Commun. Korean Math. Soc. **35** (2020), No. 2, pp. 487–497
https://doi.org/10.4134/CKMS.c190254
pISSN: 1225-1763 / eISSN: 2234-3024

ON COMPLEX REPRESENTATIONS OF THE CLIFFORD ALGEBRAS

Youngkwon Song

Abstract. In this paper, we establish a complex matrix representation of the Clifford algebra $C\ell_{p,q}$. The size of our representation is significantly smaller than the size of the elements in $L_{p,q}(\mathbb{R})$. Additionally, we give detailed information about the spectrum of the constructed matrix representation.

1. Introduction

Throughout this paper, $C\ell_{p,q}$ is the Clifford algebra on $\mathbb{R}^{p,q}$ which is the n-dimensional pseudo Euclidean space with the quadratic form $Q(v) = \sum_{i=1}^{p} v_i^2 - \sum_{i=p+1}^{p+q} v_i^2$ of signature (p, q), where $p + q = n$. Dirac introduced matrices that provided a representation of the Clifford algebra of Minkowski space.

In a series of papers, the real matrix representations and various properties of the Clifford algebra $C\ell_{p,q}$ have been established and developed [1–3,5–8]. We constructed matrix algebras $L_{p,q}(\mathbb{R})$ and $S_{2^n}(\mathbb{R})$ whose elements are real matrix representation of the Clifford algebra $C\ell_{p,q}$ for some p and q, where $p+q = n$. The size of the matrix representation of the elements in the Clifford algebra $C\ell_{p,q}$ is $2^n \times 2^n$. Thus, if we can reduce the size of the matrix representation, then it would be easier to understand.

In this paper, we will construct a subalgebra $\mathcal{R}_{2^n}(\mathbb{C})$ of the matrix algebra $M_{2^n}(\mathbb{C})$ and show that $\mathcal{R}_{2^n}(\mathbb{C})$ is isomorphic to the Clifford algebra $C\ell_{p,q}$ for some p and q. The size of the elements in $\mathcal{R}_{2^n}(\mathbb{C})$ is $2^{n-1} \times 2^{n-1}$ which is significantly smaller than the size of the elements in the algebra $L_{p,q}(\mathbb{R})$.

Also, the theory of spectrum of matrices attracts more and more attention because of its important role in various applications including quantum physics and computer sciences [4, 9, 10]. In a second part of this paper, we give information about the spectrum of the constructed matrix representation of the elements in the Clifford algebra $C\ell_{p,q}$.

Received July 23, 2019; Revised September 20, 2019; Accepted November 19, 2019.
2010 *Mathematics Subject Classification.* 15A18, 15A66.
Key words and phrases. Clifford algebra, complex matrix representation.
The present research has been conducted by the Research Grant of Kwangwoon University in 2019.

2. Complex matrix representations of the Clifford algebras

We begin by defining terms necessary to use. Let

$$T_I = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}, \quad T_R = \left\{ \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

We will call the matrix in T_I (or T_R) by the I-type matrix (or R-type matrix) [7].

Consider the following Pauli matrices.

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then,

$$\sigma_1\sigma_2 = \sigma_3 i = -\sigma_2\sigma_1, \quad \sigma_2\sigma_3 = \sigma_1 i = -\sigma_3\sigma_2, \quad \sigma_1\sigma_3 = -\sigma_2 i = -\sigma_3\sigma_1.$$

Now, we construct the subalgebra $\mathcal{R}_{2^n}(\mathbb{C})$ of the matrix algebra $M_{2^n}(\mathbb{C})$ using the Pauli matrices as follows:

For $n = 1$, let

$$A^{(1)} = A_1^1\sigma_0 + A_2^1\sigma_1, \quad B^{(1)} = B_1^1\sigma_3 - B_2^1\sigma_2 i,$$

where $A_1^1, A_2^1, B_1^1, B_2^1 \in \mathbb{R}$. Also, define $r^{(1)} = A^{(1)} + B^{(1)}i$ and

$$\mathcal{R}_{2^1}(\mathbb{C}) = \left\{ r^{(1)} \in M_2(\mathbb{C}) \mid A_t^1, B_t^1 \in \mathbb{R}, \, t = 1, 2 \right\}.$$

For $n = 2$, replace A_1^1 and A_2^1 by $A_1^2\sigma_0 + A_2^2\sigma_1$ and $A_3^2\sigma_3 - A_4^2\sigma_2 i$, respectively. Also, replace B_1^1 and B_2^1 by $B_1^2\sigma_0 + B_2^2\sigma_1$ and $B_3^2\sigma_3 - B_4^2\sigma_2 i$, respectively. Then, we obtain

$$\begin{aligned} A^{(2)} &= \begin{pmatrix} A_1^2 & A_2^2 & A_3^2 & -A_4^2 \\ A_2^2 & A_1^2 & A_4^2 & -A_3^2 \\ A_3^2 & -A_4^2 & A_1^2 & A_2^2 \\ A_4^2 & -A_3^2 & A_2^2 & A_1^2 \end{pmatrix} \\ &= \sigma_0 \otimes (A_1^2\sigma_0 + A_2^2\sigma_1) + \sigma_1 \otimes (A_3^2\sigma_3 - A_4^2\sigma_2 i), \\ B^{(2)} &= \begin{pmatrix} B_1^2 & B_2^2 & -B_3^2 & B_4^2 \\ B_2^2 & B_1^2 & -B_4^2 & B_3^2 \\ B_3^2 & -B_4^2 & -B_1^2 & -B_2^2 \\ B_4^2 & -B_3^2 & -B_2^2 & -B_1^2 \end{pmatrix} \\ &= \sigma_3 \otimes (B_1^2\sigma_0 + B_2^2\sigma_1) + (-\sigma_2 i) \otimes (B_3^2\sigma_3 - B_4^2\sigma_2 i). \end{aligned}$$

