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SOME IDENTITIES ASSOCIATED WITH 2-VARIABLE TRUNCATED EXPONENTIAL BASED SHEFFER POLYNOMIAL SEQUENCES

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Abstract. Since Sheffer introduced the so-called Sheffer polynomials in 1939, the polynomials have been extensively investigated, applied and classified. In this paper, by using matrix algebra, specifically, some properties of Pascal and Wronskian matrices, we aim to present certain interesting identities involving the 2-variable truncated exponential based Sheffer polynomial sequences. Also, we use the main results to give some interesting identities involving so-called 2-variable truncated exponential based Miller-Lee type polynomials. Further, we remark that a number of different identities involving the above polynomial sequences can be derived by applying the method here to other combined generating functions.

1. Introduction and preliminaries

Sequences of polynomials play an important role in dealing with various problems arising in many different areas of pure and applied mathematics (see, e.g., [2, 5, 17, 19–22]). Among a variety of polynomials, in 1939, Sheffer [24] introduced the so-called Sheffer polynomials. The Sheffer polynomials have been extensively investigated, applied and classified (see, e.g., [18, pp. 218–232]).

A polynomial sequence $\{s_n(x)\}_{n=0}^{\infty}$ is called Sheffer polynomial sequence if and only if its generating function is given by

$$a(t)\,\exp\left(x\,b(t)\right)=\sum_{n=0}^{\infty}s_n(x)\frac{t^n}{n!}. \tag{1.1}$$

Here $a(t)$ and $b(t)$ are formal power series

$$a(t)=\sum_{n=0}^{\infty}a_n\,t^n \quad \text{and} \quad b(t)=\sum_{n=0}^{\infty}b_n\,t^n$$

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with $a(0) = a_0 \neq 0$, $b(0) = b_0 = 0$, and $b'(0) = b_1 \neq 0$ (see, e.g., [10, 20, 24]).

The truncated exponential polynomials $e_n(x)$ defined by the series (see [3])

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \tag{1.2}$$

are the first $n+1$ terms of the Maclaurin series for e^x. Obviously, the truncated exponential polynomials $e_n(x)$ are defined by the generating function (see [6])

$$\frac{e^{xt}}{1-t} = \sum_{n=0}^{\infty} e_n(x)\, t^n. \tag{1.3}$$

The higher-order truncated exponential polynomials ${}_{[r]}e_n(x)$ defined by the series

$$ {}_{[r]}e_n(x) = \sum_{k=0}^{[\frac{n}{r}]} \frac{x^{n-rk}}{(n-rk)!} \quad (n,\, r \in \mathbb{N}) \tag{1.4}$$

are generated by the following function

$$\frac{e^{xt}}{1-t^r} = \sum_{n=0}^{\infty} {}_{[r]}e_n(x)\, t^n \tag{1.5}$$

(see [6]). Here and in the following, let $\mathbb{N}$ and $\mathbb{C}$ be the sets of positive integers and complex numbers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The 2-variable truncated exponential polynomials (2VTEP) $e_n^{(r)}(x, y)$ of order r defined by

$$e_n^{(r)}(x, y) = \sum_{k=0}^{[\frac{n}{r}]} \frac{y^k x^{n-rk}}{(n-rk)!} \tag{1.6}$$

are generated by the following function

$$\frac{e^{xt}}{1-yt^r} = \sum_{n=0}^{\infty} e_n^{(r)}(x, y) \frac{t^n}{n!} \tag{1.7}$$

(see [8, p. 174]).

Let $\mathcal{F}$ be a class of functions which are analytic at the origin. Then the generalized Pascal functional matrix $[P_n(g(t))]$ $(g(t) \in \mathcal{F})$ is a lower triangular $n+1$ by $n+1$ matrix defined by

$$P_n[g(t)]_{i,j} = \begin{cases} \binom{i}{j} g^{(i-j)}(t), & \text{if } i \geq j \\ 0, & \text{otherwise,} \end{cases} \tag{1.8}$$

for all $i, j = 0, 1, \ldots, n$ (see [25, 27]). Here and in the following, $g^{(i)}(t)$ is i^{th} derivative of $g(t)$.

The n^{th} order Wronskian matrix of analytic functions $g_1(t), g_2(t), \ldots, g_m(t) \in \mathcal{F}$ is an $n+1$ by m matrix defined by

$$W_n[g_1(t), g_2(t), \ldots, g_m(t)] = \begin{bmatrix} g_1(t) & g_2(t) & g_3(t) & \cdots & g_m(t) \\ g_1'(t) & g_2'(t) & g_3'(t) & \cdots & g_m'(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1^{(n)}(t) & g_2^{(n)}(t) & g_3^{(n)}(t) & \cdots & g_m^{(n)}(t) \end{bmatrix} \tag{1.9}$$

(see [25, 27]).

