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ESSENTIAL EXACT SEQUENCES

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Abstract. Let R be a commutative ring with identity and M a unital R-module. We give a new generalization of exact sequences called e-exact sequences. A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be e-exact if f is monic, $\mathrm{Im} f \leq_e \mathrm{Ker} g$ and $\mathrm{Im} g \leq_e C$. We modify many famous theorems including exact sequences to one includes e-exact sequences like 3×3 lemma, four and five lemmas. Next, we prove that for torsion-free module M, the contravariant functor $\mathrm{Hom}(-, M)$ is left e-exact and the covariant functor $M \otimes -$ is right e-exact. Finally, we define e-projective module and characterize it. We show that the direct sum of R-modules is e-projective module if and only if each summand is e-projective.

1. Introduction

Throughout this article R will denote a commutative ring with identity and M a unitary R-module. Here we use monic and epic to denote monomorphism and epimorphism. A submodule N of a module M is called essential (large) in M if the intersection of N with each nonzero submodule of M is nonzero and we write $N \leq_e M$. In this case, M is called essential extension of N. Equivalently, N is essential submodule of M if $N \cap Rx \neq 0$ for any nonzero element $x \in M$ ([1, p. 75]). So in particular, a nonzero ideal I of R is an essential ideal of R if $I \cap K \neq 0$ for any nonzero ideal K of R which is equivalent to the condition $I \cap Rx \neq 0$ for any nonzero element $x \in R$. An element m in an R-module M is torsion element if there is a nonzero element r in R such that $rm = 0$. The set of all torsion elements $T(M)$ is a submodule of M called torsion submodule of M. A torsion module is a module whose elements are all torsion that is $T(M) = M$. A torsion-free module is a module whose elements are not torsion, other than 0, that is $T(M) = 0$. A sequence of R-modules and R-morphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

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is said to be exact at M_i if $\mathrm{Im}(f_{i-1}) = \mathrm{Ker}(f_i)$. In this paper we generalize this notion to e-exact sequence using essential (large) submodule, that is, instead of saying $\mathrm{Im}(f_{i-1})$ is equal to $\mathrm{Ker}(f_i)$ we say $\mathrm{Im}(f_{i-1})$ is large in $\mathrm{Ker}(f_i)$. For more details on the notions used in this paper see [2] and [3].

2. Essential exact sequences

Definition. A sequence of R-modules and R-morphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

is said to be *essential exact* briefly (e-exact) at M_i if $\mathrm{Im}(f_{i-1}) \leq_e \mathrm{Ker}(f_i)$, and to be e-exact if it is e-exact at each M_i. In particular, a sequence of R-modules and R-morphisms

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

is a *short e-exact sequence* if and only if $\mathrm{Ker}(f_1) = 0$, $\mathrm{Im}(f_1) \leq_e \mathrm{Ker}(f_2)$ and $\mathrm{Im}(f_2) \leq_e A_3$.

Definition. An R-morphism $f : M \longrightarrow N$ is called e-epic if $\mathrm{Im}f \leq_e N$. In addition if $\mathrm{Ker}f = 0$, then f is called essential monic.

The next example shows that the class of all e-exact sequences is larger than the class of exact sequences.

Example 2.1. Consider the short e-exact sequence

$$0 \longrightarrow 4Z \xrightarrow{f_1} Z \xrightarrow{f_2} Z/4Z \longrightarrow 0$$

where f_1 and f_2 are defined as $f_1(4n) = 2n$ and $f_2(n) = 2n + 4Z$. Since f_2 is not epic, the sequence is not exact.

Lemma 2.2 (Four lemma with e-exact sequence)**.** *Consider the commutative diagram of an R-modules and R-morphisms with e-exact rows*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "t_3"] & A_4 \arrow[d, "t_4"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4
\end{tikzcd}$$

where R is a domain.

1. *If t_1, t_3 are e-epic, t_4 is monic and B_2 is torsion-free R-module, then t_2 is e-epic.*
2. *If t_1 is e-epic and t_2, t_4 are monic, then* $\mathrm{Ker}(t_3)$ *is torsion module. In addition, if A_3 is torsion-free R-module, then t_3 is monic.*

Proof. (1) Let b_2 be a non-zero element of B_2. Then $g_2(b_2) \in B_3$. Since $\mathrm{Im}(t_3) \leq_e B_3$, $\mathrm{Im}(t_3) \cap Rg_2(b_2) \neq 0$. So there exist $0 \neq r \in R$ and $a_3 \in A_3$ such that $t_3(a_3) = rg_2(b_2)$. Then $t_4 f_3(a_3) = g_3(t_3(a_3)) = g_3(rg_2(b_2)) = rg_3(g_2(b_2)) = 0$. As t_4 is monic, $f_3(a_3) \in \mathrm{Ker}(t_4) = 0$. So $f_3(a_3) = 0$ and $a_3 \in \mathrm{Ker}(f_3)$. Since $\mathrm{Im}(f_2) \leq_e \mathrm{Ker}(f_3)$, there exist $a_2 \in A_2$ and $s \in R$ such that $f_2(a_2) = sa_3 \neq 0$. Then $rsg_2(b_2) = st_3(a_3) = t_3(sa_3) = t_3(f_2(a_2)) = g_2 t_2(a_2)$ and $rsb_2 - t_2(a_2) \in \mathrm{Ker}(g_2)$. Since we have $\mathrm{Im}(g_1) \cap R(rsb_2 - t_2(a_2)) \neq 0$, there exist $b_1 \in B_1$ and $k \in R$ such that $g_1(b_1) = k(rsb_2 - t_2(a_2))$. Now, $b_1 \in B_1$ and $\mathrm{Im}(t_1) \leq_e B_1$. Then, there exists $0 \neq q \in R$ such that $t_1(a_1) = qb_1$. Therefore

$$t_2 f_1(a_1) = g_1 t_1(a_1) = g_1(qb_1) = qg_1(b_1) = q(krsb_2 - kt_2(a_2)),$$
$$t_2 f_1(a_1) = (qkrs)b_2 - qkt_2(a_2),$$
$$t_2(qka_2 + f_1(a_1)) = (qkrs)b_2.$$

