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SHECHTER SPECTRA AND RELATIVELY DEMICOMPACT LINEAR RELATIONS

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Abstract. In this paper, we denote by $\mathcal{L}$ the block matrix linear relation, acting on the Banach space $X \oplus Y$, of the form

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are four linear relations with dense domains. We first try to determine the conditions under which a block matrix linear relation becomes a demicompact block matrix linear relation (see Theorems 4.1 and 4.2). Second we study Shechter spectra using demicompact linear relations and relatively demicompact linear relations (see Theorem 5.1).

1. Introduction

Linear relations were introduced into functional analysis by J. von Neumann [17], motivated by the need to consider adjoints of non-densely defined linear differential operators, which were reported previously by E. A. Coddington and A. Dijksma [9] and Dikjsma et al. One main reason that linear relations are more convenient than operators is that one can define the inverse, the closure and the completion for a linear relation. Interesting works on multivalued linear operators including the treatise on partial differential relations were identified by M. Gromov [13]. The application of multivalued methods to the solutions of differential equations was demonstrated by A. Favini and A. Yagi [11]. The development of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution including many problems in fuzzy theory was set forward by researchers (see, for instance [3, 4, 12]).

The notation of demicompactness for linear operators (that is, single valued operators) was introduced into the functional analysis by W. V. Petroshyn [16] to discuss fixed points. Since much attention was paid to this notation,

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such research papers as [14, 16] used it. In 2012, W. Chaker, A. Jeribi and B. Krichen achieved some results on Fredholm and upper semi-Fredholm operators involving demicompact operators [8].

In what follows, we will present the definitions which were set forward by A. Ammar, H. Daoud and A. Jeribi in 2017 [5], who extended the concept of demicompactness of linear operators on multivalued linear operators and developed some properties.

In 2018, the concept of relatively demicompactness of linear operators was further developed by A. Ammar, S. Fakhfakh and A. Jeribi [6], who extended on multivalued linear operators and developed some properties. Indeed, they have determined the conditions under which a linear relation μT for each $\mu \in [0, 1)$ becomes a demicompact linear relation and they displayed some results on Fredholm and upper semi-Fredholm linear relations involving a demicompact linear relation. They also provided some results in which a block matrix of linear relations becomes a demicompact block matrix of linear relations.

The central objective of this work is to pursue the analysis start by [6] and [5] and extend it to more general classes, using the concept of relatively demicompact linear relations. Basically, we explore some features of relatively linear demicompact linear relations and provide a necessary and sufficient condition on matrix linear relations so as to be a demicompact matrix linear relation. Finally, we investigate Shechter spectra using demicompact linear relations and relatively demicompact linear relations.

Our paper is organize as follows: In Section 2, we recall some definitions, basic notions and notations on linear relations that shall be used in the rest of this article. In Section 3, we study some preliminary results on relatively demicompact linear relations which will be subsequently needed in our investigation. In Section 4, we give a necessary and sufficient condition on matrix linear relations so as to be a demicompact matrix linear relation. In the last Section, we investigate Shechter spectra using demicompact linear relations and relatively demicompact linear relations.

2. Auxilary results

Throughout this article, we denote as X, Y and Z complex normed linear spaces, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$L(X, Y)$ denotes the class of all linear bounded operators from X into Y.

A multivalued linear operator (or a linear relation) T from X to Y is a mapping from a subspace $\mathcal{D}(T)$ of X, called the domain of T,

$$\mathcal{D}(T) := \{x \in X : Tx \neq \emptyset\}$$

into $\mathcal{P}(Y)\backslash\{\emptyset\}$ (collection of non-empty subsets of Y) such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all non-zero scalars $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{D}(T)$.

If T maps the points of its domain to singletons, then T is said to be a single valued linear operator (or simply an operator).

In this notation, we denote as $\mathcal{LR}(X,Y)$ the class of all linear relations from X into Y, if $X = Y$, it simply notes $\mathcal{LR}(X,X) := \mathcal{LR}(X)$.

We denote by $\mathcal{LRB}(X,Y)$ the class of all bounded linear relations on X into Y and abbreviate $\mathcal{LRB}(X,X)$ to $\mathcal{LRB}(X)$.

A linear relation is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) := \{(x,y) \in X \times Y : x \in \mathcal{D}(T) \text{ and } y \in Tx\}.$$

The inverse of T is the linear relation T^{-1} defined by

$$G(T^{-1}) := \{(y,x) \in Y \times X : (x,y) \in G(T)\}.$$

According to these definitions and notations, we state the following proposition:

Proposition 2.1 ([10, Proposition I.2.3])**.** *The following properties are equivalent*:

(i) *T is a linear relation.*
(ii) *$G(T)$ is a linear subspace of $X \times Y$.*
(iii) *T^{-1} is a linear relation.*
(iv) *$G(T^{-1})$ is a linear subspace of $Y \times X$.*

Definition 2.1 ([10, Definition II.5.1])**.** Let $T \in \mathcal{LR}(X,Y)$. The closure of T, denoted $\overline{T}$, is defined in terms of its graph

$$G(\overline{T}) := \overline{G(T)}.$$

It is clear that $\overline{T} \in \mathcal{LR}(X,Y)$.

A linear relation T is said to be closed if its graph $G(T)$ is a closed subspace of $X \times Y$. We denote by $\mathcal{CR}(X,Y)$ the class of all closed linear relations on X into Y and abbreviate $\mathcal{CR}(X,X)$ to $\mathcal{CR}(X)$.

Definition 2.2 ([10, Definition II.5.2])**.** The linear relation T is said to be closable if $\overline{T}$ is an extension to T; i.e., if

$$Tx = \overline{T}x \quad \text{for all} \quad x \in \mathcal{D}(T).$$

Lemma 2.1 ([10, Corollary I.2.4])**.** *Let T be a linear relation. Then, $T(0)$ and $T^{-1}(0)$ are linear subspaces.*

Let U denotes arbitrary nonempty sets. Given a subset $M \subset U$, we write

$$T(M) := \bigcup \{T(m) : m \in M \cap \mathcal{D}(T)\}$$

called the image of M, with

$$\mathcal{R}(T) := T(U) \ (= T(\mathcal{D}(T)))$$

called the range of T.

Definition 2.3 ([10, Definition I.2.6]). (i) The subspace $T^{-1}(0)$ is called the null space (or kernel) of T, and is denoted $\mathcal{N}(T)$. We shall use both $\mathcal{N}(T)$ and $T^{-1}(0)$ throughout the sequel.