Many authors have presented some recurrence relations, differential equations and identities involving various polynomial sequences (see, e.g., [4, 9, 11–13, 16, 23]). Youn and Yang [27] obtained some identities and differential equation for Sheffer polynomial sequences by using matrix algebra (see also [1]).

Here and in the following, let $g(t)$ be an invertible analytic function, that is, $g(0) \neq 0$, and $f(t)$ be analytic function with $f(0) = 0$ and $f'(0) \neq 0$ that admits compositional inverse. Then the 2-variable truncated exponential based Sheffer polynomial sequences ${}_{e^{(r)}}s_n(x,y)$ are defined by the following generating function

$$\frac{1}{g(f^{-1}(t))\,(1 - y(f^{-1}(t))^r)} \exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} {}_{e^{(r)}}s_n(x,y) \frac{t^n}{n!}, \tag{1.10}$$

where $f^{-1}(t)$ is the compositional inverse of $f(t)$. In order to define the polynomial sequence in (1.10), the left-member of (1.10) should be analytic at $t = 0$. Then, obviously,

$$ {}_{e^{(r)}}s_n(x,y) = \frac{d^n}{dt^n} \left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1 - y(f^{-1}(t))^r)} \right) \Bigg|_{t=0} \quad (n \in \mathbb{N}_0)\,. \tag{1.11}$$

Khan et al. [14] used operational methods to present some useful identities involving the 2-variable truncated exponential based Sheffer polynomial sequences ${}_{e^{(r)}}s_n(x,y)$ and apply some results to demonstrate some special polynomials, for example, the 2-variable truncated-exponential based generalized Laguerre polynomials. Youn and Yang [27] used matrix algebra to provide a differential equation and several recursive formulas of one variable Sheffer polynomial sequences. In this paper, by employing Youn and Yang's method [27], we derive some presumably new identities involving the 2-variable truncated exponential based Sheffer polynomial sequences ${}_{e^{(r)}}s_n(x,y)$.

To do this, some properties of the Wronskian matrices and the generalized Pascal functional matrices and their relationships are recalled in the following lemma (see, e.g., [26, 27]).

Lemma 1. *Let u, $v \in \mathbb{C}$ and $g(t)$, $g_1(t)$, $\ldots$, $g_m(t)$, $h(t) \in \mathcal{F}$. Then*

(a) *Linear*

$$P_n[u\,g(t) + v\,h(t)] = u\,P_n[g(t)] + v\,P_n[h(t)],$$
$$W_n[u\,g(t) + v\,h(t)] = u\,W_n[g(t)] + v\,W_n[h(t)].$$

(b) *Multiplicative*

$$P_n[g(t)\,h(t)] = P_n[g(t)]\,P_n[h(t)] = P_n[h(t)]\,P_n[g(t)].$$

In addition, if $g(t) \neq 0$, *then* $(P_n[g(t)])^{-1} = P_n\left[g^{-1}(t)\right]$, *where* $g^{-1}(t)$ *denotes the multiplicative inverse of* $g(t)$.

(c) *Pascal and Wronskian*

$$P_n[g(t)]\,W_n[h(t)] = P_n[h(t)]\,W_n[g(t)] = W_n[(gh)(t)].$$

In addition,

$$P_n[g(t)]\,W_n[g_1(t), g_2(t), \ldots, g_m(t)] = W_n[(g\,g_1)(t), (g\,g_2)(t), \ldots, (g\,g_m)(t)].$$

(d) *Let* $g(0) = 0$ *and* $g'(0) \neq 0$. *Then*

$$W_n[h(g(t))]_{t=0} = W_n\left[1, g(t), g^2(t), g^3(t), \ldots, g^n(t)\right]_{t=0}\,\Omega_n^{-1}\,W_n[h(t)]_{t=0}.$$

Here and in the following, $\Omega_n := \mathrm{diag}[0!, 1!, 2!, \ldots, n!]$ *is the diagonal* $n+1$ *by* $n+1$ *matrix.*

2. Some identities involving 2-variable truncated exponential based Sheffer polynomial sequences

Here, we introduce a vector form of the 2-variable truncated exponential based Sheffer polynomial sequences ${}_{e^{(r)}}s_n(x, y)$ for $(g(t), f(t))$ which is defined by

$$ {}_{e^{(r)}}\vec{s}_n(x, y) := [{}_{e^{(r)}}s_0(x, y), {}_{e^{(r)}}s_1(x, y), \ldots, {}_{e^{(r)}}s_n(x, y)]^T, \tag{2.1}$$

where T denotes the transpose of a matrix. From (1.11), we have

$$ {}_{e^{(r)}}\vec{s}_n(x, y) = W_n\left[\frac{1}{g(f^{-1}(t))(1 - y(f^{-1}(t))^r)}\exp(xf^{-1}(t))\right]\Bigg|_{t=0}. \tag{2.2}$$

