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ON A NONLOCAL PROBLEM WITH INDEFINITE WEIGHTS IN ORLICZ-SOBOLEV SPACE

Mustafa Avci and Nguyen Thanh Chung

Abstract. In this paper, we consider a class of nonlocal problems with indefinite weights in Orlicz-Sobolev space. Under some suitable conditions on the nonlinearities, we establish some existence results using variational techniques and Ekeland's variational principle.

1. Introduction

In this paper, we are interested in the existence of solutions for the following nonlocal problem with indefinite weights in Orlicz-Sobolev space:

$$\begin{cases} -M\left(\int_\Omega (\Phi_1(|\nabla u|)+\Phi_2(|\nabla u|))\,dx\right)\operatorname{div}\left((a_1(|\nabla u|)+a_2(|\nabla u|))\nabla u\right) \\ =\lambda V_1(x)|u|^{q_1(x)-2}u-\mu V_2(x)|u|^{q_2(x)-2}u, \quad x\in\Omega, \\ u=0, \quad x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$; $\lambda, \mu > 0$ are two real parameters; $V_i : \Omega \to \mathbb{R}$ $(i = 1, 2)$ are two weight functions; $q_i : \overline{\Omega} \to (1, +\infty)$ are continuous functions; $M : \mathbb{R}_0^+ := [0, +\infty) \to \mathbb{R}_0^+$ is an increasing and continuous function; $a_i : (0, \infty) \to \mathbb{R}$, $i = 1, 2$, are two functions satisfying some specific conditions.

Equations of type (1.1) can be particularised to many well-known problems involving variable exponent. For example, if we let $a_i(t) = |t|^{p(x)-2}$, where $p(x)$ is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, Equation (1.1) turns into the $p(x)$-Kirchhoff-type equation. If we additionally consider the case $M(t) = 1$, Equation (1.1) becomes the $p(x)$-Laplace equation, a generalization of p-Laplace equation given by $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x, u)$, $1 < p < N$. This kind of equations have been intensively studied by many authors for the past two decades due to its significant role in many fields of mathematics, such as in the study of calculus of variations, partial differential equations [1, 20, 21], but also for their use in a variety of physical and engineering contexts: the

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modeling of electrorheological fluids [35], the analysis of Non-Newtonian fluids [38], fluid flow in porous media [4], magnetostatics [13], image restoration [11], and capillarity phenomena [5], see also, e.g., [3, 6, 8, 12, 16, 17, 19, 24, 27, 30, 37] and references therein. Therefore, Equation (1.1) may represent a variety of mathematical models corresponding to certain phenomena:
For $\varphi(t) := p|t|^{p-2}t$;

- Nonlinear elasticity: $\varphi(t) = \left(1+t^2\right)^{\alpha} - 1$, $\alpha > \frac{1}{2}$,
- Plasticity: $\varphi(t) = t^{\alpha}\left(\log\left(1+t\right)\right)^{\beta}$, $\alpha \geq 1, \beta > 0$,
- Generalized Newtonian fluids: $\varphi(t) = \int_0^t s^{1-\alpha}\left(\sinh^{-1} s\right)^{\beta} ds$, $0 \leq \alpha \leq 1, \beta > 0$.

For $\varphi(t) = \varphi(x,t) := p(x)|t|^{p(x)-2}t$;

- There is a new model for image restoration given in [14]. In this model, main aim is to recover an image u, from an observed, noisy image u_0, where the two are related by $u_0 = u + \text{noise}$. The proposed model incorporates the strengths of the various types of diffusion arising from the minimization problem

  $$E(u) = \int_\Omega \left[|\nabla u|^{p(x)} + \lambda\left(u - u_0\right)^2\right] dx$$

  for $1 \leq p\left(x\right) \leq 2$, where $\int_\Omega |\nabla u|^{p(x)}\, dx$ is a regularizing term to remove the noise and $\lambda \geq 0$.

Motivated by the results on nonhomogeneous problems in Orlicz-Sobolev spaces introduced in [26, 31, 32, 36] and some of our results on the nonlocal case for these problems in [7, 15–17, 27], we study the existence of solutions for problem (1.1) with indefinite weights and multiple parameters. In [31], Mihailescu et al. considered the following problem:

$$(1.2) \qquad \begin{cases} -\operatorname{div}\left(\left(a_1(|\nabla u|) + a_2(|\nabla u|)\right)\nabla u\right) = \lambda|u|^{q(x)-2}u, & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{cases}$$

Using variational techniques, the authors established the existence of two positive constants λ_0, λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, +\infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (1.2). In [32], the authors obtained some similar results in the case when sign-changing potentials are involved. Interested readers are referred to [36], in which the author studied the existence of solutions for (1.2) with multiple parameters. In a recent paper [26], Ge has considered the eigenvalue problem:

$$(1.3) \qquad \begin{cases} -\operatorname{div}\left(a(|\nabla u|)\nabla u\right) = \lambda V(x)|u|^{q(x)-2}u, & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases}$$

where V is an indefinite sign-changing weight and λ is a positive parameter. Using variational methods, the author proved that any $\lambda > 0$ sufficiently small is an eigenvalue of problem (1.3). The purpose of this paper is consider problem (1.1) under suitable conditions on the weights V_i, $i = 1, 2$ as well as the parameters λ and μ. As we will see, our results are natural extensions from the papers mentioned above. We believe that the obtained results are new even in the local case $M(t) \equiv 1$, see [36]. Finally, it should be noticed that the Kirchhoff function here is allowed to be degenerate at zero which makes some difficulties in applying variational methods, we refer to [15–17, 23] for more details.

2. Preliminaries

In order to study problem (1.1), let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz and Orlicz-Sobolev spaces, useful for what follows, for more details we refer the readers to the monographs [2, 33, 34], and the papers [7, 9, 10, 18, 28, 31].