(ii) T is called injective if $\mathcal{N}(T) = \{0\}$, that is, if T^{-1} is a single valued linear operator.

(iii) T is called surjective if $\mathcal{R}(T) = Y$.

(iv) If T is injective and surjective, we say that T is bijective.

Let M be a subspace of X such that $M \cap \mathcal{D}(T) \neq \emptyset$ and let $T \in \mathcal{LR}(X,Y)$. Then, the restriction $T_{|M}$, is the linear relation given by

$$G(T_{|M}) := \{(m,y) \in G(T) : m \in M\} = G(T) \cap (M \times Y).$$

For S, $T \in \mathcal{LR}(X,Y)$ and $R \in \mathcal{LR}(Y,Z)$, the sum $S+T$ and the product RS are the linear relations defined by

$$G(T+S) := \{(x, y+z) \in X \times Y : (x,y) \in G(T) \text{ and } (x,z) \in G(S)\}, \text{ and}$$

$$G(RS) := \{(x,z) \in X \times Z : (x,y) \in G(S), (y,z) \in G(R) \text{ for some } y \in Y\}$$

respectively, and if $\lambda \in \mathbb{K}$, λT is defined by

$$G(\lambda T) := \{(x, \lambda y) : (x,y) \in G(T)\}.$$

If $T \in \mathcal{LR}(X)$ and $\lambda \in \mathbb{K}$, then the linear relation $\lambda - T$ is given by

$$G(\lambda - T) := \{(x, y - \lambda x) : (x,y) \in G(T)\}.$$

The quotient map from Y into $Y/\overline{T(0)}$ is denoted $Q_T T$ and it stands for an operator (single valued). Therefore, we can define:

$$\|Tx\| := \|Q_T Tx\| \text{ for all } x \in \mathcal{D}(T)$$

$$\|T\| := \|Q_T T\|$$

called the norm of Tx and T respectively. We note that $\|Tx\|$ and $\|T\|$ are not real norms, in fact a nonzero relation can have a zero norm.

Definition 2.4 ([10, Definition V.1.1]). Let $T \in \mathcal{LR}(X,Y)$. T is said compact if $\overline{Q_T T(B_{\mathcal{D}(T)})}$ is compact in Y where $B_{\mathcal{D}(T)} := \{x \in \mathcal{D}(T) : \|x\| \leq 1\}$.

We denote by $\mathcal{KR}(X,Y)$ the class of all compact linear relations on X into Y and abbreviate $\mathcal{KR}(X,X)$ to $\mathcal{KR}(X)$.

Now, in the following definition, we define the graph operator. It is used to reduce a linear relation T to a bounded everywhere defined relation.

Definition 2.5 ([10, Definition IV.3.1]). Let $T \in \mathcal{LR}(X,Y)$, and let X_T denote the vector space $\mathcal{D}(T)$ normed by

$$\|x\|_T := \|x\| + \|Tx\| \quad \text{for all } x \in \mathcal{D}(T).$$

Let $G_T \in \mathcal{LR}(X_T, X)$ be the identity injection of $X_T = (\mathcal{D}(T), \|\cdot\|_T)$ into X, i.e.,

$$\mathcal{D}(G_T) = X_T,\ G_T(x) = x \text{ for all } x \in X_T.$$

Definition 2.6 ([10, Definition I.6.6]). Let $T \in \mathcal{LR}(X, Y)$. The quantities

$$\alpha(T) := \dim(\mathcal{N}(T)) \text{ and } \beta(T) := \operatorname{codim}(\mathcal{R}(T)) = \dim(Y/_{\mathcal{R}(T)})$$

are called the nullity (or the kernel index) and the deficiency of T, respectively. We also write $\overline{\beta}(T) := \operatorname{codim}(\overline{\mathcal{R}(T)})$. The index of T is defined by $i(T) := \alpha(T) - \beta(T)$ provided that both $\alpha(T)$ and $\beta(T)$ are not infinite. If $\alpha(T)$ and $\beta(T)$ are infinite, then T is said to have no index.

Definition 2.7 ([10, Definition V.1.1]). (i) A linear relation $T \in \mathcal{LR}(X, Y)$ is said to be upper semi-Fredholm, and denoted by $T \in \mathcal{F}_+(X, Y)$, if there exists a finite codimensional subspace M of X for which $T_{|M}$ is injective and open.

(ii) A linear relation T is said to be lower semi-Fredholm, and denoted by $T \in \mathcal{F}_-(X, Y)$, if its conjugate T' is upper semi-Fredholm.

For the case when X and Y are Banach spaces, we extend the class of closed single valued Fredholm type operators given earlier to include closed multivalued operators. Note that the definitions of $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$ are consistent with

$$\Phi_+(X, Y) := \{T \in \mathcal{CR}(X, Y) : R(T) \text{ is closed, and } \alpha(T) < \infty\},$$

$$\Phi_-(X, Y) := \{T \in \mathcal{CR}(X, Y) : R(T) \text{ is closed, and } \beta(T) < \infty\}.$$

If $X = Y$, it simply implies $\Phi_+(X, Y)$, $\Phi_-(X, Y)$, $\mathcal{F}_+(X, Y)$, and $\mathcal{F}_-(X, Y)$ by respectively $\Phi_+(X)$, $\Phi_-(X)$, $\mathcal{F}_+(X)$, and $\mathcal{F}_-(X)$.

Lemma 2.2 ([1, Lemma 2.1]). *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow Y$ be a closed linear relation. Then*

(i) *$T \in \Phi_+(X, Y)$ if and only if $Q_T T \in \Phi_+(X, Y/T(0))$.*
(ii) *$T \in \Phi_-(X, Y)$ if and only if $Q_T T \in \Phi_-(X, Y/T(0))$.*

Theorem 2.1 ([10, Theorem V.10.3]). *Let X be a Banach space, Y a normed space and $T \in \mathcal{LR}(X, Y)$. Then, the following properties are equivalent*:

(i) *$T \in \mathcal{F}_+(X, Y)$.*
(ii) *There exists a bounded linear operator A and a bounded finite rank projection operator P such that*

$$AT = I_{\mathcal{D}(T)} - P.$$

Proposition 2.2 ([10, Proposition V.2.6]). *Let $T, S \in \mathcal{LR}(X)$. If T is single valued and $S, T \in \mathcal{F}_+(X)$, then $ST \in \mathcal{F}_+(X)$.*

3. Preliminaries results on relatively demicompact linear relations

This section exhibits some definitions and auxiliary results which will be needed in the rest of this paper. Besides, it displays conditions on which a linear relation becomes a relatively demicompact linear relation. From this perspective, we start by the following lemma, to which we give an equivalence with Definition 3.2:

Definition 3.1 ([5, Definition 3.1])**.** A linear relation $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be demicompact if for every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that

$$Q_{I-T}(I-T)x_n = Q_T(I-T)x_n \to x \in X/\overline{T(0)},$$

there is a convergent subsequence of $Q_T x_n$.