Now, define

$$r^{(2)} = A^{(2)} + B^{(2)}i = \begin{pmatrix} A_1^2 + B_1^2 i & A_2^2 + B_2^2 i & A_3^2 - B_3^2 i & -A_4^2 + B_4^2 i \\ A_2^2 + B_2^2 i & A_1^2 + B_1^2 i & A_4^2 - B_4^2 i & -A_3^2 + B_3^2 i \\ A_3^2 + B_3^2 i & -A_4^2 - B_4^2 i & A_1^2 - B_1^2 i & A_2^2 - B_2^2 i \\ A_4^2 + B_4^2 i & -A_3^2 - B_3^2 i & A_2^2 - B_2^2 i & A_1^2 - B_1^2 i \end{pmatrix}$$

and

$$\mathcal{R}_{2^2}(\mathbb{C}) = \{ r^{(2)} \in M_4(\mathbb{C}) \mid A_t^2, \; B_t^2 \in \mathbb{R}, \;\; t = 1, 2, 3, 4 \}.$$

For $n = 3$, replace A_1^2 and A_3^2 by $A_1^3\sigma_0 + A_2^3\sigma_1$ and $A_5^3\sigma_0 + A_6^3\sigma_1$, respectively. Also, replace A_2^2 and A_4^2 by $A_3^3\sigma_3 - A_4^3\sigma_2 i$ and $A_7^3\sigma_3 - A_8^3\sigma_2 i$, respectively.

Furthermore, replace B_1^2 and B_3^2 by $B_1^3\sigma_0 + B_2^3\sigma_1$ and $B_5^3\sigma_0 + B_6^3\sigma_1$, respectively. Also, replace B_2^2 and B_4^2 by $B_3^3\sigma_3 - B_4^3\sigma_2 i$ and $B_7^3\sigma_3 - B_8^3\sigma_2 i$, respectively. Then, we obtain an 8×8 matrix $r^{(3)} = A^{(3)} + B^{(3)}i$, where

$$\begin{aligned} A^{(3)} &= \sigma_0 \otimes \left\{\sigma_0 \otimes (A_1^3\sigma_0 + A_2^3\sigma_1) + \sigma_1 \otimes (A_3^3\sigma_3 - A_4^3\sigma_2 i)\right\} \\ &\quad + \sigma_1 \otimes \left\{\sigma_3 \otimes (A_5^3\sigma_0 + A_6^3\sigma_1) + (-\sigma_2 i) \otimes (A_7^3\sigma_3 - A_8^3\sigma_2 i)\right\}, \\ B^{(3)} &= \sigma_3 \otimes \left\{\sigma_0 \otimes (B_1^3\sigma_0 + B_2^3\sigma_1) + \sigma_1 \otimes (B_3^3\sigma_3 - B_4^3\sigma_2 i)\right\} \\ &\quad + (-\sigma_2 i) \otimes \left\{\sigma_3 \otimes (B_5^3\sigma_0 + B_6^3\sigma_1) + (-\sigma_2 i) \otimes (B_7^3\sigma_3 - B_8^3\sigma_2 i)\right\}. \end{aligned}$$

Thus, the matrix representations of $A^{(3)}$ and $B^{(3)}$ are the following 8×8 matrices:

$$A^{(3)} = \begin{pmatrix} A_1^3 & A_2^3 & A_3^3 & -A_4^3 & A_5^3 & A_6^3 & -A_7^3 & A_8^3 \\ A_2^3 & A_1^3 & A_4^3 & -A_3^3 & A_6^3 & A_5^3 & -A_8^3 & A_7^3 \\ A_3^3 & -A_4^3 & A_1^3 & A_2^3 & A_7^3 & -A_8^3 & -A_5^3 & -A_6^3 \\ A_4^3 & -A_3^3 & A_2^3 & A_1^3 & A_8^3 & -A_7^3 & -A_6^3 & -A_5^3 \\ A_5^3 & A_6^3 & -A_7^3 & A_8^3 & A_1^3 & A_2^3 & A_3^3 & -A_4^3 \\ A_6^3 & A_5^3 & -A_8^3 & A_7^3 & A_2^3 & A_1^3 & A_4^3 & -A_3^3 \\ A_7^3 & -A_8^3 & -A_5^3 & -A_6^3 & A_3^3 & -A_4^3 & A_1^3 & A_2^3 \\ A_8^3 & -A_7^3 & -A_6^3 & -A_5^3 & A_4^3 & -A_3^3 & A_2^3 & A_1^3 \end{pmatrix},$$

$$B^{(3)} = \begin{pmatrix} B_1^3 & B_2^3 & B_3^3 & -B_4^3 & -B_5^3 & -B_6^3 & B_7^3 & -B_8^3 \\ B_2^3 & B_1^3 & B_4^3 & -B_3^3 & -B_6^3 & -B_5^3 & B_8^3 & -B_7^3 \\ B_3^3 & -B_4^3 & B_1^3 & B_2^3 & -B_7^3 & B_8^3 & B_5^3 & B_6^3 \\ B_4^3 & -B_3^3 & B_2^3 & B_1^3 & -B_8^3 & B_7^3 & B_6^3 & B_5^3 \\ B_5^3 & B_6^3 & -B_7^3 & B_8^3 & -B_1^3 & -B_2^3 & -B_3^3 & B_4^3 \\ B_6^3 & B_5^3 & -B_8^3 & B_7^3 & -B_2^3 & -B_1^3 & -B_4^3 & B_3^3 \\ B_7^3 & -B_8^3 & -B_5^3 & -B_6^3 & -B_3^3 & B_4^3 & -B_1^3 & -B_2^3 \\ B_8^3 & -B_7^3 & -B_6^3 & -B_5^3 & -B_4^3 & B_3^3 & -B_2^3 & -B_1^3 \end{pmatrix}.$$