Hence from the last equation and being B_2 a torsion-free module, we conclude that $\mathrm{Im}(t_2) \cap Rb_2 \neq 0$ and t_2 is e-epic.

(2) Suppose $a_3 \in \mathrm{Ker}(t_3)$. Then $t_3(a_3) = 0$. By commutativity of the last square of the diagram, $t_4 f_3(a_3) = g_3 t_3(a_3) = 0$, so $f_3(a_3) \in \mathrm{Ker}(t_4)$ and monicness of t_4 gives us $a_3 \in \mathrm{Ker}(f_3)$. Since $\mathrm{Im}(f_2) \leq_e \mathrm{Ker}(f_3)$, there exist $a_2 \in A_2$ and $r \in R$ such that $f_2(a_2) = ra_3 \neq 0$ and $g_2 t_2(a_2) = t_3 f_2(a_2) = t_3(ra_3) = 0$, so $t_2(a_2) \in \mathrm{Ker}(g_2)$. Again, there exist $b_1 \in B_1$ and $s \in R$ such that $g_1(b_1) = st_2(a_2) \neq 0$. But $\mathrm{Im}(t_1) \leq_e B_1$, so there exist $a_1 \in A_1$ and $k \in R$ such that $t_1(a_1) = kb_1$. Then $kst_2(a_2) = kg_1(b_1) = g_1 t_1(a_1) = t_2 f_1(a_1)$ and $t_2(ksa_2 - f_1(a_1)) = 0$. The monicness of t_2 implies $ksa_2 = f_1(a_1)$. Therefore $ksra_3 = ksf_2(a_2) = f_2(ks(a_2)) = f_2 f_1(a_1) = 0$ which means $ksra_3 = 0$. Hence $\mathrm{Ker}(t_3)$ is torsion module. Being A_3 torsion-free module gives that $\mathrm{Ker}(t_3) = 0$ and t_3 is monic. □

Lemma 2.3 (Five lemma with e-exact sequence)**.** *Consider the commutative diagram of R-modules and R-morphisms with e-exact rows*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "t_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "t_4"] & A_5 \arrow[d, "t_5"] \\ B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd}$$

where R is a domain and A_3, B_3 are torsion-free R-modules.

(1) *If t_2, t_4 are e-epic and t_5 is monic, then t_3 is e-epic.*
(2) *If t_1 is e-epic and t_2, t_4 are monic, then t_3 is monic.*

Proof. (1) Suppose that b_3 is a non-zero element of B_3. Since $g_3(b_3) \in B_4$ and $\mathrm{Im}(t_4) \leq_e B_4$, $\mathrm{Im}(t_4) \cap Rg_3(b_3) \neq 0$. So, there exist $a_4 \in A_4$ and $0 \neq r \in R$ such that $t_4(a_4) = rg_3(b_3)$. Also $t_5 f_4(a_4) = g_4 t_4(a_4) = g_4(rg_3(b_3)) = rg_4 g_3(b_3) = 0$. Thus $f_4(a_4) \in \mathrm{Ker}(t_5) = 0$ and $a_4 \in \mathrm{Ker}(f_4)$. By assumption $\mathrm{Im}(f_3) \leq_e \mathrm{Ker}(f_4)$ and hence $\mathrm{Im}(f_3) \cap Ra_4 \neq 0$. So, there exist $a_3 \in A_3$ and $0 \neq s \in R$ such that $f_3(a_3) = sa_4$ and $g_3(rsb_3 - t_3(a_3)) = g_3(rsb_3) -$

$g_3t_3(a_3) = rsg_3(b_3) - t_4f_3(a_3) = st_4(a_4) - t_4(sa_4) = 0$, that is, $rsb_3 - t_3(a_3) \in \operatorname{Ker}(g_3)$ and by e-exactness of the second row, we have $\operatorname{Im}(g_2) \leq_e \operatorname{Ker}(g_3)$. Then $\operatorname{Im}(g_2) \cap R(rsb_3 - t_3(a_3)) \neq 0$ and so $g_2(b_2) = q(rsb_3 - t_3(a_3)) \neq 0$ for some $b_2 \in B_2$ and $0 \neq q \in R$. Also $\operatorname{Im}(t_2) \leq_e B_2$ and so we can find $a_2 \in A_2$ and $0 \neq k \in R$ such that $t_2(a_2) = kb_2$. Thus $t_3(f_2(a_2) + qka_3) = t_3f_2(a_2) + t_3(qka_3) = g_2t_2(a_2) + qkt_3(a_3) = g_2(kb_2) + qkt_3(a_3) = kg_2(b_2) + kt_3(a_3) = kq(rsb_3 - t_3(a_3)) + qkt_3(a_3) = qkrsb_3 - kqt_3(a_3) + kqt_3(a_3) = (qkrs)b_3$. Therefore $\operatorname{Im}(t_3) \cap Rb_3 \neq 0$ and t_3 is e-epic.