Lemma 2. *Let* ${}_{e^{(r)}}s_n(x, y)$ *be the* 2*-variable truncated exponential based Sheffer polynomial sequences for* $(g(t), f(t))$. *Then*

$$\begin{aligned} &W_n[{}_{e^{(r)}}s_0(x, y), {}_{e^{(r)}}s_1(x, y), \ldots, {}_{e^{(r)}}s_n(x, y)]^T\,\Omega_n^{-1} \\ &= W_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0}\,\Omega_n^{-1} \\ &\times P_n\left[\frac{1}{g(t)(1 - yt^r)}\right]_{t=0}\,P_n\left[e^{xt}\right]_{t=0}. \end{aligned} \tag{2.3}$$

Proof. Applying (d) in Lemma 1 to the right member of (2.2), we obtain

$$\begin{aligned} {}_{e^{(r)}}\vec{s}_n(x, y) = &W_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0}\,\Omega_n^{-1} \\ &\times W_n\left[\frac{e^{xt}}{g(t)(1 - yt^r)}\right]_{t=0}. \end{aligned} \tag{2.4}$$

Using (c) in Lemma 1, we get

$$W_n\left[\frac{e^{xt}}{g(t)(1 - yt^r)}\right]_{t=0} = P_n\left[\frac{1}{g(t)(1 - yt^r)}\right]_{t=0}\,W_n\left[e^{xt}\right]_{t=0}. \tag{2.5}$$

Easily, we find the following well known identity

$$W_n\left[e^{xt}\right]_{t=0} = [1 \;\; x \;\; x^2 \;\; \cdots \;\; x^n]^T. \tag{2.6}$$

Using (2.6) in (2.5) and applying the resulting identity in (2.4), we have

$$\begin{aligned} {}_{e^{(r)}}\vec{s}_n(x,y) = & W_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0} \Omega_n^{-1} \\ & \times P_n\left[\frac{1}{g(t)(1-yt^r)}\right]_{t=0} [1 \;\; x \;\; x^2 \;\; \cdots \;\; x^n]^T. \end{aligned} \tag{2.7}$$

Taking k^{th} order partial derivative with respect to x on both sides of (2.7) and dividing each side of the resulting identity by $k!$, we obtain

(2.8)
$$\begin{aligned} & \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k}\,{}_{e^{(r)}}s_0(x,y), \frac{\partial^k}{\partial x^k}\,{}_{e^{(r)}}s_1(x,y), \ldots, \frac{\partial^k}{\partial x^k}\,{}_{e^{(r)}}s_n(x,y)\right]^T \\ = & W_n\left[1, (f^{-1}(t)), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0} \Omega_n^{-1}\, P_n\left[\frac{1}{g(t)(1-yt^r)}\right]_{t=0} \\ & \times \left[0 \;\; \cdots \;\; 0 \;\; 1 \;\; \binom{k+1}{k}x \;\; \binom{k+2}{k}x^2 \;\; \cdots \;\; \binom{n}{k}x^{n-k}\right]^T. \end{aligned}$$

Finally, we observe that the right member and the left member of (2.8) are equal to the kth columns of the corresponding member of (2.3), respectively. This completes the proof. □

Theorem 3. *Let ${}_{e^{(r)}}s_n(x,y)$ be the 2-variable truncated exponential based Sheffer polynomial sequences for $(g(t), f(t))$. Then*

$$\sum_{k=0}^{n} \frac{(x\alpha_k + yr\beta_k + \gamma_k)}{k!} \frac{\partial^k}{\partial x^k}\,{}_{e^{(r)}}s_n(x,y) = {}_{e^{(r)}}s_{n+1}(x,y), \tag{2.9}$$

where

$$\alpha_k = \left(\frac{1}{f'(t)}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0),$$

$$\beta_k = \left(\frac{t^{r-1}}{(1-yt^r)f'(t)}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0)$$

and

$$\gamma_k = \left(-\frac{g'(t)}{g(t)f'(t)}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0).$$

Proof. Consider

$$\mathcal{A}(t;x,y;n,r)\big|_{t=0} := W_n\left[\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right)\right]_{t=0}. \tag{2.10}$$

