B^* \setminus A^*]x \in \mathcal{H}_3.$$

Hence $[A/B]^* = [B^* \setminus A^*] = (B^*)^\dagger A^*$ and $[A/B]^{**} = [A/B]$. \square

Remark 4.4. Let us point out that Izumino, in [12], construct a right quotient operator to be an adjoint of $[A/B]$, so our objective is completely different. Here the adjoint of $[A/B]$ is always exists (in particular, as a left quotient) but its adjoint in the same nature (as a right quotient) may be not exist, for more details see [12].

5. Sums and products

Let $A, B, C, D \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Let $[B \setminus A]$ and $[D \setminus C]$ be two left quotient operators (with the respective range conditions). Then we prove that the sum of them is also a left quotient operator. To define the denominator of the sum we need the following range operator, called parallel sum of B^*B and D^*D , defined by

$$B^*B : D^*D = B^*B(B^*B + D^*D)^\dagger D^*D : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$$

with $\mathcal{R}(B^*) + \mathcal{R}(D^*)$ is closed in \mathcal{H}_1 , see [6, p. 277]. If we denotes

$$S_{B^*, D^*} = (B^*B : D^*D)^{1/2},$$

then we have, by [6, Theorem 4.2]:

- (1) $B^*B : D^*D = D^*D : B^*B$ is positive and bounded.
- (2) $\mathcal{R}(S_{B^*, D^*}) = \mathcal{R}(B^*) \cap \mathcal{R}(D^*)$.

This implies that the left operators quotient

$$B_1^* = [B^* \setminus S_{B^*, D^*}] \quad \text{and} \quad D_1^* = [D^* \setminus S_{B^*, D^*}]$$

and the right operators quotient

$$B_1 = [S_{B^*, D^*} / B] \quad \text{and} \quad D_1 = [S_{B^*, D^*} / D]$$

are always makes sense. Now, for the left quotient form of the sum we have:

Theorem 5.1. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Let $[B \setminus A]$ and $[D \setminus C]$ be two left quotient operators (with the respective range conditions). We have*

$$[B \setminus A] + [D \setminus C] = [S_{B^*, D^*} \setminus (B_1 A + D_1 C)] : \mathcal{H}_1 \longrightarrow \mathcal{N}(B)^\perp + \mathcal{N}(D)^\perp.$$

Proof. If $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(C) \subseteq \mathcal{R}(D)$, then the operators $B_1 A + D_1 C$ is well-defined from \mathcal{H}_1 to $\mathcal{R}(S_{B^*, D^*})$. Hence $\mathcal{R}(B_1 A + D_1 C) \subset \mathcal{R}(S_{B^*, D^*})$ and $[S_{B^*, D^*} \setminus B_1 C + D_1 A]$ is also well-defined from \mathcal{H}_1 to $\overline{\mathcal{R}(S_{A^*, B^*})}$. Further,

$$[S_{B^*, D^*} \setminus (B_1 A + D_1 C)] = S_{B^*, D^*}^\dagger (B_1 A + D_1 C).$$

On the other hand, $B_1 = S_{B^*, D^*} P_{\overline{\mathcal{R}(B^*)}} B^\dagger$ and $D_1 = S_{B^*, D^*} P_{\overline{\mathcal{R}(D^*)}} D^\dagger$, so

$$\begin{aligned} S_{B^*, D^*}^\dagger (B_1 A + D_1 C) &= S_{B^*, D^*}^\dagger (S_{B^*, D^*} P_{\overline{\mathcal{R}(B^*)}} B^\dagger A + S_{B^*, D^*} P_{\overline{\mathcal{R}(D^*)}} D^\dagger C) \\ &= P_{\overline{\mathcal{R}(S_{B^*, D^*})}} (P_{\overline{\mathcal{R}(B^*)}} B^\dagger A + P_{\overline{\mathcal{R}(D^*)}} D^\dagger C) \\ &= P_{\overline{\mathcal{R}(S_{B^*, D^*})}} ([B \setminus A] + [D \setminus C]). \end{aligned}$$