Assume that $a_i : (0, \infty) \to \mathbb{R}$, $i = 1, 2$, are two functions such that the odd mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi_i(t) := \begin{cases} a_i(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. For the functions φ_i above, let us define

$$\Phi_i(t) = \int_0^t \varphi_i(s)ds \quad \text{for all } t \in \mathbb{R},\ i = 1, 2,$$

on which will be imposed some suitable conditions later.

For φ_i and Φ_i defined above, we can see that Φ_i are Young functions, that is, $\Phi_i(0) = 0$, Φ_i are convex, and $\lim_{t\to\infty} \Phi_i(t) = +\infty$. Furthermore, since $\Phi_i(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{\Phi_i(t)}{t} = 0$, and $\lim_{t\to\infty} \frac{\Phi_i(t)}{t} = +\infty$, the functions Φ_i, $i = 1, 2$, are then called N-functions. Let us define the function Φ_i^* by the formula

$$\Phi_i^*(t) = \int_0^t \varphi_i^{-1}(s)ds \ \text{ for all } t \in \mathbb{R},\ i = 1, 2,$$

which are called the complementary functions of Φ_i and they satisfy

$$\Phi_i^*(t) = \sup\{st - \Phi_i(s) :\ s \geq 0\} \quad \text{for all } t \geq 0,\ i = 1, 2.$$

We observe that the functions Φ_i^* are also N-functions in the sense above and the following Young inequality holds

$$st \leq \Phi_i(s) + \Phi_i^*(t) \quad \text{for all } s, t \geq 0,\ i = 1, 2.$$

The Orlicz classes defined by the N-functions Φ_i, $i = 1, 2$, are the sets

$$K_{\Phi_i}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is measurable} : \int_\Omega \Phi_i(|u(x)|)\, dx < \infty \right\}$$

and the Orlicz spaces $L_{\Phi_i}(\Omega)$ are then defined as the linear hulls of the sets $K_{\Phi_i}(\Omega)$. The spaces $L_{\Phi_i}(\Omega)$ are Banach spaces under the following Luxemburg norms

$$\|u\|_{\Phi_i} := \inf\left\{k > 0 : \int_\Omega \Phi_i\left(\frac{u(x)}{k}\right)\, dx \le 1\right\}$$

or the equivalent Orlicz norms

$$\|u\|_{L_{\Phi_i}} := \sup\left\{\left|\int_\Omega u(x)v(x)\, dx\right| : v \in K_{\Phi_i^*}(\Omega),\ \int_\Omega \Phi_i^*(|v(x)|)\, dx \le 1\right\},$$

respectively. For Orlicz spaces, the Hölder inequality reads as follows (see [34]):

$$\int_\Omega uv\, dx \le 2\|u\|_{L_{\Phi_i}}\|u\|_{L_{\Phi_i^*}} \quad \text{for all } u \in L_{\Phi_i} \text{ and } v \in L_{\Phi_i^*}.$$

The Orlicz-Sobolev spaces $W^1L_{\Phi_i}$ building upon $L_{\Phi_i}(\Omega)$ are the spaces defined by

$$W^1L_{\Phi_i}(\Omega) := \left\{u \in L_{\Phi_i}(\Omega) : \frac{\partial u}{\partial x_l} \in L_{\Phi_i}(\Omega),\ l = 1, 2, \ldots, N\right\}$$

and they are Banach spaces with respect to the norm

$$\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \||\nabla u|\|_{\Phi_i}.$$

Now, we introduce the Orlicz-Sobolev spaces $W_0^1L_{\Phi_i}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_{\Phi_i}(\Omega)$. It turns out that the spaces $W_0^1L_{\Phi_i}(\Omega)$, $i = 1, 2$, can be renormed by using as an equivalent norms

$$\|u\|_i := \||\nabla u|\|_{\Phi_i}.$$

Throughout this paper, we assume that Φ_i and Φ_i^* satisfy the Δ_2-conditions at infinity, $i = 1, 2$, namely,

$$1 < (\varphi_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \le (\varphi_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} < \infty, \quad t \ge 0. \tag{2.1}$$

Furthermore, we also need the following conditions

$$\text{the function } t \mapsto \Phi_i(\sqrt{t}) \text{ are convex for all } t \in [0, \infty),\ \ i = 1, 2, \tag{2.2}$$

and

$$\lim_{t\to 0} \int_t^1 \frac{(\Phi_i)^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds < +\infty \text{ and } \lim_{t\to+\infty} \int_1^t \frac{(\Phi_i)^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds = +\infty \tag{2.3}$$

which help us to define the Orlicz-Sobolev conjugates $(\Phi_i)_*$ of Φ_i, $i = 1, 2$, which are given by the formula

$$(\Phi_i)_*^{-1}(t) = \int_0^t \frac{(\Phi_i)^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds. \tag{2.4}$$

We notice that Orlicz-Sobolev spaces, unlike the Sobolev spaces they generalize, are in general neither separable nor reflexive. A key tool to guarantee these properties is represented by the Δ_2-condition (2.1). Actually, condition (2.1) assures that both $L_{\Phi_i}(\Omega)$ and $W_0^1L_{\Phi_i}(\Omega)$ are separable, see [2]. Conditions

(2.1) and (2.2) assure that $L_{\Phi_i}(\Omega)$ are uniformly convex spaces and thus, reflexive Banach spaces (see [31]); consequently, the Orlicz-Sobolev spaces $W_0^1L_{\Phi_i}(\Omega)$ are also reflexive Banach spaces.

Proposition 2.1 (see [18,31]). *Let* $u \in W_0^1L_{\Phi_i}(\Omega)$, $i = 1, 2$. *Then we have*

(i) $\|u\|_i^{(\varphi_i)^0} \leq \int_\Omega \Phi_i(|\nabla u(x)|)\,dx \leq \|u\|_i^{(\varphi_i)_0}$ *if* $\|u\|_i < 1$.