Using (1.9), we get

$$\mathcal{A}(t;x,y;n,r)\big|_{t=0} = \left[{}_{e^{(r)}}s_1(x,y), {}_{e^{(r)}}s_2(x,y), \ldots, {}_{e^{(r)}}s_{n+1}(x,y)\right]^T. \tag{2.11}$$

Since $f'(0) \neq 0$, we have

$$\{f^{-1}(t)\}'\Big|_{t=0} = \frac{1}{f'\left(f^{-1}(t)\right)}\Big|_{t=0}. \tag{2.12}$$

Using (2.12), we find

$$\begin{aligned}&\mathcal{A}(t;x,y;n,r)\big|_{t=0}\\ &= W_n\left[\left(x\frac{1}{f'(f^{-1}(t))} + yr\frac{(f^{-1}(t))^{r-1}}{(1-y(f^{-1}(t))^r)f'(f^{-1}(t))} - \frac{g'(f^{-1}(t))}{g(f^{-1}(t))f'(f^{-1}(t))}\right)\right.\\ &\quad\left.\times\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right]_{t=0}.\end{aligned} \tag{2.13}$$

Using (d) in Lemma 1, we obtain

$$\begin{aligned}&\mathcal{A}(t;x,y;n,r)\big|_{t=0}\\ &= W_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0}\Omega_n^{-1}\\ &\quad\times W_n\left[\left(x\frac{1}{f'(t)} + yr\frac{t^{r-1}}{(1-yt^r)f'(t)} - \frac{g'(t)}{g(t)f'(t)}\right)\frac{\exp(xt)}{g(t)(1-yt^r)}\right]_{t=0}.\end{aligned} \tag{2.14}$$

Employing (b) and (c) in Lemma 1, we get

$$\begin{aligned}\mathcal{A}(t;x,y;n,r)\big|_{t=0} &= W_n\left[1, (f^{-1}(t)), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0}\Omega_n^{-1}\\ &\quad\times P_n\left[\frac{1}{g(t)(1-yt^r)}\right]_{t=0} P_n\left[\exp(xt)\right]_{t=0}\\ &\quad\times W_n\left[x\frac{1}{f'(t)} + yr\frac{t^{r-1}}{(1-yt^r)f'(t)} - \frac{g'(t)}{g(t)f'(t)}\right]_{t=0}.\end{aligned} \tag{2.15}$$

Using Lemma 2, we obtain

$$\begin{aligned}\mathcal{A}(t;x,y;n,r)\big|_{t=0} &= W_n[{}_{e^{(r)}}s_0(x,y), {}_{e^{(r)}}s_1(x,y), \ldots, {}_{e^{(r)}}s_n(x,y)]^T\Omega_n^{-1}\\ &\quad\times W_n\left[x\frac{1}{f'(t)} + yr\frac{t^{r-1}}{(1-yt^r)f'(t)} - \frac{g'(t)}{g(t)f'(t)}\right]_{t=0}.\end{aligned}$$

Or, equivalently,

$$\begin{aligned}&\mathcal{A}(t;x,y;n,r)\big|_{t=0}\\ &= \begin{bmatrix} {}_{e^{(r)}}s_0(x,y) & 0 & 0 & \ldots & 0\\ {}_{e^{(r)}}s_1(x,y) & \frac{{}_{e^{(r)}}{s_1}'(x,y)}{1!} & 0 & \ldots & 0\\ {}_{e^{(r)}}s_2(x,y) & \frac{{}_{e^{(r)}}{s_2}'(x,y)}{1!} & \frac{{}_{e^{(r)}}{s_2}''(x,y)}{2!} & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ {}_{e^{(r)}}s_n(x,y) & \frac{{}_{e^{(r)}}{s_n}'(x,y)}{1!} & \frac{{}_{e^{(r)}}{s_n}''(x,y)}{2!} & \ldots & \frac{{}_{e^{(r)}}{s_n}^{(n)}(x,y)}{n!}\end{bmatrix}\begin{bmatrix} x\alpha_0 + yr\beta_0 + \gamma_0\\ x\alpha_1 + yr\beta_1 + \gamma_1\\ x\alpha_2 + yr\beta_2 + \gamma_2\\ \vdots\\ x\alpha_n + yr\beta_n + \gamma_n\end{bmatrix}.\end{aligned} \tag{2.16}$$

Finally, equating nth rows of the right members of (2.11) and (2.16), we obtain the desired result (2.9). $\square$