(2) Let $a_3 \in \operatorname{Ker}(t_3)$. Then $t_3(a_3) = 0$ and by commutativity of the third square, $t_4f_3(a_3) = g_3t_3(a_3) = 0$. Thus $f_3(a_3) \in \operatorname{Ker}(t_4) = 0$ and $a_3 \in \operatorname{Ker}(f_3)$. Since $\operatorname{Im}(f_2) \leq_e \operatorname{Ker}(f_3)$, there exist $a_2 \in A_2$ and $0 \neq r \in R$ such that $f_2(a_2) = ra_3$ and $g_2t_2(a_2) = t_3f_2(a_2) = t_3(ra_3) = rt_3(a_3) = 0$. Which means that $t_2(a_2) \in \operatorname{Ker}(g_2)$. By e-exactness we have $\operatorname{Im}(g_1) \cap Rt_2(a_2) \neq 0$, so there exist $b_1 \in B_1$ and $0 \neq s \in R$ such that $g_1(b_1) = st_2(a_2)$. Again $\operatorname{Im}(t_1) \leq_e B_1$, so there exist $a_1 \in A_1$ and $0 \neq k \in R$ such that $t_1(a_1) = kb_1$. From monicness of t_2 and the fact that $kst_2(a_2) = kg_1(b_1) = g_1t_1(a_1) = t_2f_1(a_1)$ we obtain that $ksa_2 = f_1(a_1)$ and so $ksra_3 = ksf_2(a_2) = f_2f_1(a_1) = 0$. Now torsion-freeness of B_3 implies that $ksra_3 = 0$ and $\operatorname{Ker}(t_3)$ is torsion module. The torsion-freeness of A_3 gives the monicness of t_3. $\square$

We conclude from Lemma 2.3 that if t_1, t_2, t_4 and t_5 are essential monic, then so is t_3. The following is our version of 3×3 lemma using e-exact sequences instead of exact sequences.

Lemma 2.4 (3×3 lemma with e-exact sequence)**.** *Consider the commutative diagram of R-modules and R-morphisms where R is a domain and A_2, A_3 are torsion-free R-modules*:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "i_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "j_1"] & A_3 \arrow[r] \arrow[d, "p_1"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "g_1"] \arrow[d, "i_2"] & B_2 \arrow[r, "g_2"] \arrow[d, "j_2"] & B_3 \arrow[r] \arrow[d, "p_2"] & 0 \\
0 \arrow[r] & C_1 \arrow[r, "h_1"] \arrow[d] & C_2 \arrow[r, "h_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the columns and the two bottom rows are e-exact, then the top row is also e-exact.

Proof. To prove that the top row is e-exact, we have to check the following three conditions:

(1) $\mathrm{Ker}(f_1) = 0$.

Take $a_1 \in \mathrm{Ker}(f_1)$. Then $g_1 i_1(a_1) = j_1 f_1(a_1) = 0$ and $i_1(a_1) \in \mathrm{Ker}(g_1)$. Since $\mathrm{Ker}(g_1) = 0$, $i_1(a_1) = 0$ and $a_1 \in \mathrm{Ker}(i_1) = 0$. Therefore $a_1 = 0$.

(2) $\mathrm{Im}(f_1) \leq_e \mathrm{Ker}(f_2)$.

First to prove $\mathrm{Im}(f_1) \subseteq \mathrm{Ker}(f_2)$. Let $a_2 \in \mathrm{Im}(f_1)$. Then, there exists $a_1 \in A_1$ such that $f_1(a_1) = a_2$ and $g_1 i_1(a_1) = j_1 f_1(a_1) = j_1(a_2)$, which means that $j_1(a_2) \in \mathrm{Im}(g_1) \subseteq \mathrm{Ker}(g_2)$ and so $p_1 f_2(a_2) = g_2 j_1(a_2) = 0$. Hence $f_2(a_2) \in \mathrm{Ker}(p_1) = 0$ and so $a_2 \in \mathrm{Ker}(f_2)$. Now for essentiality, take a_2 to be a non-zero element of $\mathrm{Ker}(f_2)$. Then $f_2(a_2) = 0$ and $g_2 j_1(a_2) = p_1 f_2(a_2) = 0$. So $j_1(a_2) \in \mathrm{Ker}(g_2)$ and as $\mathrm{Im}(g_1) \leq_e \mathrm{Ker}(g_2)$, $\mathrm{Im}(g_1) \cap R j_1(a_2) \neq 0$. Hence, there exist $0 \neq s \in R$ and $b_1 \in B_1$ such that $g_1(b_1) = s j_1(a_2)$. Also $h_1 i_2(b_1) = j_2 g_1(b_1) = s j_2 j_1(a_2) = 0$. Thus $i_2(b_1) \in \mathrm{Ker}(h_1) = 0$. By e-exactness of the first column we get $Im(i_1) \cap R b_1 \neq 0$. Then, there exist $a_1 \in A_1$ and $k \in R$ such that $i_1(a_1) = k b_1 \neq 0$ and so $k s j_1(a_2) = g_1(k b_1) = g_1 i_1(a_1) = j_1 f_1(a_1)$. Thus $j_1(k s a_2 - f_1(a_1)) = 0$ and as $\mathrm{Ker}(j_1) = 0$, $k s a_2 = f_1(a_1)$. Therefore we have $\mathrm{Im}(f_1) \leq_e \mathrm{Ker}(f_2)$.