On the other hand; it is clear that the sum $[B \setminus A] + [D \setminus C]$ is an operator defined from \mathcal{H}_1 to $\mathcal{N}(B)^\perp + \mathcal{N}(D)^\perp = \overline{\mathcal{R}(S_{B^*, D^*})}$, which implies that

$$[B \setminus A] + [D \setminus C] = [S_{B^*, D^*} \setminus B_1 A + D_1 C]. \quad \square$$

Remark 5.2. In the case of the scalars case, that is $a, b, c, d \in \mathbb{C}$, this Theorem is nothing,

$$\frac{a}{b} + \frac{c}{d} = \frac{b_1 a + d_1 c}{|b| \left(\sqrt{|b|^2 + |d|^2} \right)^{-1} |d|},$$

with

$$b_1 = \frac{\sqrt{|b|^2 + |d|^2}}{b |d| |b|}, \quad d_1 = \frac{\sqrt{|b|^2 + |d|^2}}{d |d| |b|} \quad b \neq 0, d \neq 0.$$

For two left quotient operators with the same denominator, we have:

Theorem 5.3. *Let $A, B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $D \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ such that $\mathcal{R}(A) \subseteq \mathcal{R}(D)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. Then*

$$[D \setminus (A + B)] = [D \setminus A] + [D \setminus B].$$

Proof. By assumptions, we have $A = D[D \setminus A]$ and $B = D[D \setminus B]$, so

$$A + B = D([D \setminus A] + [D \setminus B]).$$

On the other hand; $\mathcal{R}(A+B) \subseteq \mathcal{R}(A) + \mathcal{R}(B) \subseteq \mathcal{R}(D)$, it follows that $[D \setminus A+B]$ is well defined and

$$A + B = D[D \setminus (A + B)].$$

By what we assert the desired equality. \square

We next show that the product of two left quotient operators is again a left operator.

Theorem 5.4. *Let $A, B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $C, D \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$. If $[B \setminus A]$ and $[D \setminus C]$ are two left quotient operators (with the respective range conditions), then*

$$[B \setminus A][D \setminus C] = [NB \setminus MC],$$

where $M \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$ and $N \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying the conditions $MD = NA$ and $\mathcal{N}(N)^\perp = \mathcal{R}(B)$.

To prove this theorem, we need the following lemma.

Lemma 5.5. *If $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ with $\mathcal{R}(T) = \mathcal{N}(S)^\perp$, then $(ST)^\dagger = T^\dagger S^\dagger$.*

Proof. In this case we have $\mathcal{R}(T)$ is closed and $ST \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ with

$$(ST)^\dagger = \begin{bmatrix} R_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(ST) \\ \mathcal{R}(ST)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(ST)^\perp \\ \mathcal{N}(ST) \end{bmatrix}.$$

Now, if

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{R}(T)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{bmatrix}$$

and

$$S^\dagger = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(S) \\ \mathcal{R}(S)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(S)^\perp \\ \mathcal{N}(S) \end{bmatrix},$$

then

$$T^\dagger S^\dagger = \begin{bmatrix} T_1^{-1} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(S) \\ \mathcal{R}(S)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(T)^\perp \\ \mathcal{N}(T) \end{bmatrix};$$

taking in account that $\mathcal{N}(S)^\perp = \mathcal{R}(T)$. We have $\mathcal{R}(ST) \subset \mathcal{R}(S)$. Now; let $y \in \mathcal{R}(S) = \mathcal{R}(SS^*)$, so $y = SS^*x$ for $x \in \mathcal{H}_3$. By assumption $\mathcal{R}(S^*) = \mathcal{N}(S)^\perp = \mathcal{N}(S^*S)^\perp = \mathcal{R}(S^*S) = \mathcal{R}(T) = \mathcal{R}(TT^*)$. Thus $S^*x \in \mathcal{R}(TT^*)$ and $S^*x = TT^*z$ for $z \in \mathcal{H}_3$. So $y = STT^*z$ which implies that $y \in \mathcal{R}(ST)$. Consequently, $\mathcal{R}(ST) = \mathcal{R}(S)$. This proves that $(ST)^\dagger$ and $T^\dagger S^\dagger$ have the same domain and $\mathcal{N}((T^\dagger S^\dagger)^\perp) = \mathcal{N}((ST)^\dagger) = \mathcal{R}(S)^\perp$. Let us remark that $TT^\dagger = P_{\mathcal{R}(T)} = P_{\mathcal{N}(S)^\perp} = SS^\dagger$. For all $y \in \mathcal{H}_3$,