Definition 3.2 ([7, Definition 1.4])**.** Let X and Y be Banach spaces. If $T : \mathcal{D}(T) \subseteq X \longrightarrow Y$ and $S : \mathcal{D}(S) \subseteq X \longrightarrow Y$ are two densely defined linear relations with $S(0) \subseteq T(0)$ and $\mathcal{D}(T) \subseteq \mathcal{D}(S)$, then T is said to be S-demicompact (or relative demicompact with respect to S) if for every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that

$$Q_{S-T}(S-T)x_n = Q_T(S-T)x_n \longrightarrow y \in Y/\overline{T(0)},$$

there is a convergent subsequence of $Q_T S x_n$. We denote by

$$\mathcal{DC}_S(X,Y) = \{T \in \mathcal{CR}(X,Y) \text{ such that } T \text{ is } S\text{-demicompact}\}.$$

Note that for $S = I$, we denote by

$$\mathcal{DC}(X) = \{T \in \mathcal{CR}(X) \text{ such that } T \text{ is demicompact}\}$$

and for $\mu \in \mathbb{C}$,

$$\mathcal{DC}_\mu(X,Y) = \{T \in \mathcal{CR}(X,Y) \text{ such that } \|Tx\| \leq |\mu| \|x\| \text{ for all } x \in \mathcal{D}(T) \text{ and } |\mu| < 1\}.$$

Remark 3.1. For $T = \frac{1}{2}I$, $T \in \mathcal{DC}_\mu(X)$. Therefore, $\mathcal{DC}_\mu(X) \neq \emptyset$.

When $\mathcal{D}(T)$ lies in a finite dimensional subspace of X, the condition of the relative demicompactness is automatically satisfied. As an example of an S-demicompact linear relation, we mention linear relation T such that $(Q_T(S-T))^{-1}$ exists and is continuous on its range $\mathcal{R}(Q_T(S-T))$. Note also that if $Q_T S$ is invertible and $(Q_T S)^{-1}T$ is compact, then T is an S-demicompact linear relation.

Remark 3.2. Note that for $S = I$, we recall the usual definition of demicompactness of a relation introduced by A. Ammar et al. in [5].

Lemma 3.1 ([7, Lemma 2.2])**.** *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow Y$ be a linear relation and $S : \mathcal{D}(S) \subseteq X \longrightarrow Y$ be a continuous linear relation. If $Q_S - Q_T$ is compact, then T is S-demicompact if, and only if, $Q_T T$ is $Q_S S$-demicompact if, and only if, $Q_T T$ is $Q_T S$-demicompact.*

Proposition 3.1 ([15, Theorem 2.3])**.** *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ and $S : \mathcal{D}(S) \subseteq X \longrightarrow Y$ be densely defined closed linear operators with $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ such that $S - T$ is closed. If T is S-demicompact, then $(S - T)$ is an upper semi-Fredholm operator.*

Lemma 3.2 ([5, Lemma 3.3])**.** *Let D be a closed linear subspace of a space X. If $\{x_n\}$ in X is a convergent sequence, then $\{Q_D x_n\}$ is also a convergent sequence.*

Lemma 3.3 ([5, Lemma 3.4])**.** *Let D be a compact linear subspace of a space X. If $\{x_n\}$ in X is a sequence such that $\{Q_D x_n\}$ is a convergent sequence, then $\{x_n\}$ has a convergent subsequence.*

Theorem 3.1 ([7, Theorem 3.2])**.** *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a closed linear relation. If T is demicompact and $I - Q_T$ is compact, then $I - T$ is a Fredholm relation and $i(I - T) = 0$.*

Lemma 3.4. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow Y$ and $S : \mathcal{D}(S) \subseteq X \longrightarrow Y$ are two densely defined linear relations with $S(0) \subseteq T(0)$, $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $Q_T(S-I)$ is compact. Then*

T is S-demicompact if and only if TG_S is demicompact.

Proof. Suppose that T is S-demicompact. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ such that $\{Q_{TG_S}(I - TG_S)x_n\} = \{Q_T(I - T)x_n\}$ converges. In other words,

$$Q_T(S - T)x_n = Q_T(I - T)x_n + Q_T(S - I)x_n,$$

then we have $\{Q_T(S - T)x_n\}$ and we use T as an S-demicompact multivalued linear relation. We obtain $\{Q_T S x_n\}$ which has a convergent subsequence. Finally, we get

$$Q_T x_n = Q_T(I - S)x_n + Q_T S x_n.$$

Therefore, $\{Q_T x_n\}$ has a convergent subsequence. Conversely, let TG_S be a demicompact. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ such that $\{Q_T(S - T)x_n\}$ converges. On the other side, assuming that

$$Q_T(I - TG_S)x_n = Q_T(I - T)x_n = Q_T(S - T)x_n - Q_T(S - I)x_n,$$

then $\{Q_T(I - TG_S)x_n\}$ converges. Using $\{Q_T(I - TG_S)x_n\}$ and the fact that TG_S is demicompact, we obtain $\{Q_{TG_S}x_n = Q_T x_n\}$ which has a convergent subsequence. Finally, we have

$$Q_T S x_n = Q_T x_n - Q_T(I - S)x_n.$$

As a matter of fact, $\{Q_T S x_n\}$ has a convergent subsequence. ☐

Proposition 3.2. *Let $\mu \in \mathbb{C}$ and let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$. If $T \in \mathcal{DC}_\mu(X)$, then $T \in \mathcal{DC}(X)$.*

Proof. Let $T \in \mathcal{DC}_\mu(X)$. Then

$$\|Tx\| \leq |\mu| \|x\|,$$
$$-\|Tx\| \geq -|\mu| \|x\|,$$
$$\|x\| - \|Tx\| \geq (1 - |\mu|)\|x\|.$$