Theorem 4. *Let ${}_{e^{(r)}}s_n(x,y)$ be the 2-variable truncated exponential based Sheffer polynomial sequences for $(g(t), f(t))$. Then*

$$ {}_{e^{(r)}}s_{n+1}(x,y) = \sum_{k=0}^{n} \binom{n}{k} (x\delta_{n-k} + yr\epsilon_{n-k} + \zeta_{n-k})\, {}_{e^{(r)}}s_k(x,y), \tag{2.17} $$

where

$$ \delta_k = \left(\frac{1}{f'(f^{-1}(t))} \right)^{(k)} \Bigg|_{t=0} \quad (k \in \mathbb{N}_0), $$

$$ \epsilon_k = \left(\frac{(f^{-1}(t))^{r-1}}{(1 - y(f^{-1}(t))^r) f'(f^{-1}(t))} \right)^{(k)} \Bigg|_{t=0} \quad (k \in \mathbb{N}_0) $$

and

$$ \zeta_k = \left(-\frac{g'(f^{-1}(t))}{g(f^{-1}(t)) f'(f^{-1}(t))} \right)^{(k)} \Bigg|_{t=0} \quad (k \in \mathbb{N}_0). $$

Proof. Applying (c) in Lemma 1 to (2.13), we obtain

$$ \begin{aligned} &\mathcal{A}(t;x,y;n,r)\big|_{t=0} \\ &= P_n \left[x \frac{1}{f'(f^{-1}(t))} + yr \frac{(f^{-1}(t))^{r-1}}{(1 - y(f^{-1}(t))^r) f'(f^{-1}(t))} - \frac{g'(f^{-1}(t))}{g(f^{-1}(t)) f'(f^{-1}(t))} \right]_{t=0} \\ &\quad \times W_n \left[\frac{\exp(f^{-1}(t))}{g(f^{-1}(t))(1 - y(f^{-1}(t))^r)} \right]_{t=0}. \end{aligned} $$

Using (a) in Lemma 1 together with (2.1) and (2.2), we have

$$ \begin{aligned} &\mathcal{A}(t;x,y;n,r)\big|_{t=0} \\ &= \begin{bmatrix} x\delta_0 + yr\epsilon_0 + \zeta_0 & 0 & 0 & \cdots & 0 \\ x\delta_1 + yr\epsilon_1 + \zeta_1 & x\delta_0 + yr\epsilon_0 + \zeta_0 & 0 & \cdots & 0 \\ x\delta_2 + yr\epsilon_2 + \zeta_2 & \binom{2}{1}(x\delta_1 + yr\epsilon_1 + \zeta_1) & x\delta_0 + yr\epsilon_0 + \zeta_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x\delta_n + yr\epsilon_n + \zeta_n & \binom{n}{1}(x\delta_{n-1} + yr\epsilon_{n-1} + \zeta_{n-1}) & \cdots & \cdots & x\delta_0 + yr\epsilon_0 + \zeta_0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} {}_{e^{(r)}}s_0(x,y) \\ {}_{e^{(r)}}s_1(x,y) \\ \vdots \\ {}_{e^{(r)}}s_{n-1}(x,y) \\ {}_{e^{(r)}}s_n(x,y) \end{bmatrix}. \end{aligned} \tag{2.18} $$

Finally, equating nth rows of right members of (2.11) and (2.18), we obtain the desired result. $\square$

Theorem 5. *Let* ${}_{e^{(r)}}s_n(x,y)$ *be the* 2*-variable truncated exponential based Sheffer polynomial sequences for* $(g(t), f(t))$*. Then*

$$\tag{2.19}\begin{aligned}&\sum_{k=0}^{n} {}_{e^{(r)}}s_{n-k+1}(x,y)\eta_k\\ &= x_{e^{(r)}}s_n(x,y) + \sum_{k=0}^{n}\binom{n}{k}(yr\theta_k+\vartheta_k)_{e^{(r)}}s_{n-k}(x,y),\end{aligned}$$

where

$$\eta_k = \left(f'(f^{-1}(t))\right)^{(k)}\Big|_{t=0} \quad (k \in \mathbb{N}_0),$$

$$\theta_k = \left(\frac{(f^{-1}(t))^{r-1}}{(1-y(f^{-1}(t))^r)}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0)$$

and

$$\vartheta_k = \left(-\frac{g'(f^{-1}(t))}{g(f^{-1}(t))}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0).$$

Proof. Consider

$$\tag{2.20}\begin{aligned}&\mathcal{B}(t;x,y;n,r)\big|_{t=0}\\ &:= W_n\left[f'(f^{-1}(t))\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right)\right]_{t=0}.\end{aligned}$$

Using (c) in Lemma 1, we get

$$\begin{aligned}&\mathcal{B}(t;x,y;n,r)\big|_{t=0}\\ &= P_n\left[\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right)\right]_{t=0} W_n\left[f'(f^{-1}(t))\right]_{t=0}.\end{aligned}$$