Now, if we let $r^{(3)} = A^{(3)} + B^{(3)}i$, then $r^{(3)}$ is the following matrix:

$$\begin{pmatrix} A_1^3 + B_1^3 i & A_2^3 + B_2^3 i & A_3^3 + B_3^3 i & -A_4^3 - B_4^3 i & A_5^3 - B_5^3 i & A_6^3 - B_6^3 i & -A_7^3 + B_7^3 i & A_8^3 - B_8^3 i \\ A_2^3 + B_2^3 i & A_1^3 + B_1^3 i & A_4^3 + B_4^3 i & -A_3^3 - B_3^3 i & A_6^3 - B_6^3 i & A_5^3 - B_5^3 i & -A_8^3 + B_8^3 i & A_7^3 - B_7^3 i \\ A_3^3 + B_3^3 i & -A_4^3 - B_4^3 i & A_1^3 + B_1^3 i & A_2^3 + B_2^3 i & A_7^3 - B_7^3 i & -A_8^3 + B_8^3 i & -A_5^3 + B_5^3 i & -A_6^3 + B_6^3 i \\ A_4^3 + B_4^3 i & -A_3^3 - B_3^3 i & A_2^3 + B_2^3 i & A_1^3 + B_1^3 i & A_8^3 - B_8^3 i & -A_7^3 + B_7^3 i & -A_6^3 + B_6^3 i & -A_5^3 + B_5^3 i \\ A_5^3 + B_5^3 i & A_6^3 + B_6^3 i & -A_7^3 - B_7^3 i & A_8^3 + B_8^3 i & A_1^3 - B_1^3 i & A_2^3 - B_2^3 i & A_3^3 - B_3^3 i & -A_4^3 + B_4^3 i \\ A_6^3 + B_6^3 i & A_5^3 + B_5^3 i & -A_8^3 - B_8^3 i & A_7^3 + B_7^3 i & A_2^3 - B_2^3 i & A_1^3 - B_1^3 i & A_4^3 - B_4^3 i & -A_3^3 + B_3^3 i \\ A_7^3 + B_7^3 i & -A_8^3 - B_8^3 i & -A_5^3 - B_5^3 i & -A_6^3 - B_6^3 i & A_3^3 - B_3^3 i & -A_4^3 + B_4^3 i & A_1^3 - B_1^3 i & A_2^3 - B_2^3 i \\ A_8^3 + B_8^3 i & -A_7^3 - B_7^3 i & -A_6^3 - B_6^3 i & -A_5^3 - B_5^3 i & A_4^3 - B_4^3 i & -A_3^3 + B_3^3 i & A_2^3 - B_2^3 i & A_1^3 - B_1^3 i \end{pmatrix}.$$

Let

$$\mathcal{R}_{2^3}(\mathbb{C}) = \{r^{(3)} \in M_8(\mathbb{C}) \mid A_t^3, B_t^3 \in \mathbb{R},\ \ t = 1, 2, \ldots, 8\}.$$

Continuing the process successively, we obtain

$$A^{(n)} = \sigma_0 \otimes A_1^{(n-1)} + \sigma_1 \otimes B_1^{(n-1)},$$

$$B^{(n)} = \sigma_3 \otimes A_2^{(n-1)} + (-\sigma_2 i) \otimes B_2^{(n-1)}$$

for some $A_j^{(n-1)}$ and $B_j^{(n-1)}$, $j = 1, 2$. Now, define $r^{(n)} = A^{(n)} + B^{(n)} i$ and

$$\mathcal{R}_{2^n}(\mathbb{C}) = \{r^{(n)} \in M_{2^n}(\mathbb{C}) \mid A_t^n, B_t^n \in \mathbb{R},\ \ t = 1, 2, \ldots, 2^n\}.$$

Also, let $S_{2^n}(\mathbb{R})$ be the set consisting of $A^{(n)}$ and define $T_{2^n}(\mathbb{R})$ by the set consisting of $B^{(n)}$ in the process. Then, the following properties can be proved.

Proposition 2.1. *Let $r^{(n)} \in \mathcal{R}_{2^n}(\mathbb{C})$. Then,*

(1) $r^{(n)} = \begin{pmatrix} E & F \\ F & E \end{pmatrix} + \begin{pmatrix} G & -H \\ H & -G \end{pmatrix} i$, *for some $E, F, G, H \in M_{2^{n-1}}(\mathbb{R})$.*

(2) (i) *The t-th row of 2×2 block entries of $r^{(n)}$ is of the following shape;*

$(P_{t1}, Q_{t2}, P_{t3}, \ldots, Q_{t2^{n-1}}) + (X_{t1}, Y_{t2}, X_{t3}, \ldots, Y_{t2^{n-1}})i$, *if t is an odd integer,*

$(Q_{t1}, P_{t2}, Q_{t3}, \ldots, P_{t2^{n-1}}) + (Y_{t1}, X_{t2}, Y_{t3}, \ldots, X_{t2^{n-1}})i$, *if t is an even integer,*

(ii) *The ℓ-th column of 2×2 block entries of $r^{(n)}$ is of the following shape;*

$$\begin{pmatrix} P_{1\ell} \\ Q_{2\ell} \\ P_{3\ell} \\ \vdots \\ Q_{2^{n-1}\ell} \end{pmatrix} + \begin{pmatrix} X_{1\ell} \\ Y_{2\ell} \\ X_{3\ell} \\ \vdots \\ Y_{2^{n-1}\ell} \end{pmatrix} i, \qquad \text{if } \ell \text{ is an odd integer,}$$

$$\begin{pmatrix} Q_{1\ell} \\ P_{2\ell} \\ Q_{3\ell} \\ \vdots \\ P_{2^{n-1}\ell} \end{pmatrix} + \begin{pmatrix} Y_{1\ell} \\ X_{2\ell} \\ Y_{3\ell} \\ \vdots \\ X_{2^{n-1}\ell} \end{pmatrix} i, \qquad \text{if } \ell \text{ is an even integer.}$$

Here, $P_{t\ell}, X_{t\ell} \in T_I$ and $Q_{t\ell}, Y_{t\ell} \in T_R$ for all t and ℓ.