So

$$\|x\| \leq \frac{\|(I - T)x\|}{1 - |\mu|} = \frac{\|Q_T(I - T)x\|}{1 - |\mu|}. \tag{1}$$

Now, take $\{x_n\}$ a bounded sequence of $\mathcal{D}(T)$ such that $Q_T(I-T)x_n \to y$. Applying Eq. (1), we get $\|x_n - y\| \to 0$. So, $\{x_n\}$ has a convergent subsequence. Finally, by Lemma 3.2, we get $\{Q_T x_n\}$ which has a convergent subsequence. So, $T \in \mathcal{DC}(X,Y)$. □

Proposition 3.3. *Let non-zero scalar $\mu \in \mathbb{C}$ and let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a densely defined linear relation such that $\|Tx\| \leq |\mu| \|Q_T x\|$ for all $x \in \mathcal{D}(T)$. If $|\mu| < 1$, then T is a demicompact linear relation.*

Proof. Since

$$\begin{aligned} \|Tx\| &\leq |\mu|\|Q_T x\|, \\ -\|Tx\| &\geq -|\mu|\|Q_T x\|, \\ \|Q_T x\| - \|Tx\| &\geq (1-|\mu|)\|Q_T x\|, \end{aligned}$$

then

$$\|Q_T x\| \leq \frac{\|Q_T(I-T)x\|}{1-|\mu|}. \tag{2}$$

Now, take $\{x_n\}$ a bounded sequence of $\mathcal{D}(T)$ such that $Q_T(I-T)x_n \to y$. Applying Eq. (2), we get $\|Q_T(x_n - y)\| \to 0$. Thus, $\{Q_T x_n\}$ has a convergent subsequence. □

Proposition 3.4. *Let $T, T_0 : \mathcal{D}(T) = \mathcal{D}(T_0) \subseteq X \longrightarrow Y$, be a densely defined linear relation and $S : \mathcal{D}(S) \subseteq X \longrightarrow Y$ be a densely defined closed linear relation with $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S(0) \subseteq T_0(0) \subseteq T(0)$.*

Suppose that T_0 is S-demicompact and there exist $a, b \in \mathbb{C}$ such that $|a| < 1$ and for all $x \in \mathcal{D}(T)$

$$\|Tx - T_0 x + Sx - Sx\| \leq |a|\|Sx - Tx\| + |b|\|Sx - T_0 x\|.$$

Then T is an S-demicompact linear relation.

Proof. Since, for all $x \in \mathcal{D}(T)$, we have

$$\begin{aligned} \|Sx - T_0 x\| - \|Sx - Tx\| &\leq \|Tx - T_0 x + Sx - Sx\|, \\ &\leq |a|\|Sx - Tx\| + |b|\|Sx - T_0 x\|. \end{aligned}$$

Therefore,

$$(1-|b|)\|Sx - T_0 x\| \leq (1+|a|)\|Sx - Tx\|.$$

Then,

$$\|Sx - T_0 x\| \leq \left(\frac{1+|a|}{1-|b|}\right)\|Sx - Tx\|. \tag{3}$$

Now, take $\{x_n\}$ a bounded sequence of $\mathcal{D}(T)$ such that $Q_T(S-T)x_n \to y$. Applying Eq. (3), we get $\|Q_{T_0}(S-T_0)(x_n - y)\| \to 0$. So, $Q_{T_0}(S-T_0)x_n \to y$, using the fact that T_0 is S-demicompact. We obtain $\{Q_{T_0} S x_n\}$ which has a convergent subsequence.

On the other side, we have

$$\begin{aligned}\|Q_T S x_n\| &= d(T(0), Sx_n)\\ &\le d(T_0(0), Sx_n)\\ &= \|Q_{T_0} S x_n\|.\end{aligned}$$

Then, we get $\{Q_T S x_n\}$ which has a convergent subsequence. $\square$

Proposition 3.5. *Let $T, T_0 : \mathcal{D}(T) = \mathcal{D}(T_0) \subseteq X \longrightarrow Y$, be a densely defined linear relations and $S : \mathcal{D}(S) \subseteq X \longrightarrow Y$ be a continuous and a densely defined closed linear relation with $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S(0) \subseteq T_0(0) \subseteq T(0)$.*

Suppose that T is continuous and T_0 is S-demicompact and there exist $a, b \in \mathbb{C}$ such that $|a| > 1$ and for all $x \in \mathcal{D}(T)$

$$\|Tx - T_0x + Sx - Sx\| \ge |a|\|Sx - T_0x\| - |b|\|Sx - Tx\|.$$

Then, T is an S-demicompact linear relation.

Proof. Since, for all $x \in \mathcal{D}(T)$, we have

$$\begin{aligned}\|Tx - T_0x + Sx - Sx\| &\le \|Sx - T_0x\| + \|Sx - Tx\|\\ |a|\|Sx - T_0x\| - |b|\|Sx - T_0x\| &\le \|Sx - Tx\| + \|Sx - Tx\|.\end{aligned}$$

Therefore,

$$(|a| - 1)\|Sx - T_0x\| \le (1 + |b|)\|Sx - Tx\|.$$

Then,

$$\|Sx - T_0x\| \le \left(\frac{1 + |b|}{|a| - 1}\right)\|Sx - Tx\|. \tag{4}$$

Now, take $\{x_n\}$ a bounded sequence of $\mathcal{D}(T)$ such that $Q_T(S - T)x_n \to y$. Applying Eq. (4), we get $\|Q_{T_0}(S - T_0)(x_n - y)\| \to 0$. So, $Q_{T_0}(S - T_0)x_n \to y$, using the fact that T_0 is S-demicompact. We obtain $\{Q_{T_0} S x_n\}$ which has a convergent subsequence.

$$\begin{aligned}\|Q_T S x_n\| &= d(T(0), Sx_n)\\ &\le d(T_0(0), Sx_n)\\ &= \|Q_{T_0} S x_n\|.\end{aligned}$$

Finally, we get $\{Q_T S x_n\}$ which has a convergent subsequence. $\square$

Theorem 3.2. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow Y$ be a densely defined closed linear relation and $S : \mathcal{D}(S) \subseteq X \longrightarrow Y$ be a continuous and a densely defined closed linear relation with $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S(0) \subseteq T(0)$. If $Q_S - Q_T$ is compact, then*

$$S - T \in \Phi_+(X, Y) \textit{ if and only if } T \textit{ is } S\textit{-demicompact.}$$