We find from (2.11) that

$$\tag{2.21}\begin{aligned}&\mathcal{B}(t;x,y;n,r)\big|_{t=0}\\ &= \begin{bmatrix} {}_{e^{(r)}}s_1(x,y) & 0 & 0 & \cdots & 0\\ {}_{e^{(r)}}s_2(x,y) & {}_{e^{(r)}}s_1(x,y) & 0 & \cdots & 0\\ {}_{e^{(r)}}s_3(x,y) & \binom{2}{1}{}_{e^{(r)}}s_2(x,y) & {}_{e^{(r)}}s_1(x,y)) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ {}_{e^{(r)}}s_{n+1}(x,y) & \binom{n}{1}{}_{e^{(r)}}s_n(x,y) & \binom{n}{2}{}_{e^{(r)}}s_n(x,y) & \cdots & {}_{e^{(r)}}s_1(x,y)\end{bmatrix}\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{bmatrix}.\end{aligned}$$

We also have

$$\tag{2.22}\begin{aligned}&\mathcal{B}(t;x,y;n,r)\big|_{t=0}\\ &= W_n\left[\left(x + yr\frac{(f^{-1}(t))^{r-1}}{(1-y(f^{-1}(t))^r)} - \frac{g'(f^{-1}(t))}{g(f^{-1}(t))}\right)\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right]_{t=0},\end{aligned}$$

which, upon using (a), (b) and (c) in Lemma 1, yields

$$\begin{aligned}
&\mathcal{B}(t;x,y;n,r)\big|_{t=0} \\
&= xW_n\left[\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right]_{t=0} \\
&\quad + yrP_n\left[\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right]_{t=0} W_n\left[\frac{(f^{-1}(t))^{r-1}}{(1-y(f^{-1}(t))^r)}\right]_{t=0} \\
&\quad + P_n\left[\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right]_{t=0} W_n\left[-\frac{g'(f^{-1}(t))}{g(f^{-1}(t))}\right]_{t=0}.
\end{aligned} \tag{2.23}$$

Finally, equating nth rows of the right members of (2.21) and (2.23), we obtain the desired result (2.19). ☐

Theorem 6. *Let* ${}_{e^{(r)}}s_n(x,y)$ *be the* 2*-variable truncated exponential based Sheffer polynomial sequences for* $(g(t),f(t))$. *Then*

$$n\,{}_{e^{(r)}}s_n(x,y) = \sum_{k=0}^{n} \frac{(x\kappa_k + yr\lambda_k + \mu_k)}{k!} \frac{\partial^k}{\partial x^k}\,{}_{e^{(r)}}s_{n-k}(x,y), \tag{2.24}$$

where

$$\kappa_k = \left(\frac{f(t)}{f'(t)}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0),$$

$$\lambda_k = \left(\frac{t^{r-1}f(t)}{(1-yt^r)f'(t)}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0)$$

and

$$\mu_k = \left(-\frac{g'(t)f(t)}{g(t)f'(t)}\right)^{(k)}\Bigg|_{t=0} \quad (k \in \mathbb{N}_0).$$

Proof. Consider

$$\mathcal{C}(t;x,y;n,r)\big|_{t=0} := W_n\left[t\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right)\right]_{t=0}. \tag{2.25}$$

Using (c) in Lemma 1, we have

$$\mathcal{C}(t;x,y;n,r)\big|_{t=0} = P_n\left[t\right]\Big|_{t=0} W_n\left[\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right)\right]_{t=0}.$$

Or, equivalently,

$$\mathcal{C}(t;x,y;n,r)\big|_{t=0} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & n+1 & 0 \end{bmatrix} \begin{bmatrix} {}_{e^{(r)}}s_1(x,y) \\ {}_{e^{(r)}}s_2(x,y) \\ {}_{e^{(r)}}s_3(x,y) \\ \vdots \\ {}_{e^{(r)}}s_n(x,y) \\ {}_{e^{(r)}}s_{n+1}(x,y) \end{bmatrix}. \tag{2.26}$$