Proposition 2.2. *Let $r_1^{(n)}, r_2^{(n)} \in \mathcal{R}_{2^n}(\mathbb{C})$ and let $M_{t\ell} + N_{t\ell}\, i$ be the (t, ℓ)-th 2×2 block entry of $r_1^{(n)} r_2^{(n)}$. Then,*

(1) *$M_{t\ell}$, $N_{t\ell} \in T_I$ if $t + \ell$ is an even integer.*

(2) *$M_{t\ell}$, $N_{t\ell} \in T_R$ if $t + \ell$ is an odd integer.*

Proof. We will prove (1) in the case that t and ℓ are all odd integers and the other cases can be proved similarly. Let $r_1^{(n)} = A_1^{(n)} + B_1^{(n)} i$ and $r_2^{(n)} = A_2^{(n)} + B_2^{(n)} i$ for some $A_1^{(n)}, A_2^{(n)} \in S_{2^n}(\mathbb{R})$ and $B_1^{(n)}, B_2^{(n)} \in T_{2^n}(\mathbb{R})$. Note that

$$\begin{aligned} M_{t\ell} = {} & (t\text{-th row of } 2 \times 2 \text{ blocks of } A_1^{(n)})\ (\ell\text{-th column of } 2 \times 2 \text{ blocks of } A_2^{(n)}) \\ & -(t\text{-th row of } 2 \times 2 \text{ blocks of } B_1^{(n)})\ (\ell\text{-th column of } 2 \times 2 \text{ blocks of } B_2^{(n)}), \\ N_{t\ell} = {} & (t\text{-th row of } 2 \times 2 \text{ blocks of } A_1^{(n)})\ (\ell\text{-th column of } 2 \times 2 \text{ blocks of } B_2^{(n)}) \\ & +(t\text{-th row of } 2 \times 2 \text{ blocks of } B_1^{(n)})\ (\ell\text{-th column of } 2 \times 2 \text{ blocks of } A_2^{(n)}). \end{aligned}$$

Now, let

$$\begin{aligned} t\text{-th row of } 2\times 2 \text{ blocks of } A_1^{(n)} &= (P_{t1}, Q_{t2}, P_{t3}, \ldots, Q_{t2^{n-1}}), \\ t\text{-th row of } 2\times 2 \text{ blocks of } B_1^{(n)} &= (X_{t1}, Y_{t2}, X_{t3}, \ldots, Y_{t2^{n-1}}), \\ \ell\text{-th column of } 2\times 2 \text{ blocks of } A_2^{(n)} &= (P'_{1\ell}, Q'_{2\ell}, P'_{3\ell}, \ldots, Q'_{2^{n-1}\ell})^T, \\ \ell\text{-th column of } 2\times 2 \text{ blocks of } B_2^{(n)} &= (X'_{1\ell}, Y'_{2\ell}, X'_{3\ell}, \ldots, Y'_{2^{n-1}\ell})^T, \end{aligned}$$

for some $P_{t\ell}, P'_{t\ell}, X_{t\ell}, X'_{t\ell} \in T_I$ and $Q_{t\ell}, Q'_{t\ell}, Y_{t\ell}, Y'_{t\ell} \in T_R$. Then,

$$\begin{aligned} M_{t\ell} &= (P_{t1}, Q_{t2}, P_{t3}, \ldots, Q_{t2^{n-1}})(P'_{1\ell}, Q'_{2\ell}, P'_{3\ell}, \ldots, Q'_{2^{n-1}\ell})^T \\ &\quad - (X_{t1}, Y_{t2}, X_{t3}, \ldots, Y_{t2^{n-1}})(X'_{1\ell}, Y'_{2\ell}, X'_{3\ell}, \ldots, Y'_{2^{n-1}\ell})^T, \\ N_{t\ell} &= (P_{t1}, Q_{t2}, P_{t3}, \ldots, Q_{t2^{n-1}})(X'_{1\ell}, Y'_{2\ell}, X'_{3\ell}, \ldots, Y'_{2^{n-1}\ell})^T \\ &\quad + (X_{t1}, Y_{t2}, X_{t3}, \ldots, Y_{t2^{n-1}})(P'_{1\ell}, Q'_{2\ell}, P'_{3\ell}, \ldots, Q'_{2^{n-1}\ell})^T. \end{aligned}$$

Note that $P_{ts}(P'_{s\ell})^T$, $Q_{ts}(Q'_{s\ell})^T$, $X_{ts}(X'_{s\ell})^T$, $Y_{ts}(Y'_{s\ell})^T$, $P_{ts}(X'_{s\ell})^T$, $Q_{ts}(Y'_{s\ell})^T$, $X_{ts}(P'_{s\ell})^T$, $Y_{ts}(Q'_{s\ell})^T$ are all in T_I and hence $M_{t\ell}, N_{t\ell} \in T_I$. □

Lemma 2.3. *Let $r^{(n)} = A^{(n)} + B^{(n)}i$ for some $A^{(n)} \in S_{2^n}(\mathbb{R})$ and $B^{(n)} \in T_{2^n}(\mathbb{R})$. Then, $A^{(n)}B^{(n)} \in T_{2^n}(\mathbb{R})$.*