(ii) $\|u\|_i^{(\varphi_i)_0} \leq \int_\Omega \Phi_i(|\nabla u(x)|)\,dx \leq \|u\|_i^{(\varphi_i)^0}$ *if* $\|u\|_i > 1$.

Next, we recall in what follows some definitions and basic properties of the generalized Lebesgue space $L^{p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [21,35], the paper of Kováčik et al. [29].
Set

$$C_+(\overline{\Omega}) := \{h;\ h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

It is said that $h(x) \in L_+^\infty(\Omega)$ when

$$1 < h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x) \text{ and } h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x) < \infty.$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u : \text{a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}$$

with respect to the following so-called *Luxemburg norm* defined by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)} \tag{2.5}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations hold

$$|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} \tag{2.6}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \tag{2.7}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow \ \rho_{p(x)}(u_n - u) \to 0. \tag{2.8}$$

Proposition 2.2. *Let $p(x)$ and $q(x)$ be measurable functions such that $p \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq +\infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ and $u \neq 0$. Then we have*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|_{p(x)q(x)}^{p^+} \leq ||u|^{p(x)}|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-},$$

$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|_{p(x)q(x)}^{p^-} \leq ||u|^{p(x)}|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+}.$$

In particular, if $p(x) = p$ is a constant, then $||u|^p|_{q(x)} = |u|_{pq(x)}^p$.

3. Main results

In this section, we will state and prove our main results. The solutions of problem (1.1) will be found in the space $X = W_0^1 L_{\Phi_1}(\Omega)$. Throughout this paper, we denote by c_i general positive number whose value may change from place to place.

Let $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing and continuous function. Assume that the functions $q_i, s_i \in L_+^\infty(\Omega) \cap C_+(\overline{\Omega})$, $i = 1, 2$. Set $q_{\max}(x) := \max\{q_1(x), q_2(x)\}$ and $s_{\min}(x) := \min\{s_1(x), s_2(x)\}$, $x \in \overline{\Omega}$ and let us introduce the following conditions:

- (M_0) $m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}, \quad \forall t \geq 0, \quad m_2 \geq m_1 > 0, \quad \alpha > 1.$
- (H_1) $1 < q_{\max}(x) < \alpha(\varphi_2)_0 \leq \alpha(\varphi_2)^0 < \alpha(\varphi_1)_0 \leq \alpha(\varphi_1)^0 < s_{\min}(x)$ for all $x \in \overline{\Omega}$;
- (H_2) $\lim_{t \to +\infty} \frac{|t|^{\frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha q_{\max}^+}}}{(\Phi_2)_*(kt)} = 0$ for all $k > 0$;
- (H_3) $V_i \in L^{\frac{s_i(x)}{\alpha}}(\Omega)$, $i = 1, 2$, and there exists a measurable set $\Omega_0 \subset\subset \Omega$ of positive measure such that $V_1(x) > 0$ for all $x \in \Omega_0$, and $V_2(x) \geq 0$ for all $x \in \Omega$;
- (H_4) $\inf_{x \in \overline{\Omega}_0} q_1(x) < \min\left\{\alpha(\varphi_2)_0, \inf_{x \in \overline{\Omega}_0} q_2(x)\right\}$.

Remark 3.1. *Assume that $q_1(x), q_2(x), s_1(x), s_2(x) \in L_+^\infty(\Omega) \cap C_+(\overline{\Omega})$. From (H_1) and (2.3), (2.4), it is clear that for all $u \in X$,*

$$\left|\int_\Omega \frac{V_i(x)}{q_i(x)} |u|^{q_i(x)}\, dx\right| \leq \frac{1}{q_i^-} |V|_{\frac{s_i(x)}{\alpha}} ||u|^{q_i(x)}|_{\frac{s_i(x)}{s_i(x) - \alpha}}$$

$$= \begin{cases} \dfrac{1}{q_i^-}|V|_{\frac{s_i(x)}{\alpha}}|u|^{q_i^-}_{\frac{s_i(x)q_i(x)}{s_i(x)-\alpha}} & \textit{if } |u|_{\frac{s_i(x)q_i(x)}{s_i(x)-\alpha}} \leq 1, \\ \dfrac{1}{q_i^-}|V|_{\frac{s_i(x)}{\alpha}}|u|^{q_i^+}_{\frac{s_i(x)q_i(x)}{s_i(x)-\alpha}} & \textit{if } |u|_{\frac{s_i(x)q_i(x)}{s_i(x)-\alpha}} \geq 1, \quad i = 1, 2. \end{cases}$$

We set $h_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha}$ *and* $g_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha q_i(x)}$, *then* $h_i(x) < g_i(x)$ *for all* $x \in \overline{\Omega}$. *By the condition* (H_1), *it follows that the embedding* $X = W_0^1 L_{\Phi_1}(\Omega) \hookrightarrow W_0^1 L_{\Phi_2}(\Omega)$ *is continuous. Moreover, by the condition* (H_2) *and the fact that* $\frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha} < \frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha q_{\max}^+}$, *the embeddings* $W_0^1 L_{\Phi_2}(\Omega) \hookrightarrow L^{\frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha}}(\Omega)$ *and* $W_0^1 L_{\Phi_2}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha q_{\max}^+}}(\Omega)$ *are continuous and compact. As a result, we deduce that the embeddings* $X \hookrightarrow\hookrightarrow L^{\frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha}}(\Omega)$ *and* $X \hookrightarrow\hookrightarrow L^{\frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha q_{\max}^+}}(\Omega)$ *are continuous and compact. On the other hand, we have* $h_i(x) < g_i(x) < \frac{s_{\min}^- q_{\max}^+}{s_{\min}^- - \alpha q_{\max}^+}$ *for all* $x \in \overline{\Omega}$, $i = 1, 2$. *Therefore, the embeddings* $X \hookrightarrow\hookrightarrow L^{h_i(x)}(\Omega)$ *and* $X \hookrightarrow\hookrightarrow L^{g_i(x)}(\Omega)$ *are continuous and compact.*

Definition 3.2. We say that $u \in X$ is a weak solution of problem (1.1) if it holds that

$$\begin{aligned} M\left(\int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\,dx\right) \int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\,\nabla u \cdot \nabla v\,dx \\ -\lambda \int_\Omega V_1(x)|u|^{q_1(x)-2}uv\,dx + \mu \int_\Omega V_2(x)|u|^{q_2(x)-2}uv\,dx = 0 \end{aligned}$$

for all $v \in X$.