We also have

$$\begin{aligned} &\mathcal{C}(t;x,y;n,r)\big|_{t=0} \\ &= W_n\left[\left(x\frac{f(f^{-1}(t))}{f'(f^{-1}(t))} + yr\frac{(f^{-1}(t))^{r-1}f(f^{-1}(t))}{(1-y(f^{-1}(t))^r)f'(f^{-1}(t))} - \frac{g'(f^{-1}(t))f(f^{-1}(t))}{g(f^{-1}(t))f'(f^{-1}(t))}\right)\right. \\ &\quad \left.\times \frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right]_{t=0}, \end{aligned}$$

which, upon using (d) in Lemma 1, yields

$$\begin{aligned} &\mathcal{C}(t;x,y;n,r)\big|_{t=0} \\ &= W_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \dots, (f^{-1}(t))^n\right]_{t=0} \Omega_n^{-1} \\ &\quad \times W_n\left[\left(x\frac{f(t)}{f'(t)} + yr\frac{t^{r-1}f(t)}{(1-yt^r)f'(t)} - \frac{g'(t)f(t)}{g(t)f'(t)}\right)\frac{\exp\left(xt\right)}{g(t)(1-yt^r)}\right]_{t=0}. \end{aligned} \tag{2.27}$$

Applying (b) and (c) in (2.27), we get

$$\begin{aligned} &\mathcal{C}(t;x,y;n,r)\big|_{t=0} \\ &= W_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \dots, (f^{-1}(t))^n\right]\Big|_{t=0} \Omega_n^{-1} P_n\left[\frac{1}{g(t)(1-yt^r)}\right]_{t=0} \\ &\quad \times P_n\left[\exp\left(xt\right)\right]_{t=0} W_n\left[x\frac{f(t)}{f'(t)} + yr\frac{t^{r-1}f(t)}{(1-yt^r)f'(t)} - \frac{g'(t)f(t)}{g(t)f'(t)}\right]\Big|_{t=0}, \end{aligned}$$

which, in view of Lemma 2, leads to

$$\begin{aligned} &\mathcal{C}(t;x,y;n,r)\big|_{t=0} \\ &= W_n\left[{}_{e^{(r)}}s_0(x,y), {}_{e^{(r)}}s_1(x,y), {}_{e^{(r)}}s_2(x,y), \dots, {}_{e^{(r)}}s_n(x,y)\right]^T \Omega_n^{-1} \\ &\quad \times W_n\left[x\frac{f(t)}{f'(t)} + yr\frac{t^{r-1}f(t)}{(1-yt^r)f'(t)} - \frac{g'(t)f(t)}{g(t)f'(t)}\right]\Big|_{t=0}. \end{aligned} \tag{2.28}$$

Finally, equating nth rows of the right members of (2.26) and (2.28), we obtain the desired result (2.24). □

3. An application

Andrews [3, p. 320] commented that the Miller-Lee polynomials $G_n^{(m)}(x)$ given by

$$G_n^{(m)}(x) = \sum_{k=0}^{n} \binom{m+n-k}{m} \frac{x^k}{k!} \tag{3.1}$$

arise in the problem of finding the probability density function for the output of a cross correlator, by referring to two related papers.

Miller-Lee polynomials $G_n^{(m)}(x)$ are given by the following generating function (see [7, p. 21, Eq. (1.11)] and [15, p. 760, Eq. (2.14)])

$$\frac{1}{(1-t)^{m+1}} \exp(xt) = \sum_{n=0}^{\infty} G_n^{(m)}(x)\, t^n \quad (|t|<1). \tag{3.2}$$

Dattoli et al. [7, p. 21, Eq. (1.9)], among integral representations of some other polynomials, presented the following integral representation

$$G_n^{(m)}(x) = \frac{1}{m!\, n!} \int_0^{\infty} e^{-t}\, t^m\, (x+t)^n\, dt. \tag{3.3}$$

Khan et al. [15, p. 760, Eq. (2.15)]) introduced Hermite-Miller-Lee polynomials ${}_HG_n^{(m)}(x,y,z)$ defined by the following generating function

$$\frac{1}{(1-t)^{m+1}} \exp\left(xt+yt^2+zt^3\right) = \sum_{n=0}^{\infty} {}_HG_n^{(m)}(x,y,z)\, t^n \quad (|t|<1). \tag{3.4}$$

They also [15, p. 760, Eqs. (2.16) and (2.17)]) pointed out that the cases $m=0$ and $m=\alpha-1$ of (3.4) give the following generating functions for, respectively, Hermite-truncated exponential polynomials ${}_He_n^{(m)}(x,y,z)$ and Hermite-modified Laguerre polynomials ${}_Hf_n^{(\alpha)}(x,y,z)$:

$$\frac{1}{1-t} \exp\left(xt+yt^2+zt^3\right) = \sum_{n=0}^{\infty} {}_He_n^{(m)}(x,y,z)\, t^n \quad (|t|<1) \tag{3.5}$$

and

$$\frac{1}{(1-t)^{\alpha}} \exp\left(xt+yt^2+zt^3\right) = \sum_{n=0}^{\infty} {}_Hf_n^{(\alpha)}(x,y,z)\, t^n \quad (|t|<1). \tag{3.6}$$

From Section 2, let ${}_{e^{(r)}}s_n(x,y)$ be the 2-variable truncated exponential based Sheffer polynomial sequences for $\left(\frac{1}{(1-t)^{m+1}}, t\right)$, which we call 2-variable truncated exponential based Miller-Lee type polynomials ${}_{e^{(r)}}G_n^{(m)}(x,y)$.