(3) $\mathrm{Im}(f_2) \leq_e A_3$.

Let a_3 be a non-zero element of A_3 and $p_1(a_3) \in B_3$. There exist $b_2 \in B_2$ and $r \in R$ such that $g_2(b_2) = r p_1(a_3) \neq 0$. By commutativity of the diagram

$$h_2 j_2(b_2) = p_2 g_2(b_2) = r p_2 p_1(a_3) = 0.$$

That means $j_2(b_2) \in \mathrm{Ker}(h_2)$ and by e-exactness of the bottom row, there exist $c_1 \in C_1$ and $s \in R$ such that $h_1(c_1) = s j_2(b_2)$. Again by e-exactness of the first column there exist $m \in R$ and $b_1 \in B_1$ such that $i_2(b_1) = m c_1$. Then $m s j_2(b_2) = m h_1(c_1) = h_1(m c_1) = h_1 i_2(b_1) = j_2 g_1(b_1)$. Therefore $j_2(g_1(b_1) - s m b_2) = 0$, so $g_1(b_1) - s m b_2 \in \mathrm{Ker}(j_2)$. Since $\mathrm{Im}(j_1) \leq_e \mathrm{Ker}(j_2)$, there exist $a_2 \in A_2$ and $n \in R$ such that $j_1(a_2) = n(g_1(b_1) - s m b_2) \neq 0$. It is clear that

$$p_1 f_2(a_2) = g_2 j_1(a_2) = n g_2(g_1(b_1) - s m b_2) = -n s m g_2(b_2).$$

So $p_1 f_2(a_2) = -n s m r p_1(a_3)$ which is equivalent to $p_1(f(a_2) + n m r a_3) = 0$. But p_1 is monic, so $f_2(a_2) = -n s m r a_3$. Hence $\mathrm{Im}(f_2) \leq_e A_3$. □

A functor F is called covariant left e-exact if for every short e-exact sequence

$$0 \longrightarrow A \xrightarrow{f_1} B \xrightarrow{f_2} C \longrightarrow 0\ ,$$

the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f_1)} F(B) \xrightarrow{F(f_2)} F(C)$$

is e-exact and called covariant right e-exact if the sequence

$$F(A) \xrightarrow{F(f_1)} F(B) \xrightarrow{F(f_2)} F(C) \longrightarrow 0$$

is e-exact whenever $0 \longrightarrow A \xrightarrow{f_1} B \xrightarrow{f_2} C \longrightarrow 0$ is e-exact. A functor F is called covariant e-exact functor if it is both covariant left e-exact functor and covariant right e-exact functor. Now, suppose S is a multiplicatively closed subset of a ring R. In what follows, we show that the localization functor is e-exact functor.

Proposition 2.5. *If a sequence of R-modules and R-morphisms*

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

is e-exact, then so is the sequence

$$0 \longrightarrow S^{-1}A_1 \xrightarrow{S^{-1}f_1} S^{-1}A_2 \xrightarrow{S^{-1}f_2} S^{-1}A_3 \longrightarrow 0 .$$

Proof. First of all we show that $\mathrm{Im}(S^{-1}f_1) \leq_e \mathrm{Ker}(S^{-1}f_2)$. Let a_2/s be a non-zero element of $\mathrm{Ker}(S^{-1}f_2)$. Then $S^{-1}f_2(a_2/s) = f_2(a_2)/s = 0$ and so there exists $t \in S$ such that $tf_2(a_2) = f_2(ta_2) = 0$. Thus $ta_2 \in \mathrm{Ker}(f_2)$. Since $\mathrm{Im}(f_1) \leq_e \mathrm{Ker}(f_2)$, there exist $r \in R$ and $a_1 \in A_1$ such that $f_1(a_1) = rta_2 \neq 0$. Then $f_1(a_1)/s = (rt/1)(a_2/s)$, which means $S^{-1}f_1(a_1/s) = (rt/1)(a_2/s)$. Therefore $\mathrm{Im}(S^{-1}f_1) \cap S^{-1}R(a_2/s) \neq 0$.

Now to prove $\mathrm{Ker}(S^{-1}f_1) = 0$ take $a_1/s \in \mathrm{Ker}(S^{-1}f_1)$. Then $S^{-1}f_1(a_1/s) = f_1(a_1)/s = 0$. Then there exists $t \in S$ such that $tf_1(a_1) = f_1(ta_1) = 0$. By hypotheses $\mathrm{Ker}(f_1) = 0$, so $ta_1 = 0$ and $a_1/s = 0$.