For each $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, let us consider the functional $J_{\lambda,\mu} : X \to \mathbb{R}$ associated to problem (1.1) as follows

$$\begin{aligned} J_{\lambda,\mu}(u) = \widehat{M}\left(\int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\,dx\right) - \lambda \int_\Omega \frac{V_1(x)}{q_1(x)}|u|^{q_1(x)}\,dx \\ + \mu \int_\Omega \frac{V_2(x)}{q_2(x)}|u|^{q_2(x)}\,dx, \end{aligned}$$

we then, by applying standard arguments, get $J_{\lambda,\mu} \in C^1(X, \mathbb{R})$ and its derivative is

$$\begin{aligned} & J'_{\lambda,\mu}(u)(v) \\ & = M\left(\int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\,dx\right) \int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\,\nabla u \cdot \nabla v\,dx \\ & \quad - \lambda \int_\Omega V_1(x)|u|^{q_1(x)-2}uv\,dx + \mu \int_\Omega V_2(x)|u|^{q_2(x)-2}uv\,dx \end{aligned}$$

for all $u, v \in X$. Hence, weak solutions of (1.1) are exactly the critical points of $J_{\lambda,\mu}$ and they will be found in X by using variational methods.

The main results of the present paper are the following:

Theorem 3.3. *Assume that the conditions* (M_0) *and* (H_1)-(H_3) *hold. Then, for all* $\mu > 0$*, there exists* $\overline{\lambda} > 0$ *such that for any* $\lambda \in [\overline{\lambda}, +\infty)$*, that is, when* λ *is large enough, problem* (1.1) *has at least one nontrivial weak solution.*

Theorem 3.4. *Assume that the conditions* (M_0) *and* (H_1)-(H_4) *hold. Then, for all* $\mu > 0$*, there exists* $\underline{\lambda}$ *such that for all* $\lambda \in (0, \underline{\lambda})$*, that is, when* λ *is small enough, problem* (1.1) *has at least one non-trivial weak solution with negative energy.*

Now, we give an auxiliary result.

Lemma 3.5. *The functional* $J_{\lambda,\mu}$ *is coercive on* X*.*

Proof. Let $\|u\|_1 > 1$. By the conditions (H_1)-(H_2) and Remark 3.1, there exists $c_1 > 0$ such that

$$|u|_{h_i(x)} \leq c_1 \|u\|_1, \quad \forall u \in X, \tag{3.1}$$

where $h_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha}$, $i = 1, 2$. Hence, by the condition (M_0), the Hölder inequality, Proposition 2.1, we deduce that

$$\begin{aligned} J_{\lambda,\mu}(u) = {} & \widehat{M}\left(\int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\, dx\right) - \lambda \int_\Omega \frac{V_1(x)}{q_1(x)} |u|^{q_1(x)}\, dx \\ & + \mu \int_\Omega \frac{V_2(x)}{q_2(x)} |u|^{q_2(x)}\, dx \\ \geq {} & \frac{m_1}{\alpha}\left(\int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\, dx\right)^\alpha - \frac{2\lambda}{q_1^-} |V_1|_{\frac{s_1(x)}{\alpha}} \left||u|^{q_1(x)}\right|_{\frac{s_1(x)}{s_1(x)-\alpha}} \\ \geq {} & \frac{m_1}{\alpha} \|u\|_1^{\alpha(\varphi_1)_0} - \frac{2\lambda}{q_1^-} |V_1|_{\frac{s_1(x)}{\alpha}} \min\left\{ |u|_{h_1(x)}^{q_1^-}, |u|_{h_1(x)}^{q_1^+} \right\} \\ \geq {} & \frac{m_1}{\alpha} \|u\|_1^{\alpha(\varphi_1)_0} - \frac{2\lambda}{q_1^-} |V_1|_{\frac{s_1(x)}{\alpha}} \min\left\{ c_1^{q_1^-} \|u\|_1^{q_1^-}, c_1^{q_1^+} \|u\|_1^{q_1^+} \right\}. \end{aligned}$$

Since $q_1^+ \leq q_{\max}^+ < \alpha(\varphi_2)_0 < \alpha(\varphi_1)_0$, we infer that $J_{\lambda,\mu}(u) \to +\infty$ as $\|u\|_1 \to +\infty$, which means that the functional $J_{\lambda,\mu}$ is coercive on X. ☐

Proof of Theorem 3.3. Set

$$\Theta(u) = \int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\, dx \tag{3.2}$$

and

$$\Upsilon(u) = -\lambda \int_\Omega \frac{V_1(x)}{q_1(x)} |u|^{q_1(x)}\, dx + \mu \int_\Omega \frac{V_2(x)}{q_2(x)} |u|^{q_2(x)}\, dx.$$

Then

$$J_{\lambda,\mu} = \widehat{M}(\Theta) + \Upsilon.$$

Let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u \in X$. Notice that due to the growth condition $(\varphi_2)^0 < (\varphi_1)_0$ (see (H_1)), we have the continuous embedding $X \hookrightarrow W_0^1 L_{\Phi_2}(\Omega)$ (see Remark 3.1) which means that $u_n \rightharpoonup u \in W_0^1 L_{\Phi_2}(\Omega)$.