Proof. Suppose that $S-T \in \Phi_+(X,Y)$, by Lemma 2.2, we get

$$Q_{S-T}(S-T) = Q_T(S-T) \in \Phi_+(X,Y).$$

Using Theorem 2.1, there exist a bounded linear operator A and a bounded finite rank projection operator P such that

$$AQ_T(S-T) = I_{\mathcal{D}(T)} - P.$$

Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $\{Q_T(S-T)x_n\}$ converges to $Q_T x$. Then $\{AQ_T(S-T)x_n\}$ converges to $AQ_T x$. So, $\{x_n - Px_n\}$ converges to $AQ_T x$. Since P is compact, we obtain $\{x_n\}$ which has a convergent subsequence. Finally, $\{Q_T S x_n\}$ has a convergent subsequence. Conversely, let T be an S-demicompact and $Q_S - Q_T$ be a compact linear relation. Using Lemma 3.1, we find that $Q_T T$ is $Q_T S$-demicompact. The latter implies and using Proposition 3.1, we obtain $Q_T S - Q_T T$ which is an upper semi-Fredholm single valued. On the other side,

$$Q_{S-T}(S-T) = Q_T(S-T).$$

We notice that $Q_{S-T}(S-T)$ is an upper single valued linear operator semi-Fredholm. Using Lemma 2.2, we obtain $S-T$ which is an upper semi-Fredholm relation. □

4. Demicompact block matrix of linear relations

In this section, we identify some definitions and notations about the block matrix linear relations and provide conditions on the entries of a $\mathcal{L}$ to have a demicompact block matrix of linear relations.

Let X, Y, and Z be three Banach spaces and $T \in \mathcal{LR}(X,Y)$, $S \in \mathcal{LR}(X,Z)$. Then, S is called T-bounded (or relatively bounded with respect to T) if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist non-negative constants a and b, such that

$$\|Sx\|^2 \leq a\|x\|^2 + b\|Tx\|^2 \quad \text{for all } x \in \mathcal{D}(T). \tag{5}$$

In that case, the infimum of the constants a and b which satisfy (5) are called the 1-bound and T-bound, respectively, of S.

We denote the block matrix linear relation by $\mathcal{L}$:

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ \mathcal{D}(\mathcal{L}) := \Big(\mathcal{D}(A) \cap \mathcal{D}(C)\Big) \oplus \Big(\mathcal{D}(B) \cap \mathcal{D}(D)\Big),$$

where $A \in \mathcal{LR}(X)$, $B \in \mathcal{LR}(Y,X)$, $C \in \mathcal{LR}(X,Y)$ and $D \in \mathcal{LR}(Y)$ are four closable linear relations with dense domains.

Definition 4.1. Let $\varepsilon \geq 0$ and $\delta \geq 0$. The block matrix of the linear relation $\mathcal{L}$ is called:

(i) Diagonally dominant of order ε and δ if C is A-bounded with 1-bound ε_C and A-bound δ_C, B is D-bounded with 1-bound ε_D and D-bound δ_B, $\varepsilon = \max(\varepsilon_C, \varepsilon_B)$ and $\delta = \max(\delta_C, \delta_B)$.

(ii) Off-diagonally dominant of order ε and δ if A is C-bounded with 1-bound ε_A and C-bound δ_A, D is B-bounded with 1-bound ε_D and B-bound δ_D, $\varepsilon = \max(\varepsilon_A, \varepsilon_D)$ and $\delta = \max(\delta_A, \delta_D)$.

Lemma 4.1 ([6, Corollary 3.1])**.** *The block matrix linear relation $\mathcal{L}$ is closed if one of the following conditions holds*:

(i) *$\mathcal{L}$ is diagonally dominant of order ε and $\delta < 1$, A and D are closed, $B(0) \subset A(0)$ and $C(0) \subset D(0)$.*

(ii) *If $\mathcal{L}$ is off-diagonally dominant of order ε and $\delta < 1$, B and C are closed, $A(0) \subset B(0)$ and $D(0) \subset C(0)$.*

Lemma 4.2 ([6, Corollary 3.2])**.** *The block matrix linear relation $\mathcal{L}$ is closed if one of the following conditions holds*:

(i) *$\mathcal{L}$ is diagonally dominant, A and D are closed, $B(0) \subset A(0)$, $C(0) \subset D(0)$ and the relative bounds δ_C and δ_B of C and B, respectively satisfy*
$$\delta_C^2(1+\delta_B^2) < 1 \text{ or } \delta_B^2(1+\delta_C^2) < 1.$$

(ii) *$\mathcal{L}$ is off-diagonally dominant, B and C are closed, $A(0) \subset B(0)$, $D(0) \subset C(0)$ and the relative bounds δ_A and δ_D of A and D, respectively satisfy*
$$\delta_A^2(1+\delta_D^2) < 1 \text{ or } \delta_D^2(1+\delta_A^2) < 1.$$

Lemma 4.3. *Let the block matrix linear relation $\mathcal{L}$ and $\mathcal{M}$ be defined by*:
$$\mathcal{L} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \text{ and } \mathcal{M} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$
where $A_1, A_2 \in \mathcal{LR}(X)$, $B_1, B_2 \in \mathcal{LR}(Y, X)$, $C_1, C_2 \in \mathcal{LR}(X, Y)$ and $D_1, D_2 \in \mathcal{LR}(Y)$ such that $A_1(0) \subset B_1(0)$ and $D_1(0) \subset C_1(0)$. Then
$$Q_{\mathcal{L}}\mathcal{M} = \begin{pmatrix} Q_{B_1}A_2 & Q_{B_1}B_2 \\ Q_{C_1}C_2 & Q_{C_1}D_2 \end{pmatrix}.$$

Proof. Let $(x, y) \in (\mathcal{D}(A_2) \cap \mathcal{D}(C_2)) \oplus (\mathcal{D}(B_2) \cap \mathcal{D}(D_2))$ and $\binom{a}{b} \in \mathcal{M}\binom{x}{y}$. $\binom{a}{b} \in \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\binom{x}{y} = \begin{pmatrix} A_2x+B_2y \\ C_2x+D_2y \end{pmatrix}$. Then, $a \in A_2x + B_2y$ and $b \in C_2x + D_2y$, i.e., there exist $a_1 \in A_2x$, $a_2 \in B_2y$, $b_1 \in C_2x$ and $b_2 \in D_2y$ such that $a = a_1 + a_2$ and $b = b_1 + b_2$.
$$Q_{\mathcal{L}}\mathcal{M}\begin{pmatrix} x \\ y \end{pmatrix} = Q_{\mathcal{L}}\begin{pmatrix} a \\ b \end{pmatrix}.$$
Now, let us examine the expression $Q_{\mathcal{L}}\binom{a}{b}$. Let $\binom{u}{v} \in Q_{\mathcal{L}}\binom{a}{b}$ if and only if $\binom{u}{v} - \binom{a}{b} \in \overline{\mathcal{L}(0)}$.