Finally, we want to show that $\mathrm{Im}(S^{-1}f_2) \leq_e S^{-1}A_3$. Suppose that a_3/s is a non-zero element of $S^{-1}A_3$ where $a_3 \in A_3$ and since $\mathrm{Im}(f_2) \leq_e A_3$, there exist $a_2 \in A_2$ and $r \in R$ such that $f_2(a_2) = ra_3 \neq 0$. Also $f_2(a_2)/s = (r/1)(a_3/s)$ which implies that $S^{-1}f_2(a_2/s) = (r/1)(a_3/s)$. Hence $\mathrm{Im}(S^{-1}f_2) \cap S^{-1}R(a_3/s) \neq 0$ and the sequence

$$0 \longrightarrow S^{-1}A_1 \xrightarrow{f_1} S^{-1}A_2 \xrightarrow{f_2} S^{-1}A_3 \longrightarrow 0$$

is e-exact. □

A functor G is called contravariant left e-exact if for every short e-exact sequence

$$0 \longrightarrow A \xrightarrow{f_1} B \xrightarrow{f_2} C \longrightarrow 0 ,$$

the sequence

$$0 \longrightarrow G(C) \xrightarrow{G(f_2)} G(B) \xrightarrow{G(f_1)} G(A)$$

is e-exact. In the following two results, we discuss the left e-exactness of the covariant functor $\mathrm{Hom}(M,-)$ and the contravariant functor $\mathrm{Hom}(-,M)$.

Theorem 2.6. *The sequence of R-modules and R-morphisms*

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

is e-exact if and only if for all R-module B, the sequence

$$0 \longrightarrow \mathrm{Hom}(B, A_1) \xrightarrow{{f_1}^*} \mathrm{Hom}(B, A_2) \xrightarrow{{f_2}^*} \mathrm{Hom}(B, A_3)$$

is e-exact.

Proof. ($\Rightarrow$) First to show that $\mathrm{Ker}(f_1^*) = 0$ take $h \in \mathrm{Ker}(f_1^*)$. Then $f_1^*(h) = f_1 \circ h = 0$ so $f_1(h(a_1)) = 0$ for all $a_1 \in A_1$. Thus $h(a_1) \in \mathrm{Ker}(f_1) = 0$. Hence $h(a_1) = 0$ for all $a_1 \in A_1$ and so $h = 0$. Also to prove $\mathrm{Im}(f_1^*) \leq_e \mathrm{Ker}(f_2^*)$. We take a non-zero element $g \in \mathrm{Ker}(f_2^*)$ and show that $\mathrm{Im}(f_1^*) \cap Rg \neq 0$. Since $g \in \mathrm{Ker}(f_2^*)$, $f_2^*(g) = 0 = f_2 \circ g$. Then $\mathrm{Im}(g) \subseteq \mathrm{Ker}(f_2)$, and by e-exactness, $\mathrm{Im}(f_1) \leq_e \mathrm{Ker}(f_2)$, Since $g : B \to A_2$, for all $b \in B, g(b) \in \mathrm{Ker}(f_2)$ and $\mathrm{Im}(f_1) \cap Rg(b) \neq 0$. Then, there exist $a_1 \in A_1$ and $r \in R$ such that $f_1(a_1) = rg(b) \neq 0$ and so $g(rb) \in \mathrm{Im}(f_1)$. We define $l : B \to A_1$ by $l(b) = a_1$. Now to show that l is well-defined. Let $b_1 = b_2$ so $g(b_1) = g(b_2)$ and $rg(b_1) = rg(b_2)$. Hence $f_1(a_1) = f_1(a_2)$ which implies that $a_1 - a_2 \in \mathrm{Ker}(f_1) = 0$. Thus $a_1 = a_2$ and $l(b_1) = l(b_2)$. Now, we have $(f_1 \circ l)(b) = f_1(l(b)) = f_1(a_1) = rg(b)$, which gives $f_1 \circ l = rg$. Therefore $f_1^*(l) = rg \neq 0$ and $\mathrm{Im}(f_1^*) \cap Rg \neq 0$.

($\Leftarrow$) Let a_2 be a non-zero element of $\mathrm{Ker}(f_2)$. We set $B = \mathrm{Ker}(f_2)$ and define $i : B \to A_2$ to be the identity map. So $[f_2^*(i)](a_2) = f_2(i(a_2)) = f_2(a_2) = 0$, for all $a_2 \in \mathrm{Ker}(f_2)$. Then $f_2^*(i) = 0$ and $0 \neq i^* \in \mathrm{Ker}(f_2^*)$. Since $\mathrm{Im}(f_1^*) \leq_e \mathrm{Ker}(f_2^*)$, there exist $l \in \mathrm{Hom}(B, A_1)$ and $r \in R$ such that $f_1^*(l) = ri$ that is $f_1 \circ l = ri \neq 0$. Therefore

$$0 \neq ra_2 = ri(a_2) = f_1 \circ l(a_2) = f_1(l(a_2))$$

and $\mathrm{Im}(f_1) \cap Ra_2 \neq 0$. Now to prove $\mathrm{Ker}(f_1) = 0$, we take $B = \mathrm{Ker}(f_1)$ and define $i : B \to A_1$ to be the identity map. Then for all $a_1 \in \mathrm{Ker}(f_1)$, $[f_1^* i](a_1) = f_1 \circ i(a_1) = f_1(i(a_1)) = f_1(a_1) = 0$. So $f_1^*(i(a_1)) = 0$ and $i(a_1) \in \mathrm{Ker}(f_1^*) = 0$. Thus $a_1 = 0$ and hence $\mathrm{Ker}(f_1) = 0$. $\square$

Theorem 2.7. *If a sequence of R-modules and R-morphisms*

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

is e-exact, then for all torsion-free R-module B, the sequence

$$0 \longrightarrow \mathrm{Hom}(A_3, B) \xrightarrow{{f_2}^*} \mathrm{Hom}(A_2, B) \xrightarrow{{f_1}^*} \mathrm{Hom}(A_1, B)$$

is e-exact. The converse is true if $A_3/\mathrm{Im}(f_2)$ and $A_2/\mathrm{Im}(f_1)$ are torsion-free R-modules.