On the other hand, since Φ_i are convex, the functional $\Theta(u)$ is weakly lower semi-continuous, namely

$$\Theta(u) \leq \liminf_{n\to\infty} \Theta(u_n). \tag{3.3}$$

If we consider (M_0), which means that $\widehat{M}$ is a continuous and monotone function, along with (3.3), it reads

$$\liminf_{n\to\infty} \widehat{M}(\Theta(u_n)) = \widehat{M}\left(\liminf_{n\to\infty} \Theta(u_n\right) \geq \widehat{M}(\Theta(u). \tag{3.4}$$

From (H_2), it holds that $\lim_{t\to+\infty} \frac{|t|^{q^+_{\max}}}{(\Phi_2)_*(kt)} = 0$ for all $k > 0$. If we consider this fact along with (2.3), we obtain that $W^1_0 L_{\Phi_2}(\Omega)$ is embedded compactly in $L^{q^+_{\max}}(\Omega)$ (see [25]). It is well known that $L^{q^+_{\max}}(\Omega)$ is embedded continuously in $L^{q_1(x)}(\Omega)$ and $L^{q_2(x)}(\Omega)$. As a result, by the continuous embedding $X \hookrightarrow W^1_0 L_{\Phi_2}(\Omega)$, we have the following compact embeddings

$$X \hookrightarrow\hookrightarrow L^{q_1(x)}(\Omega)$$

and

$$W^1_0 L_{\Phi_2}(\Omega) \hookrightarrow\hookrightarrow L^{q_2(x)}(\Omega).$$

On the other hand, by Remark 3.1, we have the compact embedding

$$W^1_0 L_{\Phi_2}(\Omega) \hookrightarrow\hookrightarrow L^{h_i(x)}(\Omega).$$

Then, applying the Young's inequality to $\frac{V_i(x)}{q_i(x)}|u_n(x)|^{q_i(x)}$ for the conjugate exponents $\delta(x) = \frac{s_i(x)}{\alpha}$ and $\delta(x)^* = \frac{s_i(x)}{s_i(x)-\alpha}$, and considering Remark 3.1 once more we get

$$\begin{aligned} \left|\frac{V_i(x)}{q_i(x)}|u_n(x)|^{q_i(x)}\right| &\leq \frac{1}{q_i(x)}\left(\frac{1}{\delta(x)}|V_i(x)|^{\delta(x)} + \frac{1}{\delta(x)^*}||u_n(x)|^{q_i(x)}|^{\delta(x)^*}\right) \\ &\leq K(x) \in L^1(\Omega) \end{aligned} \tag{3.5}$$

for all $x \in \Omega$ and $n \in \mathbb{N}$, $i = 1, 2$. Therefore, by the Lebesgue convergence theorem, up to a subsequence still denoted by (u_n), we have

$$\lim_{n\to\infty} \int_\Omega \frac{V_1(x)}{q_1(x)}|u_n|^{q_1(x)}dx = \int_\Omega \frac{V_1(x)}{q_1(x)}|u|^{q_1(x)}dx,$$

$$\lim_{n\to\infty} \int_\Omega \frac{V_2(x)}{q_2(x)}|u_n|^{q_2(x)}dx = \int_\Omega \frac{V_2(x)}{q_2(x)}|u|^{q_2(x)}dx,$$

that is,

$$\Upsilon(u) = \lim_{n\to\infty} \Upsilon(u_n). \tag{3.6}$$

From (3.4) and (3.6), we conclude that

$$J_{\lambda,\mu} \leq \liminf_{n\to\infty} J_{\lambda,\mu}(u_n), \tag{3.7}$$

that is, functional $J_{\lambda,\mu}$ is weakly lower semi-continuous on X. By Lemma 3.5, $J_{\lambda,\mu}$ is coercive, so it has a global minimum point $u_{\lambda,\mu} \in X$, which in turn becomes a weak solution of problem (1.1). Next, we show that $u_{\lambda,\mu}$ is not trivial. Let $t_0 > 1$ be a fixed real number, and let $u_* \in C_0^\infty(\Omega)$ such that $u_*(x) = t_0$ in $\overline{\Omega}_1$ and $0 \leq u_*(x) \leq t_0$ in $\Omega \setminus \Omega_1$, where Ω_1 is an open subset of Ω such that $\Omega_1 \subseteq \Omega_0$. Therefore, it reads

$$\begin{aligned} J_{\lambda,\mu}(u_*) &= \widehat{M}\left(\int_\Omega (\Phi_1(|\nabla u_*|) + \Phi_2(|\nabla u_*|))\,dx\right) - \lambda \int_\Omega \frac{V_1(x)}{q_1(x)} |u_*|^{q_1(x)}\,dx \\ &\quad + \mu \int_\Omega \frac{V_2(x)}{q_2(x)} |u_*|^{q_2(x)}\,dx \\ &\leq \frac{m_2}{\alpha}\left(\int_\Omega (\Phi_1(|\nabla u_*|) + \Phi_2(|\nabla u_*|))\,dx\right)^\alpha - \lambda \int_{\Omega_1} \frac{V_1(x)}{q_1(x)} |u_*|^{q_1(x)}\,dx \\ &\quad + \frac{\mu c_2}{q_2^-} \\ &\leq \frac{c_3 m_2}{\alpha} - \frac{\lambda t_0^{q_1^+} c_4}{q_1^+} + \frac{\mu c_1}{q_2^-} \leq c_5 - \frac{\lambda t_0^{q_1^+} c_4}{q_1^+}. \end{aligned}$$

Thus, $J_{\lambda,\mu}(u_*) < 0$ provided λ is large enough, that is, there exists $\overline{\lambda} > 0$ such that for any $\lambda \in [\overline{\lambda}, +\infty)$, $J_{\lambda,\mu}(u_{\lambda,\mu}) < 0$, and hence, $u_{\lambda,\mu}$ is not trivial. □

In the rest of the paper, we will prove Theorem 3.4 by using variational techniques and Ekeland's variational principle. We first have to obtain the following result.