Proof. Obviously $A^{(1)}B^{(1)} \in T_{2^1}(\mathbb{R})$. Assume that $A^{(m)}B^{(m)} \in T_{2^m}(\mathbb{R})$. From the construction, $A^{(m+1)} \in S_{2^m}(T_I \cup T_R)$ and $B^{(m+1)} \in T_{2^m}(T_I \cup T_R)$ as $2^m \times 2^m$ matrices with 2×2 block matrix entries. Note also that (p,q)-th 2×2 block matrix entry of $A^{(m+1)}B^{(m+1)}$ can be obtained by virtue of the rule to get (p,q)-th real entry of $A^{(m)}B^{(m)}$. Thus, by the mathematical induction hypothesis, the 2×2 block entries of $A^{(m+1)}B^{(m+1)}$ preserve the relationships about the structure between row and column entries of $T_{2^m}(T_I \cup T_R)$. Also, $(p,1)$-th 2×2 block entries of $A^{(m+1)}B^{(m+1)}$ are in T_I if p is an odd integer and $(p,1)$-th 2×2 block entries of $A^{(m+1)}B^{(m+1)}$ are in T_R if p is an even integer by proposition 2.2. Therefore, the lemma is proved. □

Theorem 2.4. *$\mathcal{R}_{2^n}(\mathbb{C})$ is a subalgebra of the matrix algebra $M_{2^n}(\mathbb{C})$.*

Proof. In order to prove the theorem, it is enough to show that $\mathcal{R}_{2^n}(\mathbb{C})$ is closed under the multiplication. Let $r_1^{(n)}, r_2^{(n)} \in \mathcal{R}_{2^n}(\mathbb{C})$. We will show that $r_1^{(n)}r_2^{(n)} \in \mathcal{R}_{2^n}(\mathbb{C})$ by the mathematical induction for n.

If $n = 1$, then $r_1^{(1)} = A_1^{(1)} + B_1^{(1)}i$ and $r_2^{(1)} = A_2^{(1)} + B_2^{(1)}i$ for some $A_1^{(1)}, A_2^{(1)} \in S_{2^1}(\mathbb{R})$ and $B_1^{(1)}, B_2^{(1)} \in T_{2^1}(\mathbb{R})$. Thus,

$$r_1^{(1)}r_2^{(1)} = (A_1^{(1)}A_2^{(1)} - B_1^{(1)}B_2^{(1)}) + (A_1^{(1)}B_2^{(1)} + B_1^{(1)}A_2^{(1)})i.$$

Since $A_1^{(1)}A_2^{(1)} - B_1^{(1)}B_2^{(1)} \in T_I$ and $A_1^{(1)}B_2^{(1)} + B_1^{(1)}A_2^{(1)} \in T_R$, we have $r_1^{(1)}r_2^{(1)} \in \mathcal{R}_{2^1}(\mathbb{C})$.

Assume that it is true for $n = m$. That is, if $r_1^{(m)} = A_1^{(m)} + B_1^{(m)} i$ and $r_2^{(m)} = A_2^{(m)} + B_2^{(m)} i$ for some $A_1^{(m)}, A_2^{(m)} \in S_{2^m}(\mathbb{R})$ and $B_1^{(m)}, B_2^{(m)} \in T_{2^m}(\mathbb{R})$, then

$$r_1^{(m)} r_2^{(m)} = (A_1^{(m)} A_2^{(m)} - B_1^{(m)} B_2^{(m)}) + (A_1^{(m)} B_2^{(m)} + B_1^{(m)} A_2^{(m)}) i \in \mathcal{R}_{2^m}(\mathbb{C}).$$

Now, let $r_1^{(m+1)}, r_2^{(m+1)} \in \mathcal{R}_{2^{m+1}}(\mathbb{C})$ and

$$r_1^{(m+1)} = A_1^{(m+1)} + B_1^{(m+1)} i, \quad r_2^{(m+1)} = A_2^{(m+1)} + B_2^{(m+1)} i$$

for some $A_1^{(m+1)}$, $A_2^{(m+1)} \in S_{2^{m+1}}(\mathbb{R})$ and $B_1^{(m+1)}$, $B_2^{(m+1)} \in T_{2^{m+1}}(\mathbb{R})$. Since

$$\begin{aligned} r_1^{(m+1)} r_2^{(m+1)} = {} & (A_1^{(m+1)} A_2^{(m+1)} - B_1^{(m+1)} B_2^{(m+1)}) \\ & + (A_1^{(m+1)} B_2^{(m+1)} + B_1^{(m+1)} A_2^{(m+1)}) i, \end{aligned}$$

it is enough to show that

$$A_1^{(m+1)} A_2^{(m+1)} - B_1^{(m+1)} B_2^{(m+1)} \in S_{2^{m+1}}(\mathbb{R}),$$

$$A_1^{(m+1)} B_2^{(m+1)} + B_1^{(m+1)} A_2^{(m+1)} \in T_{2^{m+1}}(\mathbb{R}).$$