Proof. We have to prove that ${f_2}^*$ is monic and $\mathrm{Im}(f_2^*) \leq_e \mathrm{Ker}(f_1^*)$. For this purpose, take $g, h \in \mathrm{Hom}(A_3, B)$ and $f_2^*(h) = f_2^*(g)$. Then $h \circ f_2 = g \circ f_2$. Since $\mathrm{Im}(f_2) \leq_e A_3$, for all non-zero element $a_3 \in A_3$ there exist $a_2 \in A_2$ and $r \in R$ such that $f_2(a_2) = ra_3 \neq 0$. So

$$h(ra_3) = h(f_2(a_2)) = h \circ f_2(a_2) = g \circ f_2(a_2) = g(f_2(a_2)) = g(ra_3)$$

and $rh(a_3) = rg(a_3)$ that is $r(h(a_3) - g(a_3)) = 0$. Since B is torsion-free module, $h = g$. Therefore f_2^* is monic. Now let h be a non-zero element of $\mathrm{Ker}(f_1^*)$. Then $f_1^*(h) = 0$ and $h \circ f_1 = 0$. Since $\mathrm{Im}(f_2) \leq_e A_3$, for all non-zero $a_3 \in A_3$, there exist $a_2 \in A_2$ and $r \in R$ such that $f_2(a_2) = ra_3 \neq 0$. Define $g \in \mathrm{Hom}(A_3, B)$ by sending a_3 to $h(a_2)$ for all $a_3 \in A_3$. The map g is well-defined since for $a_3, \hat{a}_3 \in A_3$ with $a_3 = \hat{a}_3$, we have $ra_3 = r\hat{a}_3$, that is $f_2(a_2) = f_2(\hat{a}_2)$. Hence $a_2 - \hat{a}_2 \in \mathrm{Ker}(f_2)$ and by e-exactness there exist $a_1 \in A_1$ and $s \in R$ such that $f_1(a_1) = s(a_2 - \hat{a}_2) \neq 0$. Therefore

$$\begin{aligned} h(sa_2) &= h(sa_2 - s\hat{a}_2) + h(s\hat{a}_2) = h(f_1(a_1)) + h(s\hat{a}_2) \\ &= f_1^*(h(a_1)) + h(s\hat{a}_2) = h(s\hat{a}_2) \end{aligned}$$

and $h(a_2) = h(\hat{a}_2)$ by torsion-freeness of B. As $a_2 \in A_2$,

$$f_2^*(g(a_2)) = g \circ f_2(a_2) = g(f_2(a_2)) = g(ra_3) = rh(a_2)$$

and so $f_2^*(g) = rh \neq 0$. Hence $\mathrm{Im}(f_2^*) \leq_e \mathrm{Ker}(f_1^*)$.

Conversely, we have to show that $\mathrm{Im}(f_2) \leq_e A_3$. To do this, we define $l : A_3 \to B$ where $B = A_3/\mathrm{Im}(f_2)$ by $l(a_3) = a_3 + \mathrm{Im}(f_2)$ for all $a_3 \in A_3$. So

$$[f_2^*(l)](a_2) = l \circ f_2(a_2) = l(f_2(a_2)) = f_2(a_2) + \mathrm{Im}(f_2) = \mathrm{Im}(f_2).$$

Then $f_2^*(l) = 0$ which implies that $l \in \mathrm{Ker}(f_2^*) = 0$. So $l(a_3) = 0$ for all $a_3 \in A_3$, that is $a_3 + \mathrm{Im}(f_2) = \mathrm{Im}(f_2)$. Then $\mathrm{Im}(f_2) \cap Ra_3 \neq 0$ and $\mathrm{Im}(f_2) \leq_e A_3$.

To prove $\mathrm{Im}(f_1) \leq_e \mathrm{Ker}(f_2)$, we take a_2 to be a non-zero element of $\mathrm{Ker}(f_2)$ and $f_2(a_2) = 0$. We define $l : A_2 \longrightarrow B$ by $l(a_2) = a_2 + \mathrm{Im}(f_1)$ where $B = A_2/\mathrm{Im}(f_1)$. For all $a_1 \in A_1$ we have $l \circ f_1(a_1) = l(f_1(a_1)) = f(a_1) + \mathrm{Im}(f_1) = \mathrm{Im}(f_1)$. Thus $l \circ f_1 = 0$ and $f_1^*(l) = 0$. That means $l \in \mathrm{Ker}(f_1^*)$. By hypothesis, $\mathrm{Im}(f_2^*) \leq_e \mathrm{Ker}(f_1^*)$, so there exist $0 \neq r \in R$ and $h \in \mathrm{Hom}(A_3, B)$ such that $f_2^*(h) = h \circ f_2 = rl$ and $rl(a_2) = h \circ f_2(a_2) = 0$. This implies that $rl(a_2) = r(a_2 + \mathrm{Im}(f_1)) = \mathrm{Im}(f_1)$ and so $ra_2 \in \mathrm{Im}(f_1)$. Therefore $\mathrm{Im}(f_1) \cap Ra_2 \neq 0$. □

The following two examples show that the covariant functor $\mathrm{Hom}(M, -)$ and the contravariant functor $\mathrm{Hom}(-, M)$ are not right e-exact and hence not e-exact.