Lemma 3.6. *Assume that the conditions* (M_0) *and* (H_1)-(H_3) *hold. Then for all* $\rho \in (0,1)$ *there exist* $\underline{\lambda} > 0$ *and a constant* $a > 0$ *such that for all* $u \in X$ *with* $\|u\|_1 = \rho$ *we have* $J_{\lambda,\mu}(u) \geq a$ *for any* $\lambda \in (0, \underline{\lambda})$.

Proof. Let us assume that $\|u\|_1 < \min\left\{1, \frac{1}{c_1}\right\}$, where c_1 is given by (3.1). It follows that $|u|_{h_i(x)} < 1$, where $h_i(x) = \frac{s_i(x)q_i(x)}{s_i(x)-\alpha}$, $i = 1, 2$. Using relations (2.1), (3.1), the condition (M_0) and Remark 3.1, we deduce that for any $u \in X$ with $\|u\|_1 = \rho \in (0,1)$ the following inequalities hold true

$$\begin{aligned} J_{\lambda,\mu}(u) &= \widehat{M}\left(\int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\,dx\right) - \lambda \int_\Omega \frac{V_1(x)}{q_1(x)} |u|^{q_1(x)}\,dx \\ &\quad + \mu \int_\Omega \frac{V_2(x)}{q_2(x)} |u|^{q_2(x)}\,dx \\ &\geq \frac{m_1}{\alpha}\left(\int_\Omega (\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|))\,dx\right)^\alpha - \lambda \int_\Omega \frac{V_1(x)}{q_1(x)} |u|^{q_1(x)}\,dx \\ &\geq \frac{m_1}{\alpha} \|u\|_1^{\alpha(\varphi_1)^0} - \frac{2\lambda}{q_1^-} |V_1|_{\frac{s_1(x)}{\alpha}} \left||u|^{q_1(x)}\right|_{\frac{s_1(x)}{s_1(x)-\alpha}} \\ &\geq \frac{m_1}{\alpha} \|u\|_1^{\alpha(\varphi_1)^0} - \frac{2\lambda}{q_1^-} |V_1|_{\frac{s_1(x)}{\alpha}} |u|_{h_1(x)}^{q_1^-} \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{m_1}{\alpha}\|u\|_1^{\alpha(\varphi_1)^0} - \frac{2\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}} c_1^{q_1^-}\|u\|_1^{q_1^-}\\
&= \rho^{q_1^-}\left(\frac{m_1}{\alpha}\rho^{\alpha(\varphi_1)^0-q_1^-} - \frac{2\lambda}{q_1^-}c_1^{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}\right).
\end{aligned}
$$

This inequality shows that if we choose

$$
\underline{\lambda} = \frac{m_1 q_1^-}{4\alpha c_1^{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}}\rho^{\alpha(\varphi_1)^0-q_1^-}, \tag{3.8}
$$

then for all $\lambda \in (0, \underline{\lambda})$ and for all $u \in X$ with $\|u\|_1 = \rho$, there exists $a > 0$ such that $J_{\lambda,\mu}(u) \geq a > 0$. The proof of Lemma 3.6 is complete. ☐

Lemma 3.7. *Assume that the conditions* (M_0) *and* (H_1)*-*(H_4) *hold. Then, there exists* $u_0 \in X$ *such that* $u_0 \geq 0$*,* $u_0 \neq 0$ *and* $J_{\lambda,\mu}(tu_0) < 0$ *for all* $t > 0$ *small enough.*

Proof. Set $q_{i,0} := \inf_{x\in\overline{\Omega}_0} q_i(x)$, $i = 1, 2$ and $\theta_0 := \min\{\alpha(\varphi_2)_0, q_{2,0}\}$. Since $q_{1,0}^- < \theta_0$, let $\epsilon_0 > 0$ be such that $q_{1,0}^- + \epsilon_0 < \theta_0$. Since $q_1 \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_2 \subset\subset \Omega_0$ such that $|q_1(x) - \theta_0| < \epsilon_0$ for all $x \in \Omega_2$. Thus, $q_1(x) \leq q_{1,0}^- + \epsilon_0 < \theta_0$ for all $x \in \Omega_2$.

Let $u_0 \in C_0^\infty(\Omega_0)$ be such that $supp(u_0) \subset \Omega_2 \subset\subset \Omega_0$, $u_0 = 1$ in a subset $\Omega_2' \subset supp(u_0)$, $0 \leq u_0 \leq 1$ in Ω_2. Therefore, applying the well-known inequality

$$
(s+t)^\gamma \leq 2^{\gamma-1}(s^\gamma + t^\gamma), \quad \forall s, t \geq 0, \quad \gamma \geq 1,
$$