Notice that $u - a \in \overline{A_1(0) + B_1(0)} = \overline{B_1(0)}$ and $v - b \in \overline{C_1(0) + D_1(0)} = \overline{C_1(0)}$. $\begin{cases} u - a \in \overline{B_1(0)}, \\ v - b \in \overline{C_1(0)}. \end{cases}$ Therefore, $\begin{cases} u \in Q_{B_1}a = Q_{B_1}(a_1 + a_2), \\ v \in Q_{C_1}b = Q_{C_1}(b_1 + b_2), \end{cases}$

so, $\begin{cases} u \in Q_{B_1}(A_2x + B_2y) = Q_{B_1}A_2x + Q_{B_1}B_2y, \\ v \in Q_{C_1}(C_2x + D_2y) = Q_{C_1}C_2x + Q_{C_1}D_2y. \end{cases}$

Finally, $\binom{u}{v} \in \begin{pmatrix} Q_{B_1}A_2 & Q_{B_1}B_2 \\ Q_{C_1}C_2 & Q_{C_1}D_2 \end{pmatrix}\binom{x}{y}$, i.e., $Q_{\mathcal{L}}\mathcal{M} = \begin{pmatrix} Q_{B_1}A_2 & Q_{B_1}B_2 \\ Q_{C_1}C_2 & Q_{C_1}D_2 \end{pmatrix}$. □

Remark 4.1. Unfortunately, there is no uniqueness in the expression $Q_{\mathcal{L}}\mathcal{M}$.

Remark 4.2. Let $A_1, A_2 \in \mathcal{LR}(X)$, $B_1, B_2 \in \mathcal{LR}(Y, X)$, $C_1, C_2 \in \mathcal{LR}(X, Y)$ and $D_1, D_2 \in \mathcal{LR}(Y)$.

(i) If $A_1(0) \subset B_1(0)$ and $C_1(0) \subset D_1(0)$, then

$$Q_{\mathcal{L}}\mathcal{M} = \begin{pmatrix} Q_{B_1}A_2 & Q_{B_1}B_2 \\ Q_{D_1}C_2 & Q_{D_1}D_2 \end{pmatrix}.$$

(ii) If $B_1(0) \subset A_1(0)$ and $C_1(0) \subset D_1(0)$, then

$$Q_{\mathcal{L}}\mathcal{M} = \begin{pmatrix} Q_{A_1}A_2 & Q_{A_1}B_2 \\ Q_{D_1}C_2 & Q_{D_1}D_2 \end{pmatrix}.$$

(iii) If $B_1(0) \subset A_1(0)$ and $D_1(0) \subset C_1(0)$, then

$$Q_{\mathcal{L}}\mathcal{M} = \begin{pmatrix} Q_{A_1}A_2 & Q_{A_1}B_2 \\ Q_{C_1}C_2 & Q_{C_1}D_2 \end{pmatrix}.$$

Theorem 4.1. *Let us define the block matrix linear relation:*

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

If A, B, C and D are four bounded linear relations that satisfy:

(i) $\|Af\|^2 \leq a^2\|f\|^2$ *and* $\|Cf\|^2 \leq c^2\|f\|^2$ *for all* $f \in \mathcal{D}(A) \cap \mathcal{D}(C)$,

(ii) $\|Bg\|^2 \leq b^2\|g\|^2$ *and* $\|Dg\|^2 \leq d^2\|g\|^2$ *for all* $g \in \mathcal{D}(B) \cap \mathcal{D}(D)$, *where* $\delta = \max(a + c, b + d) < \frac{1}{2}$.

Then, $\mathcal{L}$ is demicompact.

Proof. Let $z^t = (f, g)^t \in \Big(\mathcal{D}(A) \cap \mathcal{D}(C)\Big) \oplus \Big(\mathcal{D}(B) \cap \mathcal{D}(D)\Big)$. Then

$$\begin{aligned}
\|\mathcal{L}z\|^2 &= \|\mathcal{T}z + \mathcal{S}z\|^2 \\
&\leq 2\|\mathcal{T}z\|^2 + 2\|\mathcal{S}z\|^2 \\
&= 2\|Q_{\mathcal{T}}\mathcal{T}z\|^2 + 2\|Q_{\mathcal{S}}\mathcal{S}z\|^2 \\
&= 2\|Q_A Af\|^2 + 2\|Q_B Bg\|^2 + 2\|Q_C Cf\|^2 + 2\|Q_D Dg\|^2 \\
&= 2\|Af\|^2 + 2\|Bg\|^2 + 2\|Cf\|^2 + 2\|Dg\|^2 \\
&\leq 2a^2\|f\|^2 + 2b^2\|g\|^2 + 2c^2\|f\|^2 + 2d^2\|g\|^2 \\
&= 2(a^2 + c^2)\|f\|^2 + 2(b^2 + d^2)\|g\|^2 \\
&\leq 2\delta(\|f\|^2 + \|g\|^2) \\
&< (\|f\|^2 + \|g\|^2) \\
&= \|z\|^2.
\end{aligned}$$

So, applying Proposition 3.2, we get $\mathcal{L}$ which is demicompact. ☐

Theorem 4.2. *Let A, B, C and D be four closable linear relations and let us define the block matrix linear relation as follows*:

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

The block matrix linear relation $\mathcal{L}$ is demicompact if one of the following conditions holds:

(i) *$\mathcal{L}$ is diagonally dominant of order ε and δ, A and D are closed, $B(0) \subset A(0)$, $C(0) \subset D(0)$, $\|Af\|^2 \le a^2\|f\|^2$ for all $f \in \mathcal{D}(A)$, $\|Dg\|^2 \le d^2\|g\|^2$ for all $g \in \mathcal{D}(D)$, and $\alpha := \max(a+\varepsilon_C+\delta_A a, d+\varepsilon_B+\delta_D d) < \frac{1}{2}$.*

(ii) *If $\mathcal{L}$ is off-diagonally dominant of order ε and $\delta < 1$, B and C are closed, $A(0) \subset B(0)$, $D(0) \subset C(0)$, $\|Cf\|^2 \le c^2\|f\|^2$ for all $f \in \mathcal{D}(C)$, $\|Bg\|^2 \le b^2\|g\|^2$ for all $g \in \mathcal{D}(B)$, and $\alpha := \max(c+\varepsilon_A+\delta_C c, b+\varepsilon_D+\delta_B b) < \frac{1}{2}$.*

Proof. We prove (i), and the proof of (ii) is similar. By Lemmas 4.1 and 4.2, we get the block matrix linear relation $\mathcal{L}$ that is closed.