Example 2.8. Consider the sequence

$$0 \longrightarrow 2Z \xrightarrow{f} Z \xrightarrow{g} Z/2Z \longrightarrow 0$$

where $f(x) = 2x$ and $g(y) = y + 2Z$.

It is clear that the above sequence is e-exact (not exact sequence). Suppose that $B = Z/2Z$. We apply $\mathrm{Hom}(B, \ )$ in the above short e-exact sequence to get

$$0 \longrightarrow \mathrm{Hom}(Z/2Z, 2Z) \xrightarrow{f_*} \mathrm{Hom}(Z/2Z, Z) \xrightarrow{g_*} \mathrm{Hom}(Z/2Z, Z/2Z) \longrightarrow 0.$$

Since $Z/2Z$ is torsion and Z, $2Z$ are torsion-free, there is no non-zero map from torsion module to torsion-free module. Therefore $\mathrm{Hom}(Z/2Z, 2Z) = 0$,

$\mathrm{Hom}(Z/2Z, Z) = 0$ and $\mathrm{Hom}(Z/2Z, Z/2Z) = Z/2Z$. Hence $\mathrm{Hom}(Z/2Z, -)$ is not e-exact.

Example 2.9. Consider the e-exact sequence of Z-modules

$$0 \longrightarrow Z \xrightarrow{f} Q \xrightarrow{g} Q/Z \longrightarrow 0$$

where f is the identity map and g is the natural morphism. By applying $\mathrm{Hom}(\ , Z)$, we get

$$0 \longrightarrow \mathrm{Hom}(Q/Z, Z) \xrightarrow{g^*} \mathrm{Hom}(Q, Z) \longrightarrow \mathrm{Hom}(Z, Z) \longrightarrow 0\ .$$

Since the second term $\mathrm{Hom}(Q, Z)$ is zero and the last term is isomorphic to Z, the functor $\mathrm{Hom}(\ , Z)$ is not right e-exact.

Theorem 2.10. *Let* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$ *be an e-exact sequence. Then for any torsion-free R-module B, the sequence*

$$B \otimes A_1 \xrightarrow{1\otimes f_1} B \otimes A_2 \xrightarrow{1\otimes f_2} B \otimes A_3 \longrightarrow 0$$

is e-exact.

Proof. To prove $1 \otimes f_2$ is e-epic, let $b \otimes a_2 \in B \otimes A_3$. Since $a_3 \in A_3$, there exist $r \in R$and $a_2 \in A_2$ such that $f_2(a_2) = ra_3$ and so, $r(b \otimes a_3) = (b \otimes ra_3) = b \otimes f_2(a_2) = (1 \otimes f_2)(b \otimes a_2)$. Therefore $1 \otimes f_2$ is e-epic. Now, to prove $\mathrm{Im}(1 \otimes f_1) \leq_e \mathrm{Ker}(1 \otimes f_2)$, let $b \otimes a_2$ be a non-zero element of $\mathrm{Hom}(1 \otimes f_2)$. Then $(1 \otimes f_2)(b \otimes a_2) = b \otimes f_2(a_2) = 0$. Since B is torsion-free, $f_2(a_2) = 0$ and $a_2 \in \mathrm{Ker}(f_2)$. By e-exactness of above sequence, there exist $r \in R$ and $a_1 \in A_1$ such that $f_1(a_1) = ra_2$. Therefore $r(b \otimes a_2) = b \otimes ra_2 = b \otimes f_1(a_1) = (1 \otimes f_1)(b \otimes a_1)$. Hence $\mathrm{Im}(1 \otimes f_1) \leq_e \mathrm{Ker}(1 \otimes f_2)$. □

The tensor functor fails to preserve monic as it explained in the following example.

Example 2.11. The sequence of Z-modules and Z-morphisms

$$0 \longrightarrow Z \xrightarrow{f} Z$$

where $f(x) = 2x$. It is clear that f is monic and the sequence is e-exact. However the sequence $0 \longrightarrow Z \otimes (Z/2Z) \xrightarrow{f\otimes 1} Z \otimes (Z/2Z)$ is not, since $f \otimes 1$ is not monic.

3. e-projective modules

We say that an R-module P is *e-projective* if satisfies the following condition: for any e-epic map $f_1 : A_1 \to A_2$, and any map $f_2 : P \to A_2$, there exist

$0 \neq r \in R$ and $f_3 : P \to A_1$ such that $f_1 f_3 = r f_2$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "f_3"'] \arrow[d, "f_2"] & \\ A_1 \arrow[r, "f_1"] & A_2 \arrow[r] & 0 \end{tikzcd}$$

Theorem 3.1. *An R-module P is e-projective if and only if* $\mathrm{Hom}(P,\ )$ *is an e-exact functor.*

Proof. ($\Rightarrow$) Suppose that P is e-projective then by Theorem 2.6 $\mathrm{Hom}(P, -)$ is left e-exact functor. It remains to show that $\mathrm{Hom}(P,\ )$ is right exact functor. Suppose $A_1 \xrightarrow{f_1} A_2 \longrightarrow 0$ is an e-exact sequence and we have to show the e-exactness of $\mathrm{Hom}(P, A_1) \xrightarrow{{f_1}^*} \mathrm{Hom}(P, A_2) \longrightarrow 0$. For every non-zero map $f_2 \in \mathrm{Hom}(P, A_2)$, by definition of e-projective there exist $0 \neq r \in R$ and $f_3 : P \to A_1$ such that $f_1 f_3 = r f_2$ or $f_1^*(f_3) = r f_2$ Therefore $\mathrm{Im}(f_1^*) \cap R f_2 \neq 0$ and we have $\mathrm{Im}(f_1^*) \leq_e \mathrm{Hom}(P, A_2)$.