$$\begin{aligned} T^\dagger S^\dagger STT^\dagger S^\dagger y &= T^\dagger S^\dagger SS^\dagger SS^\dagger y \\ &= T^\dagger P_{\overline{\mathcal{R}(S^\dagger)}} S^\dagger y \\ &= T^\dagger S^\dagger y. \end{aligned}$$

Hence $T^\dagger S^\dagger STx = P_{\overline{\mathcal{R}(T^\dagger S^\dagger)}} x$ for all $y \in \mathcal{H}_1$.

Similarly, for all $x \in \mathcal{H}_1$,

$$\begin{aligned} STT^\dagger S^\dagger STx &= SS^\dagger SS^\dagger STx \\ &= SP_{\overline{\mathcal{R}(S)}} Tx \\ &= SP_{\overline{\mathcal{R}(ST)}} Tx \\ &= STx. \end{aligned}$$

Hence $STT^\dagger S^\dagger z = P_{\overline{\mathcal{R}(ST)}} z$ for all $z \in \mathcal{D}(T^\dagger S^\dagger) = \mathcal{R}(S) \oplus \mathcal{R}(S)^\perp$. Hence, by the uniqueness of Moore-Penrose inverse, $(ST)^\dagger = T^\dagger S^\dagger$. \square

Proof of Theorem 5.4. We have $[B \setminus A][D \setminus C] = B^\dagger A D^\dagger C$ is an operator well defined from \mathcal{H}_1 to $\mathcal{N}(B)^\perp$ and the operator $[NB \setminus MC]$ is defined from \mathcal{H}_1 to $\mathcal{N}(NB)^\perp \subset \mathcal{N}(B)^\perp$. Now, by assumption, $MD = NA$, so $N^\dagger M D D^\dagger = N^\dagger N A D^\dagger$; Thus $N^\dagger M P_{\overline{\mathcal{R}(D)}} = P_{\overline{\mathcal{N}(N)^\perp}} A D^\dagger = P_{\overline{\mathcal{R}(B)}} A D^\dagger = A D^\dagger$ ($\mathcal{R}(A) \subset \mathcal{R}(B)$). This implies that $B^\dagger A D^\dagger C = B^\dagger N^\dagger M P_{\overline{\mathcal{R}(D)}} C = B^\dagger N^\dagger M C$. Now by Lemma 5.5, $B^\dagger N^\dagger = (NB)^\dagger$; then

$$[B \setminus A][D \setminus C] = B^\dagger A D^\dagger C = (NB)^\dagger M C = [NB \setminus MC]. \quad \square$$

Now using Theorem 4.1, we get the following results for the sum and product for the right quotient operators.

Theorem 5.6. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. If $[A/B]$ and $[C/D]$ are two right quotient operators (with the respective kernel conditions) such that $\mathcal{R}(B) + \mathcal{R}(D)$ is closed in \mathcal{H}_2 , then*

$$[A/B] + [C/D] = [AB_1 + CD_1/S_{B,D}] : \mathcal{R}(B) \cap \mathcal{R}(D) \longrightarrow \mathcal{R}(A) \cap \mathcal{R}(C),$$

with

$$B_1 = [B \setminus S_{B,D}] \quad \text{and} \quad D_1 = [D \setminus S_{B,D}].$$

Theorem 5.7. *Let $A, B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ such that $\mathcal{N}(D) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(D) \subseteq \mathcal{N}(B)$. Then*

$$[(A+B)/D] = [A/D] + [B/D].$$

Theorem 5.8. *Let $A, B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $C, D \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$. If $[A/B]$ and $[C/D]$ are two right quotient operators (with the respective kernel conditions), then*

$$[A/B][C/D] = [AP_{\mathcal{N}(B)^\perp}M/DN],$$

where $M \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$ and $N \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$ satisfying the conditions $BM = CN$ and $\mathcal{N}(D)^\perp = \mathcal{R}(N)$.