Let $z^t = (f,g)^t \in \Big(\mathcal{D}(A) \cap \mathcal{D}(C)\Big) \oplus \Big(\mathcal{D}(B) \cap \mathcal{D}(D)\Big)$. Then

$$\begin{aligned}
\|\mathcal{L}z\|^2 &= \|\mathcal{T}z + \mathcal{S}z\|^2 \\
&\le 2\|\mathcal{T}z\|^2 + 2\|\mathcal{S}z\|^2 \\
&= 2\|Q_{\mathcal{T}}\mathcal{T}z\|^2 + 2\|Q_{\mathcal{S}}\mathcal{S}z\|^2 \\
&= 2\|Q_A Af\|^2 + 2\|Q_B Bg\|^2 + 2\|Q_C Cf\|^2 + 2\|Q_D Dg\|^2 \\
&= 2\|Af\|^2 + 2\|Bg\|^2 + 2\|Cf\|^2 + 2\|Dg\|^2 \\
&\le 2\|Af\|^2 + 2\varepsilon_B\|g\|^2 + 2\delta_D\|Dg\|^2 + 2\varepsilon_C\|f\|^2 + 2\delta_A\|Af\|^2 + 2\|Dg\|^2 \\
&\le (2+2\delta_A)\|Af\|^2 + (2+2\delta_D)\|Dg\|^2 + 2\varepsilon_C\|f\|^2 + 2\varepsilon_B\|g\|^2 \\
&\le (2+2\delta_A)a\|f\|^2 + (2+2\delta_D)d\|g\|^2 + 2\varepsilon_C\|f\|^2 + 2\varepsilon_B\|g\|^2 \\
&\le 2(a+\delta_A a+\varepsilon_C)\|f\|^2 + 2(d+\delta_D d+\varepsilon_B)\|g\|^2 \\
&\le 2\alpha(\|f\|^2 + \|g\|^2) \\
&< (\|f\|^2 + \|g\|^2) \\
&= \|z\|^2.
\end{aligned}$$

So, applying Proposition 3.2, we get $\mathcal{L}$ that is demicompact. □

5. Schechter essential spectra of linear relations

In this section, we present the Schechter essential spectrum definition on relatively demicompact linear relations. The main results of this section are Theorems 5.1, 5.2 and 5.3.

Definition 5.1 ([2, Definition, 4.1])**.** Let X be a Banach space and $T \in \mathcal{CR}(X)$. The resolvent set of T is defined by

$$\rho(T) := \left\{ \lambda \in \mathbb{C} : (\lambda - T)^{-1} \text{ is everywhere defined, and single valued} \right\}.$$

The spectrum of T is $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

In this research paper, we concern with the following essential spectrum:

$$\sigma_e(T) := \bigcap_{K \in \mathcal{K}_T(X)} \sigma(T+K),$$

where $\mathcal{K}_T(X) := \{K \in \mathcal{KR}(X) : \mathcal{D}(T) \subset \mathcal{D}(K), K(0) \subset T(0)\}$.

We define these sets in terms of,

$$\begin{aligned} \Theta_{T,S}(X) := \big\{ & K \in \mathcal{LRB}(X) \text{ such that } \forall\ \lambda \in \rho(T+K) \text{ and } K(0) \subseteq T(0), \\ & -(\lambda - T - K)^{-1} K \in \mathcal{DC}_S(X) \big\}, \\ \Gamma_{T,S}(X) := \big\{ & K \text{ is } T\text{-bounded such that } \forall\ \lambda \in \rho(T+K) \text{ and } K(0) \subseteq T(0), \\ & -K(\lambda - T - K)^{-1} \in \mathcal{DC}_S(X) \big\}. \end{aligned}$$

We denote

$$\sigma_r(T) = \bigcap_{K \in \Theta_{T,S}(X)} \sigma(T+K)$$

and

$$\sigma_l(T) = \bigcap_{K \in \Gamma_{T,S}(X)} \sigma(T+K).$$

The following theorem is the main result of this section:

Theorem 5.1. *Let $T, S \in \mathcal{CR}(X)$ with $S(0) \subseteq T(0)$, $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $Q_T(S-I)$ is compact. Then*

$$\sigma_e(T) = \sigma_r(T) = \sigma_l(T).$$

Proof. Let $\lambda \notin \sigma_r(T)$ (resp. $\lambda \notin \sigma_l(T)$), then there exists $K \in \Theta_{T,S}(X)$ (resp. $K \in \Gamma_{T,S}(X)$) such that $-(\lambda - T - K)^{-1}K \in \mathcal{DC}_S(X)$ (resp. $-K(\lambda - T - K)^{-1} \in \mathcal{DC}_S(X)$) where $\lambda \in \rho(T+K)$. Applying Lemma 3.4, we obtain $-(\lambda - T - K)^{-1}KG_S$ is demicompact (resp. $-K(\lambda - T - K)^{-1}G_S$ is demicompact) where $\lambda \in \rho(T+K)$. Hence, applying Theorem 3.1, we get

$$[I + (\lambda - T - K)^{-1}KG_S] \in \Phi(X) \quad \text{and} \quad i[I + (\lambda - T - K)^{-1}KG_S] = 0.$$

$$(\text{resp. } [I + K(\lambda - T - K)^{-1}G_S] \in \Phi(X) \quad \text{and} \quad i[I + K(\lambda - T - K)^{-1}G_S] = 0).$$

Moreover, we have

$$\lambda - T = (\lambda - T - K)[I + (\lambda - T - K)^{-1}KG_S]$$

$$(\text{resp. } \lambda - T = [I + K(\lambda - T - K)^{-1}G_S](\lambda - T - K).)$$

Then, by Proposition 2.2, one gets

$$(\lambda - T) \in \Phi(X) \text{ and } i(\lambda - T) = 0.$$

So, $\lambda \notin \sigma_e(T)$.