for any $t \in (0, 1)$ we have

$$
\begin{aligned}
J_{\lambda,\mu}(tu_0) &= \widehat{M}\left(\int_\Omega (\Phi_1(|\nabla tu_0|) + \Phi_2(|\nabla tu_0|))\,dx\right)\\
&\quad - \lambda\int_\Omega \frac{t^{q_1(x)}}{q_1(x)}V_1(x)|tu_0|^{q_1(x)}\,dx + \mu\int_\Omega \frac{t^{q_2(x)}}{q_2(x)}V_2(x)|tu_0|^{q_2(x)}\,dx\\
&\leq \frac{m_2}{\alpha}\left(\int_\Omega (\Phi_1(|\nabla tu_0|) + \Phi_2(|\nabla tu_0|))\,dx\right)^\alpha\\
&\quad - \lambda\int_{\Omega_2} \frac{t^{q_1(x)}}{q_1(x)}V_1(x)|u_0|^{q_1(x)}\,dx + \mu\int_{\Omega_0} \frac{t^{q_2(x)}}{q_2(x)}V_2(x)|u_0|^{q_2(x)}\,dx\\
&\leq \frac{m_2 2^{\alpha-1}}{\alpha}\left[\left(\int_\Omega \Phi_1(|\nabla tu_0|)\,dx\right)^\alpha + \left(\int_\Omega \Phi_2(|\nabla tu_0|)\,dx\right)^\alpha\right]\\
&\quad - \frac{\lambda t^{q_{1,0}^-+\epsilon_0}}{q_{1,0}^-}\int_{\Omega_2} V_1(x)|u_0|^{q_1(x)}\,dx + \frac{\mu t^{q_{2,0}^-}}{q_{2,0}^-}\int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx\\
&\leq \frac{m_2 2^{\alpha-1}}{\alpha}\left[t^{\alpha(\varphi_1)_0}\|u_0\|_1^{\alpha(\varphi_2)_0} + t^{\alpha(\varphi_2)_0}\|u_0\|_2^{\alpha(\varphi_2)_0}\right]\\
&\quad - \frac{\lambda t^{q_{1,0}^-+\epsilon_0}}{q_{1,0}^-}\int_{\Omega_2} V_1(x)|u_0|^{q_1(x)}\,dx + \frac{\mu t^{q_{2,0}^-}}{q_{2,0}^-}\int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx
\end{aligned}
$$

$$\leq t^{\theta_0}\left[\frac{m_2 2^{\alpha-1}}{\alpha}\left(\|u_0\|_1^{\alpha(\varphi_1)_0}+\|u_0\|_2^{\alpha(\varphi_2)_0}\right)+\frac{\mu}{q_{2,0}^-}\int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx\right]$$

$$-\frac{\lambda t^{q_{1,0}^-+\epsilon_0}}{q_{1,0}^-}\int_{\Omega_2} V_1(x)|u_0|^{q_1(x)}\,dx. \tag{3.9}$$

It follows from relation (3.9) that $J_{\lambda,\mu}(tu_0)<0$ for all $0<t<\delta^{\overline{\frac{1}{\theta_0-q_{1,0}^--\epsilon_0}}}$ with $0<\delta<\min\{1,\delta_0\}$ and

$$\delta_0 := \frac{\lambda\int_{\Omega_2} V_1(x)|u_0|^{q_1(x)}\,dx}{q_{1,0}^+\left[\frac{m_2 2^{\alpha-1}}{\alpha}\left(\|u_0\|_1^{\alpha(\varphi_1)_0}+\|u_0\|_2^{\alpha(\varphi_2)_0}\right)+\frac{\mu}{q_{2,0}^-}\int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx\right]}.$$

Finally, we point out that

$$\frac{m_2 2^{\alpha-1}}{\alpha}\left(\|u_0\|_1^{\alpha(\varphi_1)_0}+\|u_0\|_2^{\alpha(\varphi_2)_0}\right)+\frac{\mu}{q_{2,0}^-}\int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx>0.$$

In fact, if it is not true, then

$$\|u_0\|_1=\|u_0\|_2=\int_{\Omega_0} V_2(x)|u_0|^{q_2(x)}\,dx=0,$$

hence $u_0=0$ in Ω_0. This is a contradiction and thus the proof of Lemma 3.6 is now complete. ☐

Proof of Theorem 3.4. Let $\underline{\lambda}>0$ be defined as in (3.3) and let $\lambda\in(0,\underline{\lambda})$ and $\mu>0$. By Lemma 3.6, we have

$$\inf_{\partial B_\rho(0)} J_{\lambda,\mu}>0, \tag{3.10}$$

where $B_\rho(0)$ is the boundary of the ball centered at the origin and of radius ρ in X.

On the other hand, by Lemma 3.6, there exists $u_0\in X$ such that $J_{\lambda,\mu}(tu_0)<0$ for all $t>0$ small enough. Moreover, by hypothesis (M_0) and the proof of Lemma 3.6 we deduce that for any $u\in B_\rho(0)$,

$$J_{\lambda,\mu}(u)\geq\frac{m_1}{\alpha}\|u\|_1^{\alpha(\varphi_1)_0}-\frac{2\lambda}{q_1^-}|V_1|_{\frac{s_1(x)}{\alpha}}c_1^{q_1^-}\|u\|_1^{q_1^-}.$$

It follows that

$$-\infty<\underline{c}:=\inf_{\overline{B_\rho(0)}} J_{\lambda,\mu}<0.$$

Let $0<\epsilon<\inf_{\partial B_\rho(0)} J_{\lambda,\mu}-\inf_{B_\rho(0)} J_{\lambda,\mu}$. Using the above information, the functional $J_{\lambda,\mu}:\overline{B_\rho(0)}\longrightarrow\mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $J_{\lambda,\mu}\in C^1(\overline{B_\rho(0)},\mathbb{R})$. Then by Ekeland's variational principle [22], there exists $u_\epsilon\in\overline{B_\rho(0)}$ such that

$$\begin{cases}\underline{c}\leq J_{\lambda,\mu}(u_\epsilon)\leq\underline{c}+\epsilon,\\ 0<J_{\lambda,\mu}(u)-J_{\lambda,\mu}(u_\epsilon)+\epsilon\|u-u_\epsilon\|_1, \quad u\neq u_\epsilon.\end{cases}$$