Note that

$$A_1^{(m+1)} = \sigma_0 \otimes A_1^{(m)} + \sigma_1 \otimes B_1^{(m)}, \quad A_2^{(m+1)} = \sigma_0 \otimes A_2^{(m)} + \sigma_1 \otimes B_2^{(m)},$$

$$B_1^{(m+1)} = \sigma_3 \otimes A_3^{(m)} - \sigma_2 i \otimes B_3^{(m)}, \quad B_2^{(m+1)} = \sigma_3 \otimes A_4^{(m)} - \sigma_2 i \otimes B_4^{(m)}$$

for some $A_t^{(m)}, B_t^{(m)}$, $t = 1, 2, 3, 4$. Thus,

$$A_1^{(m+1)} A_2^{(m+1)} = \sigma_0 \otimes (A_1^{(m)} A_2^{(m)} + B_1^{(m)} B_2^{(m)}) + \sigma_1 \otimes (A_1^{(m)} B_2^{(m)} + B_1^{(m)} A_2^{(m)}),$$

$$B_1^{(m+1)} B_2^{(m+1)} = \sigma_0 \otimes (A_3^{(m)} A_4^{(m)} - B_3^{(m)} B_4^{(m)}) - \sigma_1 \otimes (A_3^{(m)} B_4^{(m)} - B_3^{(m)} A_4^{(m)}),$$

$$A_1^{(m+1)} B_2^{(m+1)} = \sigma_3 \otimes (A_1^{(m)} A_4^{(m)} + B_1^{(m)} B_4^{(m)}) - \sigma_2 i \otimes (A_1^{(m)} B_4^{(m)} + B_1^{(m)} A_4^{(m)}),$$

$$B_1^{(m+1)} A_2^{(m+1)} = \sigma_3 \otimes (A_3^{(m)} A_2^{(m)} - B_3^{(m)} B_2^{(m)}) + \sigma_2 i \otimes (A_3^{(m)} B_2^{(m)} - B_3^{(m)} A_2^{(m)}).$$

By the mathematical induction hypothesis, $A_t^{(m)} A_s^{(m)} - B_t^{(m)} B_s^{(m)} \in S_{2^m}(\mathbb{R})$ and hence $B_t^{(m)} B_s^{(m)} \in S_{2^m}(\mathbb{R})$ since $A_t^{(m)} A_s^{(m)} \in S_{2^m}(\mathbb{R})$. Thus, $A_1^{(m)} A_2^{(m)} + B_1^{(m)} B_2^{(m)}$, $A_3^{(m)} A_4^{(m)} - B_3^{(m)} B_4^{(m)}$, $A_1^{(m)} A_4^{(m)} + B_1^{(m)} B_4^{(m)}$, and $A_3^{(m)} A_2^{(m)} - B_3^{(m)} B_2^{(m)}$ are all in $S_{2^m}(\mathbb{R})$.

On the other hand, $A_t^{(m)} B_s^{(m)} + B_t^{(m)} A_s^{(m)} \in T_{2^m}(\mathbb{R})$ by the mathematical induction hypothesis. Thus, $B_t^{(m)} A_s^{(m)} \in T_{2^m}(\mathbb{R})$ by Lemma 2.3 and we obtain $A_1^{(m)} B_2^{(m)} + B_1^{(m)} A_2^{(m)}$, $A_3^{(m)} B_4^{(m)} - B_3^{(m)} A_4^{(m)}$, $A_1^{(m)} B_4^{(m)} + B_1^{(m)} A_4^{(m)}$, $A_3^{(m)} B_2^{(m)} - B_3^{(m)} A_2^{(m)}$ are all in $T_{2^m}(\mathbb{R})$. Thus, $A_1^{(m+1)} A_2^{(m+1)} - B_1^{(m+1)} B_2^{(m+1)} \in S_{2^{m+1}}(\mathbb{R})$ and $A_1^{(m+1)} B_2^{(m+1)} + B_1^{(m+1)} A_2^{(m+1)} \in T_{2^{m+1}}(\mathbb{R})$. Therefore, $r_1^{(m+1)} r_2^{(m+1)} \in \mathcal{R}_{2^{m+1}}(\mathbb{C})$ and the theorem is proved. □

Theorem 2.5. *The subalgebra $\mathcal{R}_{2^n}(\mathbb{C})$ of $M_{2^n}(\mathbb{C})$ is isomorphic to the Clifford algebra $C\ell_{p,q}$ for some p and q. Concretely,*

(1) $\mathcal{R}_{2^n}(\mathbb{C}) \cong C\ell_{[\frac{n}{2}]+2,[\frac{n}{2}]}$ *if n is an odd integer.*

(2) $\mathcal{R}_{2^n}(\mathbb{C}) \cong C\ell_{\frac{n}{2},\frac{n}{2}+1}$ *if n is an even integer.*

Here, $[x]$ is the greatest integer less than or equal to the real number x.

Proof. Define $A^{(m)} \in S_{2^n}(\mathbb{R})$ and $B^{(n)} \in T_{2^n}(\mathbb{R})$ as follows:

$$(t,1)\text{-th entry of } A^{(m)} = \begin{cases} 1, & t = 2^m \\ 0, & \text{otherwise} \end{cases}$$

$$(t,1)\text{-th entry of } B^{(n)} = \begin{cases} 1, & t = 2^n \\ 0, & \text{otherwise.} \end{cases}$$

Also, define $\alpha_m \in \mathcal{R}_{2^n}(\mathbb{C})$ as follows:

$$\alpha_m = \begin{cases} I_{2^n}, & m = 0 \\ A^{(m)}, & m = 1, 2, \ldots, n \\ B^{(n)}\, i, & m = n+1. \end{cases}$$

Since the entries in the first column of $\alpha_m \in \mathcal{R}_{2^n}(\mathbb{C})$ determines the other entries of α_m, we can express α_m as follows:

Let $K_1 = -\sigma_2 i$ and for $m \geq 2$, let

$$K_m = \begin{pmatrix} O_{2^{m-1}} & -K_{m-1} \\ K_{m-1} & O_{2^{m-1}} \end{pmatrix} \in M_{2^m}(\mathbb{R})$$

and

$$T_{m-1} = \begin{pmatrix} O_{2^{m-1}} & K_{m-1} \\ K_{m-1} & O_{2^{m-1}} \end{pmatrix} \in M_{2^m}(\mathbb{R}).$$