($\Leftarrow$) Let $f_1 : A_1 \to A_2$ be e-epic. Since $A_1 \xrightarrow{f_1} A_2 \longrightarrow 0$ is an e-exact sequence and $\mathrm{Hom}(P, -)$ is an e-exact functor, then

$$\mathrm{Hom}(P, A_1) \xrightarrow{{f_1}^*} \mathrm{Hom}(P, A_2) \longrightarrow 0$$

is e-exact. By e-exactness $\mathrm{Im}(f_1^*) \leq_e \mathrm{Hom}(P, A_2)$, so for every $f_2 \in \mathrm{Hom}(P, A_2)$ there exist $f_3 \in \mathrm{Hom}(P, A_1)$ and $0 \neq r \in R$ such that $f_1^*(f_3) = r f_2$ or $f_1 f_3 = r f_2$. Hence P is e-projective. $\square$

Recall that an exact sequence

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

of R-modules is split if there exist a morphism $g : A_3 \to A_2$ and $r \in R$ such that $f_1 g = r 1_{A_3}$, where 1_{A_3} is the identity map on A_3.

Proposition 3.2. *An e-exact sequence*

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} P \longrightarrow 0$$

splits if P is e-projective module.

Proof. Suppose that the sequence $0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} P \longrightarrow 0$ is e-exact and P is e-projective, by Theorem 2.6 the sequence

$$0 \longrightarrow \mathrm{Hom}(P, A_1) \xrightarrow{f_1^*} \mathrm{Hom}(P, A_2) \xrightarrow{f_2^*} \mathrm{Hom}(P, P) \longrightarrow 0$$

is e-exact. Since $1_P \in \mathrm{Hom}(P, P)$ and f_2^* is e-epic, there exist a map $g : P \longrightarrow A_2$ and $r \in R$ such that $f_1 g = r 1_P$. Therefore the sequence is split. $\square$

Proposition 3.3. *Every summand of an e-projective R-module is e-projective.*

Proof. Suppose that P_1 be any e-projective R-module and P_2 a summand of P_1. We define a projection and an injection maps $p : P_1 \longrightarrow P_2$ and $i : P_2 \longrightarrow P_1$, with $pi = 1_{P_2}$. Consider the diagram:

$$\begin{tikzcd}
P_1 \arrow[r, shift left, "p"] \arrow[d, dashed, "f_3"'] & P_2 \arrow[l, shift left, "i"] \arrow[d, "f_2"] & \\
A_1 \arrow[r, "f_1"] & A_2 \arrow[r] & 0
\end{tikzcd}$$

since P_1 is e-projective, for the composite map f_2p there exist $0 \neq r \in R$ and a map $f_3 : P_1 \to A_1$ such that $f_1f_3 = rf_2p$. Now we define $f_4 : P_2 \longrightarrow A_1$ by $f_4 = f_3i$ so we have

$$f_1f_4 = f_1f_3i = rf_2pi = rf_2.$$

Therefore P_2 is e-projective. □

Theorem 3.4. *Let $\{P_j \colon j \in J\}$ be a family of e-projective modules. Then $\coprod_{j\in J} P_j$ is also e-projective.*

Proof. Consider the diagram

$$\begin{tikzcd}
P_j \arrow[r, shift left, "\lambda_j"] \arrow[d, dashed, "g_j"'] & \coprod_{j\in J} P_j \arrow[l, shift left, "p_j"] \arrow[d, "f_2"] & \\
A_1 \arrow[r, "f_1"] & A_2 \arrow[r] & 0
\end{tikzcd}$$

where λ_j and p_j are injective and projective maps respectively. Since for all e-projective module P_j we have a map $f_2\lambda_j : P_j \longrightarrow A_2$, there exist a map $g_j : P_j \longrightarrow A_1$ and $0 \neq r_j \in R$ such that

$$f_1g_j = r_jf_2\lambda_j.$$

We define $h : \coprod_{j\in J} P_j \longrightarrow A_1$ as $h = g_jp_j$. Therefore

$$f_1h = f_1g_jp_j = r_jf_2\lambda_jp_j = r_jf_2$$

that is $f_1h = r_jf_2$ for all $0 \neq r_j \in R$. Hence $\coprod_{j\in J} P_j$ is e-projective. □

We conclude from Proposition 3.3 and Theorem 3.4 that a direct sum of a family of R-modules is an e-projective module if and only if each of the direct summands in the family is e-projective.

Definition. An *e-projective resolution* of an R-module A is an e-exact sequence

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

in which each P_n is e-projective.

Definition. An *e-injective resolution* of an R-module A is an e-exact sequence

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow E^{n+1} \longrightarrow \cdots$$

in which each E^n is injective.

Question 1. One can use the above two definitions to redefine the homology, using the left e-exact functors $\mathrm{Hom}(M, -)$, $\mathrm{Hom}(-, M)$ and right e-exact functor $M \otimes -$ to define their derived functors and study properties of them.

Question 2. Dually to e-projective modules, one can define e-injective and e-flat modules and discuss their characterizations and properties.

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