Applying the results in Theorems 3-6 to ${}_{e^{(r)}}G_n^{(m)}(x,y)$, we, respectively, get the following identities:

(3.7)
$$\begin{aligned}{}_{e^{(r)}}G_{n+1}^{(m)}(x,y) = &\left(x + ry^{\frac{1}{r}} - m - 1\right) {}_{e^{(r)}}G_n^{(m)}(x,y)\\ &+ \sum_{k=1}^{n} \left(ry^{\frac{1+k}{r}} - m - 1\right) \frac{\partial^k}{\partial x^k} {}_{e^{(r)}}G_n^{(m)}(x,y) \quad (n \in \mathbb{N}_0)\,;\end{aligned}$$

(3.8)
$$\begin{aligned}&{}_{e^{(r)}}G_{n+1}^{(m)}(x,y)\\ &= \left(ry^{\frac{1}{r}} - m - 1\right) {}_{e^{(r)}}G_n^{(m)}(x,y)\\ &\quad + \sum_{k=0}^{n-1} \binom{n}{k} \left(x + r(n-k)!y^{\frac{1+n-k}{r}} - (m+1)(n-k)!\right) {}_{e^{(r)}}G_{n-k}^{(m)}(x,y) \quad (n \in \mathbb{N}_0)\,;\end{aligned}$$

$$\begin{aligned}&{}_{e^{(r)}}G_{n+1}^{(m)}(x,y) - x\,{}_{e^{(r)}}G_n^{(m)}(x,y) + \sum_{k=1}^{n} {}_{e^{(r)}}G_{n-k+1}^{(m)}(x,y)\\ &= \sum_{k=0}^{n} \binom{n}{k} \left(ry^{\frac{1+k}{r}} - (m+1)\right) k!\,{}_{e^{(r)}}G_{n-k}^{(m)}(x,y) \quad (n \in \mathbb{N}_0)\,;\end{aligned} \tag{3.9}$$

$$\begin{aligned}&(n-r)\,{}_{e^{(r)}}G_n^{(m)}(x,y)\\ &= \left(x + ry^{\frac{1}{r}} - m - 1\right) \frac{\partial}{\partial x} {}_{e^{(r)}}G_{n-1}^{(m)}(x,y)\\ &\quad + \sum_{k=2}^{n} \frac{1}{k!} \left(x + ry^{\frac{k}{r}} k! - (m+1)k!\right) \frac{\partial^k}{\partial x^k} {}_{e^{(r)}}G_{n-k}^{(m)}(x,y) \quad (n \in \mathbb{N}_0)\,.\end{aligned} \tag{3.10}$$

4. Remarks

In addition to the results in Theorems 3-6, we can obtain a number of identities involving the 2-variable truncated exponential based Sheffer polynomial sequences for $(g(t), f(t))$ by considering some combinations different from $\mathcal{A}(t;x,y;n,r)\big|_{t=0}$, $\mathcal{B}(t;x,y;n,r)\big|_{t=0}$, and $\mathcal{C}(t;x,y;n,r)\big|_{t=0}$, for example,

$$W_n \left[f'(f^{-1}(t))g(f^{-1}(t)) \frac{d}{dt} \left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1 - y(f^{-1}(t))^r)} \right) \right]_{t=0},$$

$$\begin{aligned}&W_n \left[\frac{1}{g(f^{-1}(t))(1 - y(f^{-1}(t))^r)} \right]_{t=0}\\ &= W_n \left[\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1 - y(f^{-1}(t))^r)} \exp\left(-xf^{-1}(t)\right) \right]_{t=0},\end{aligned}$$

$$W_n \left[g(f^{-1}(t)) \frac{d}{dt} \left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1 - y(f^{-1}(t))^r)} \right) \right]_{t=0},$$

$$\begin{aligned} & W_n\left[\frac{\exp\left(xf^{-1}(t)\right)}{(1-y(f^{-1}(t))^r)}\right]_{t=0} \\ &= W_n\left[g(f^{-1}(t))\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right]_{t=0}, \end{aligned}$$

and

$$W_n\left[tf'(f^{-1}(t))\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)\right)}{g(f^{-1}(t))(1-y(f^{-1}(t))^r)}\right)\right]_{t=0}.$$

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