Then,

$$\alpha_1 = \begin{pmatrix} \sigma_1 & O_2 & \cdots & O_2 & O_2 \\ O_2 & \sigma_1 & \cdots & O_2 & O_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O_2 & O_2 & \cdots & \sigma_1 & O_2 \\ O_2 & O_2 & \cdots & O_2 & \sigma_1 \end{pmatrix} \in M_{2^n}(\mathbb{R})$$

and, for $2 \leq m \leq n$,

$$\alpha_m = \begin{pmatrix} T_{m-1} & O_{2^m} & \cdots & O_{2^m} \\ O_{2^m} & T_{m-1} & \cdots & O_{2^m} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^m} & O_{2^m} & \cdots & T_{m-1} \end{pmatrix} \in M_{2^n}(\mathbb{R}).$$

Also, for $n \geq 2$,

$$\alpha_{n+1} = \begin{pmatrix} O_2 & O_2 & \cdots & O_2 & -\sigma_2 \\ O_2 & O_2 & \cdots & \sigma_2 & O_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O_2 & -\sigma_2 & \cdots & O_2 & O_2 \\ \sigma_2 & O_2 & \cdots & O_2 & O_2 \end{pmatrix} \in M_{2^n}(\mathbb{C})$$

if n is an even integer and

$$\alpha_{n+1} = \begin{pmatrix} O_2 & O_2 & \cdots & O_2 & \sigma_2 \\ O_2 & O_2 & \cdots & -\sigma_2 & O_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O_2 & -\sigma_2 & \cdots & O_2 & O_2 \\ \sigma_2 & O_2 & \cdots & O_2 & O_2 \end{pmatrix} \in M_{2^n}(\mathbb{C})$$

if n is an odd integer. Note that we can express α_m in tensor form as follows:

$$\begin{aligned} \alpha_1 &= \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_1, \\ \alpha_2 &= \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes \sigma_1 \otimes (-\sigma_2 i), \\ \alpha_3 &= \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes \sigma_1 \otimes (-\sigma_2 i) \otimes (-\sigma_2 i), \\ &\vdots \\ \alpha_n &= \sigma_1 \otimes (-\sigma_2 i) \otimes \cdots \otimes (-\sigma_2 i) \otimes (-\sigma_2 i), \\ \alpha_{n+1} &= (-\sigma_2 i) \otimes (-\sigma_2 i) \otimes \cdots \otimes (-\sigma_2 i) \otimes (-\sigma_2 i)\, i. \end{aligned}$$

Since $\sigma_0^2 = \sigma_1^2 = I_2$ and $(-\sigma_2 i)^2 = -I_2$ for all m with $1 \leq m \leq n$, we obtain

$$\alpha_m^2 = \begin{cases} -I_{2^n}, & \text{if } m \text{ is an even integer} \\ I_{2^n}, & \text{if } m \text{ is an odd integer} \end{cases}$$

and

$$\alpha_{n+1}^2 = \begin{cases} -I_{2^n}, & \text{if } n \text{ is an even integer} \\ I_{2^n}, & \text{if } n \text{ is an odd integer.} \end{cases}$$

Moreover, for all m and ℓ with $1 \leq m, \ell \leq n+1$ and $m \neq \ell$, $\alpha_m \alpha_\ell = -\alpha_\ell \alpha_m$ since $\sigma_1\sigma_2 = -\sigma_2\sigma_1$. Hence $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}$ can be considered as the vector generators of a Clifford algebra. Since $S_{2^n}(\mathbb{R}) \cong C\ell_{[\frac{n}{2}]+1,[\frac{n}{2}]}$ if n is an odd integer and $S_{2^n}(\mathbb{R}) \cong C\ell_{\frac{n}{2},\frac{n}{2}}$ if n is an even integer [6], we now can conclude that $\mathcal{R}_{2^n}(\mathbb{C}) \cong C\ell_{[\frac{n}{2}]+2,[\frac{n}{2}]}$ if n is an odd integer and $\mathcal{R}_{2^n}(\mathbb{C}) \cong C\ell_{\frac{n}{2},\frac{n}{2}+1}$ if n is an even integer. □

Example 2.6. For $n = 3$, $\mathcal{R}_{2^3}(\mathbb{C}) \cong C\ell_{3,1}$ and the vector generators are

$$\alpha_1 = \sigma_0 \otimes \sigma_0 \otimes \sigma_1, \quad \alpha_2 = \sigma_0 \otimes \sigma_1 \otimes (-\sigma_2 i),$$

$$\alpha_3 = \sigma_1 \otimes (-\sigma_2 i) \otimes (-\sigma_2 i), \quad \alpha_4 = (-\sigma_2 i) \otimes (-\sigma_2 i) \otimes (-\sigma_2 i) i.$$

Also, the corresponding matrix representations are the following 8×8 matrices.

$$\alpha_1 = \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}, \ \alpha_4 = \begin{pmatrix} 0&0&0&0&0&0&0&-i\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&i&0&0\\ 0&0&0&0&-i&0&0&0\\ 0&0&0&i&0&0&0&0\\ 0&0&-i&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0\\ i&0&0&0&0&0&0&0 \end{pmatrix}.$$

As one can see, the matrix representations of the vector generators have simple and regular patterns and so it makes it easy to investigate a lot of the algebraic properties. For example, $tr(\alpha_m)$ and $\det(\alpha_m)$ can be calculated automatically for all $m = 1, 2, \ldots, n+1$.

Theorem 2.7. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ be the matrix representations of vector generators of the Clifford algebra constructed in the proof of theorem 2.5. Then,*

(1) $tr(\alpha_m) = 0$, $m = 1, 2, \ldots, n+1$.