Since

$$J_{\lambda,\mu}(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} J_{\lambda,\mu} + \epsilon \leq \inf_{B_\rho(0)} J_{\lambda,\mu} + \epsilon < \inf_{\partial B_\rho(0)} J_{\lambda,\mu},$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $\overline{J}_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $\overline{J}_{\lambda,\mu}(u) = J_{\lambda,\mu}(u) + \epsilon\|u - u_\epsilon\|_1$. It is clear that u_ϵ is a minimum point of $\overline{J}_{\lambda,\mu}$ and thus

$$\frac{\overline{J}_{\lambda,\mu}(u_\epsilon + t \cdot v) - \overline{J}_{\lambda,\mu}(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. Hence,

$$\frac{J_{\lambda,\mu}(u_\epsilon + t \cdot v) - J_{\lambda,\mu}(u_\epsilon)}{t} + \epsilon\|v\|_1 \geq 0.$$

Letting $t \to 0$ it follows that $J'_{\lambda,\mu}(u_\epsilon)(v) + \epsilon\|v\|_1 \geq 0$ and we infer that $\|J'_{\lambda,\mu}(u_\epsilon)\|_1 \leq \epsilon$.

From the above information, we deduce that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_{\lambda,\mu}(u_n) \longrightarrow \underline{c} < 0 \quad \text{and} \quad J'_{\lambda,\mu}(u_n) \longrightarrow 0_{X^*}. \tag{3.11}$$

It is clear that $\{u_n\}$ is bounded in X. Thus, there exists u in X such that, up to a subsequence, $\{u_n\}$ converges weakly to u in X. By Remark 3.1, the embedding $X \hookrightarrow\hookrightarrow L^{g_i(x)}(\Omega)$ is continuous and compact, hence the sequence $\{u_n\}$ converges strongly to u in $L^{g_i(x)}(\Omega)$, $i = 1, 2$. Using Hölder's inequality (2.2) we have

$$\begin{aligned}\int_\Omega V_1(x)|u_n|^{q_1(x)-2}u_n(u_n - u)\,dx &\leq |V_1|_{\frac{s_1(x)}{\alpha}} \left||u_n|^{q_1(x)-2}u_n(u_n - u)\right|_{h_1(x)} \\ &\leq 2|V_1|_{\frac{s_1(x)}{\alpha}} \left||u_n|^{q_1(x)-2}u_n\right|_{\frac{q_1(x)}{q_1(x)-1}} |u_n - u|_{g_1(x)}.\end{aligned}$$

Now if $\left||u_n|^{q_1(x)-2}u_n\right|_{\frac{q_1(x)}{q_1(x)-1}} > 1$, then we get

$$\left||u_n|^{q_1(x)-2}u_n\right|_{\frac{q_1(x)}{q_1(x)-1}} \leq |u_n|^{q_1^+}_{q_1(x)}.$$

The compact embedding $X \hookrightarrow\hookrightarrow L^{q_1(x)}(\Omega)$ helps us to show that

$$\lim_{n\to\infty} \int_\Omega V_1(x)|u_n|^{q_1(x)-2}u_n(u_n - u)\,dx = 0. \tag{3.12}$$

Similarly, we get

$$\lim_{n\to\infty} \int_\Omega V_2(x)|u_n|^{q_2(x)-2}u_n(u_n - u)\,dx = 0. \tag{3.13}$$

Moreover, by (3.11) we have

$$\lim_{n\to\infty} J'_{\lambda,\mu}(u_n)(u_n - u) = 0$$

or

$$M\left(\Theta(u_n)\right) \int_\Omega \left(a_1(|\nabla u_n|) + a_2(|\nabla u_n|)\right) \nabla u_n \cdot (\nabla u_n - \nabla u)\,dx$$

$$-\lambda\int_\Omega V_1(x)|u_n|^{q_1(x)-2}u_n(u_n-u)\,dx+\mu\int_\Omega V_2(x)|u_n|^{q_2(x)-2}u_n(u_n-u)\,dx\to 0.$$

Combining this with relations (3.12)-(3.13) it follows that

$$M\left(\Theta(u_n)\right)\int_\Omega\left(a_1(|\nabla u_n|)+a_2(|\nabla u_n|)\right)\nabla u_n\cdot(\nabla u_n-\nabla u)\,dx\to 0. \tag{3.14}$$

If $\Theta(u_n)\to 0$ as $n\to\infty$, then $\int_\Omega \Phi_1(|\nabla u_n|)\,dx\to 0$, it follows from Proposition 2.1 that $u_n\to 0$ strongly in X and the proof is finished. If $\Theta(u_n)\to t_0>0$, then for n large enough, we have

$$M\left(\Theta(u_n)\right)\to M(t_0)\geq m_1t_0^{\alpha-1}>0,$$

so that

$$\lim_{n\to\infty}\int_\Omega\left(a_1(|\nabla u_n|)+a_2(|\nabla u_n|)\right)\nabla u_n\cdot(\nabla u_n-\nabla u)\,dx=0.$$

Combining this with similar arguments as those presented in [30, Proposition 4.5] or [18, Page 50], we deduce that $\{u_n\}$ converges strongly to u in X. Since $J_{\lambda,\mu}\in C^1(X,\mathbb{R})$, we conclude that

$$J'_{\lambda,\mu}(u_n)\to J'_{\lambda,\mu}(u)\text{ as }n\to\infty. \tag{3.15}$$

Relations (3.11) and (3.15) show that $J'_{\lambda,\mu}(u)=0$ and thus u is a weak solution for problem (1.1). Moreover, by relation (3.11), it follows that $J_{\lambda,\mu}(u)<0$ and thus, u is a nontrivial weak solution for (1.1). The proof of Theorem 3.4 is complete. □

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Mustafa Avci
Department of Science
Grande Prairie Regional College
Grande Prairie, AB T8W 0A8, Canada
Email address: `mavci@gprc.ab.ca`

Nguyen Thanh Chung
Department of Mathematics
Quang Binh University
312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam
Email address: `ntchung82@yahoo.com`