Conversely, since $\mathcal{KR}(X) \subseteq \Theta_{T,S}(X)$ (resp. $\mathcal{KR}(X) \subseteq \Gamma_{T,S}(X)$), we infer that $\sigma_r(T) \subseteq \sigma_e(T)$ (resp. $\sigma_l(T) \subseteq \sigma_e(T)$). $\square$

Theorem 5.2. *Define the block matrix of linear relation*:

$$\mathcal{L} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$
$$\mathcal{T} = \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} \ \textit{and} \ \mathcal{S} = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}.$$

Let $\mathcal{L}$ and $\mathcal{M}$ be diagonally dominant of the order < 1, A_1, A_2, D_1 and D_2 be closed, $B_1(0) \subset A_1(0)$, $C_1(0) \subset D_1(0)$, $\mathcal{M}(0) \subseteq \mathcal{L}(0)$, $\mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(\mathcal{M})$ and $Q_{\mathcal{L}}(\mathcal{M} - I)$ is compact.
If $G(\mathcal{S}) \subset G(\mathcal{T})$ and $\dim D(\mathcal{S}) = \infty$, then

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) \subseteq \sigma_r(A_1) \cup \sigma_r(D_1) = \sigma_l(A_1) \cup \sigma_l(D_1).$$

If in addition, $\dim \mathcal{S}(0) < \infty$, then we have

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) \subseteq \sigma_r(A_1) = \sigma_r(D_1) = \sigma_l(A_1) = \sigma_l(D_1).$$

Proof. By Lemma 4.1, we prove that $\mathcal{L}$ and $\mathcal{M}$ are two closed block matrix linear relations. Applying [6, Theorem 4.1] and Theroem 5.1, we get the result. $\square$

Referring to this Theorem 5.2, we infer the next corollary:

Corollary 5.1. *Define the block matrix of linear relation*:

$$\mathcal{L} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$
$$\mathcal{T} = \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} \ \textit{and} \ \mathcal{S} = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}.$$

Let $\mathcal{L}$ and $\mathcal{M}$ be off-adiagonally dominant of the order < 1, B_1, B_2, C_1 and C_2 be closed, $A_1(0) \subset B_1(0)$, $D_1(0) \subset C_1(0)$, $\mathcal{M}(0) \subseteq \mathcal{L}(0)$, $\mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(\mathcal{M})$ and $Q_{\mathcal{L}}(\mathcal{M} - I)$ is compact.
If $G(\mathcal{T}) \subset G(\mathcal{S})$ and $\dim D(\mathcal{T}) = \infty$, then

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) \subseteq \sigma_r(B_1) \cup \sigma_r(C_1) = \sigma_l(B_1) \cup \sigma_l(C_1).$$

If in addition, $\dim \mathcal{S}(0) < \infty$, then we have

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) \subseteq \sigma_r(B_1) \cup \sigma_r(C_1) = \sigma_l(B_1 \cup \sigma_l(C_1).$$

Based on Theorem 5.2 in addition to Lemma 4.2, we deduce the next theorem:

Theorem 5.3. *Define the block matrix of linear relation*:

$$\mathcal{L} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$
$$\mathcal{T} = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \text{ and } \mathcal{S} = \begin{pmatrix} 0 & 0 \\ C_1 & 0 \end{pmatrix}.$$

Let $\mathcal{L}$ and $\mathcal{M}$ be diagonally dominant such that the relative bounds δ_{C_1},δ_{C_2} and δ_{B_1} δ_{B_2} of C_1, C_2, B_1 and B_2, respectively satisfy $\delta_{C_1}^2(1+\delta_{B_1}^2) < 1$ or $\delta_{B_1}^2(1+\delta_{C_1}^2) < 1$ and $\delta_{C_2}^2(1+\delta_{B_2}^2) < 1$ or $\delta_{B_2}^2(1+\delta_{C_2}^2) < 1$. Let A_1, A_2, D_1 and D_2 be closed, $B_1(0) \subset A_1(0)$, $C_1(0) \subset D_1(0)$, $\mathcal{M}(0) \subseteq \mathcal{L}(0)$, $\mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(\mathcal{M})$ and $Q_{\mathcal{L}}(\mathcal{M}-I)$ is compact.
If $G(\mathcal{S}) \subset G(\mathcal{T})$ and $\dim D(\mathcal{S}) = \infty$, then

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) \subseteq \sigma_r(A_1) \cup \sigma_r(D_1) = \sigma_l(A_1) \cup \sigma_l(D_1).$$

If in addition, $\dim \mathcal{S}(0) < \infty$, then we have

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) \subseteq \sigma_r(A_1) \cup \sigma_r(D_1) = \sigma_l(A_1) = \sigma_l(D_1).$$

Hence, this Theorem 5.3 immediately implies the next Corollary:

Corollary 5.2. *Define the block matrix of linear relation*:

$$\mathcal{L} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$
$$\mathcal{T} = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \text{ and } \mathcal{S} = \begin{pmatrix} 0 & 0 \\ C_1 & 0 \end{pmatrix}.$$

Let $\mathcal{L}$ and $\mathcal{M}$ be diagonally dominant such that the relative bounds δ_{D_1},δ_{D_2} and δ_{A_1} δ_{A_2} of D_1, D_2, A_1 and A_2, respectively satisfy $\delta_{D_1}^2(1+\delta_{A_1}^2) < 1$ or $\delta_{A_1}^2(1+\delta_{D_1}^2) < 1$ and $\delta_{D_2}^2(1+\delta_{A_2}^2) < 1$ or $\delta_{A_2}^2(1+\delta_{D_2}^2) < 1$. Let B_1, B_2, C_1 and C_2 be closed, $A_1(0) \subset B_1(0)$, $D_1(0) \subset C_1(0)$, $\mathcal{M}(0) \subseteq \mathcal{L}(0)$, $\mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(\mathcal{M})$ and $Q_{\mathcal{L}}(\mathcal{M}-I)$ is compact.
If $G(\mathcal{T}) \subset G(\mathcal{S})$ and $\dim D(\mathcal{T}) = \infty$, then

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) \subseteq \sigma_r(\mathcal{S}) = \sigma_l(\mathcal{S}).$$

If in addition, $\dim \mathcal{S}(0) < \infty$, then we get

$$\sigma_l(\mathcal{L}) = \sigma_r(\mathcal{L}) = \sigma_r(\mathcal{S}) = \sigma_l(\mathcal{S}).$$

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