(2) $\det(\alpha_m) = 1$ *or* $\det(\alpha_m) = -1$, $m = 1, 2, \ldots, n+1$.

3. Spectrum of matrix representations of the Clifford algebras

In this section, we give some information about the spectrum of the constructed complex matrix representation. The spectrum of A is denoted by $spec(A)$.

Theorem 3.1. *Let $A = \sum_{m=1}^{n+1} b_m\alpha_m$. Then, A has 2^n complex eigenvalues.*

Proof. Note that $\det(A - \lambda I_{2^n}) = 0$ generates 2^n degree equations. Since $\mathbb{C}$ is an algebraically closed field, the result follows. □

Theorem 3.2. *Let $A = \sum_{m=1}^{n+1} b_m\alpha_m$. Then,*

$$spec(A) \subset \left\{ z \in \mathbb{C} \,|\, |z| \le \sum_{\ell=1}^{n-1} |b_\ell| + |b_n + b_{n+1}i| \right\}.$$

Proof. Let $A = (a_{ts})_{2^n\times 2^n}$. Then, $R_1(A) = \sum_{s\neq 1} |a_{1s}| = \sum_{\ell=1}^{n} |a_{12^\ell}|$ and $R_1(A) = R_m(A)$ for all $m = 1, 2, \ldots, n+1$. Thus,

$$spec(A) \subset \bigcup_{m=1}^{2^n} \left\{ z \in \mathbb{C} \,|\, |z - a_{mm}| \le \sum_{\ell=1}^{n} |a_{12^\ell}| \right\}$$

by the Geršgorin theorem [5]. But, $a_{mm} = 0$ for all $m = 1, 2, \ldots, n+1$ and $|a_{12^\ell}| = |b_\ell|$ for all $1 \leq \ell \leq n-1$ and $|a_{12^n}| = |b_n + b_{n+1}i|$. Hence

$$\left\{ z \in \mathbb{C} \,\middle|\, |z - a_{mm}| \leq \sum_{\ell=1}^{n} |a_{12^\ell}| \right\} = \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sum_{\ell=1}^{n-1} |b_\ell| + |b_n + b_{n+1}i| \right\}$$

and we prove the theorem. □

Corollary 3.3. (1) *spec*$(\alpha_m) \subset \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ *for all* $1 \leq m \leq n+1$.
(2) *Let* $A = \sum_{m=1}^{n+1} \alpha_m$. *Then, spec*$(A) \subset \left\{ z \in \mathbb{C} \,|\, |z| \leq n - 1 + \sqrt{2} \right\}$.

Specially, we can easily obtain the spectrum of the pure imaginary generator α_{n+1}.

Example 3.4. If n is an odd integer, then *spec*$(\alpha_{n+1}) \subset \{-1, 1\}$.

Proof. Let $\lambda \in$ *spec*(α_{n+1}). Then, $(\alpha_{n+1} - \lambda I_{2^n})X = O$ for some $X \neq O$. Note that $(\alpha_{n+1} - \lambda I_{2^n})(\alpha_{n+1} - \lambda I_{2^n})^T = (\lambda^2 - 1)I_{2^n}$ and so $\det(\alpha_{n+1} - \lambda I_{2^n})^2 = (\lambda^2 - 1)^{2^n}$. Thus, we obtain $\lambda = -1$ or $\lambda = 1$. □

Example 3.4 shows that eigenvalues of the pure imaginary generator occur on the boundary of the Geršgorin disc.

References

[1] A. H. Bilge, Koçak, and S. Uğuz, *Canonical bases for real representations of Clifford algebras*, Linear Algebra Appl. **419** (2006), no. 2-3, 417–439. https://doi.org/10.1016/j.laa.2006.01.034
[2] Y. Brihaye, P. Maslanka, S. Giler, and P. Kosinsk, *Real representations of Clifford algebras*, J. Math. Phys. **33** (1992), no. 5, 1579–1581. https://doi.org/10.1063/1.529682
[3] G. N. Hile and P. Lounesto, *Matrix representations of Clifford algebras*, Linear Algebra Appl. **128** (1990), 51–63. https://doi.org/10.1016/0024-3795(90)90282-H
[4] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. https://doi.org/10.1017/CBO9780511810817
[5] D. Lee and Y. Song, *Extension of algebras and its applications to the real Clifford algebras*, Linear Multilinear Algebra **64** (2016), no. 11, 2241–2251. https://doi.org/10.1080/03081087.2016.1147525
[6] C. P. Poole, Jr., and H. A. Farach, *Pauli-Dirac matrix generators of Clifford algebras*, Found. Phys. **12** (1982), no. 7, 719–738. https://doi.org/10.1007/BF00729808
[7] Y. Song and D. Lee, *A construction of matrix representation of Clifford algebras*, Adv. Appl. Clifford Algebr. **25** (2015), no. 3, 719–731. https://doi.org/10.1007/s00006-014-0521-x
[8] ———, *Vector generators of the real Clifford algebras*, Linear Multilinear Algebra **63** (2015), no. 12, 2448–2460. https://doi.org/10.1080/03081087.2015.1018812
[9] Y. Tian, *Universal similarity factorization equalities over real Clifford algebras*, Adv. Appl. Clifford Algebras **8** (1998), no. 2, 365–402. https://doi.org/10.1007/BF03043105
[10] F. Zhang, *Geršgorin type theorems for quaternionic matrices*, Linear Algebra Appl. **424** (2007), no. 1, 139–153. https://doi.org/10.1016/j.laa.2006.08.004

Youngkwon Song
Department of Mathematics
Kwangwoon University
Seoul 01897, Korea
Email address: yksong@kw.ac.kr