Proof. Since the domain of $[A/B][C/D]$ is $\mathcal{R}(D)$ and those of $[AM/DN]$ is $\mathcal{R}(DN)$, as seen in Lemma 5.5, the assumption $\mathcal{N}(D)^\perp = \mathcal{R}(N)$ guaranties that this two operators have the same domain. Now, by assumption, $BM = CN$, so $B^\dagger BMN^\dagger = B^\dagger CNN^\dagger$; Thus $P_{\mathcal{N}(B)^\perp}MN^\dagger = B^\dagger CP_{\mathcal{N}(D)^\perp}$. This implies that for $y \in \mathcal{R}(D)$, we have

$$\begin{aligned} [A/B][C/D]y &= AB^\dagger CD^\dagger y = AB^\dagger CP_{\mathcal{N}(D)^\perp} D^\dagger y \\ &= AP_{\mathcal{N}(B)^\perp} MN^\dagger D^\dagger y \\ &= AMN^\dagger D^\dagger y \\ &= AM(DN)^\dagger y \\ &= [AM/DN]y. \end{aligned}$$

Here we use the fact that $AP_{\mathcal{N}(B)^\perp} = A$ because $\mathcal{N}(B) \subset \mathcal{N}(A)$ and $(DN)^\dagger = N^\dagger D^\dagger$ by Lemma 5.5. \square

Remark 5.9. Theorems 5.6 and 5.8 are a generalization of [12, Theorem 3.1] and [12, Theorem 3.2], respectively.

We finally turn to Lemma 5.5, to give some results like a simplification in a quotient as in the case of scalars.

Corollary 5.10. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ and $[B \setminus A]$ is a left quotient operator. Then we have*

$$[MB \setminus MA] = [B \setminus A]$$

for all $M \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ such that $\mathcal{N}(M)^\perp = \mathcal{R}(B)$. In particular, if $M \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$ and $M = B^*$, we have

$$[B^* B \setminus B^* A] = [B \setminus A].$$

Proof. We have $\mathcal{R}(AM) \subset \mathcal{R}(A) \subset \mathcal{R}(B) = \mathcal{R}(BM)$; then $[MB \setminus MA]$ is well defined. Further,

$$\begin{aligned} [MB \setminus MA] &= (MB)^\dagger MA = B^\dagger M^\dagger MA \\ &= B^\dagger P_{\mathcal{N}(M)^\perp} A = B^\dagger P_{\mathcal{R}(B)} A = B^\dagger A = [B \setminus A]. \quad \square \end{aligned}$$

Corollary 5.11. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $[A/B]$ is a right quotient operator. Then we have*

$$[AM/BM] = [A/B]$$

for all $M \in \mathcal{B}(\mathcal{H}_4, \mathcal{H}_1)$ such that $\mathcal{N}(B)^\perp = \mathcal{R}(M)$. In particular, if $M \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$ and $M = B^*$, we have

$$[AB^*/BB^*] = [A/B].$$

Proof. Since $\mathcal{N}(B)^\perp = \mathcal{R}(M)$ and $\mathcal{N}(B) \subset \mathcal{N}(A)$; then $\mathcal{N}(BM) \subset \mathcal{N}(AM)$; then $[AM/BM]$ is well defined. Further,

$$\begin{aligned} [AM/BM] &= AM(BM)^\dagger = AMM^\dagger B^\dagger \\ &= AP_{\mathcal{R}(M)} B^\dagger = AP_{\mathcal{N}(B)^\perp} B^\dagger = AB^\dagger = [A/B]. \quad \square \end{aligned}$$

Corollary 5.12. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with closed range. Then we have*

$$A = [A^* \setminus A^* A] \quad \text{and} \quad A^\dagger = [A^* A \setminus A^*].$$

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