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ON $g(x)$-INVO CLEAN RINGS

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Abstract. An element in a ring R with identity is called invo-clean if it is the sum of an idempotent and an involution and R is called invo-clean if every element of R is invo-clean. Let $C(R)$ be the center of a ring R and $g(x)$ be a fixed polynomial in $C(R)[x]$. We introduce the new notion of $g(x)$-invo clean. R is called $g(x)$-invo if every element in R is a sum of an involution and a root of $g(x)$. In this paper, we investigate many properties and examples of $g(x)$-invo clean rings. Moreover, we characterize invo-clean as $g(x)$-invo clean rings where $g(x) = (x-a)(x-b)$, $a, b \in C(R)$ and $b - a \in Inv(R)$. Finally, some classes of $g(x)$-invo clean rings are discussed.

1. Introduction and preliminaries

Everywhere in the text of the current paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0. As usual, for such a ring R, the symbol $U(R)$ stands for the group of units, $Inv(R)$ for the set of all involutions (= square roots of 1), $Id(R)$ for the set of all idempotents and $Nil(R)$ for the set of all nilpotents. Following Han and Nicholson [14], an element $r \in R$ is called clean if $r = u + e$ for some $u \in U(R)$ and $e \in Id(R)$. A ring R is called clean if every element of R is clean. The notion of clean rings was first introduced by Nicholson [17] in 1977 in his study of lifting idempotents and exchange rings. Since then, some stronger concepts have been considered (e.g. uniquely clean, strongly clean and some special clean rings), see [4, 7, 18, 20–23], as well as some weaker ones (e.g. almost clean and weakly clean rings), see [1]. Recently, in 2017, Danchev [9] studied the following special case of cleanness, namely, invo-clean rings. They are rings in which every element is a sum of an idempotent element and an involution element.

Let $C(R)$ denotes the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. Then following Camillo and Simón [5], R is called $g(x)$-clean if for each $r \in R$,

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$r = u + s$ where $u \in U(R)$ and $g(s) = 0$. Of course $(x^2 - x)$-clean rings are precisely the clean rings.

Nicholson and Zhou [19] proved that if $g(x) \in (x - a)(x - b)C(R)[x]$ with $a, b \in C(R)$ and $b, b - a \in U(R)$ and ${}_RM$ is a semisimple left R-module, then $End({}_RM)$ is $g(x)$-clean. Recently, Fan and Yang [13], studied more properties of $g(x)$-clean rings. Among many results, they prove that if $b - a \in U(R)$ with $a, b \in C(R)$, then R is a clean ring if and only if R is $(x - a)(x - b)$-clean.

This work is motivated by the notions of $g(x)$-cleanness and invo-cleanness and we will combine them into a new concept. In this way, we define and study $g(x)$-invo clean rings as a special class of $g(x)$-clean rings. For a ring R and $g(x) \in C(R)[x]$, an element $r \in R$ is called $g(x)$-invo clean if $r = v + s$ for some $v \in Inv(R)$ and $g(s) = 0$. Moreover, R is called $g(x)$-invo clean if every element in R is $g(x)$-invo clean.

The paper is organized as follows: In Section 1, we already have given the main definitions of the used concepts. In Section 2, we define $g(x)$-invo clean rings and determine the relation between $g(x)$-invo clean rings and invo-clean rings; in Section 3, some general properties of $g(x)$-invo clean rings are given; and in Section 4, for a commutative ring A, we give a characterization for the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$) (see for instance [11]) to be $g(x)$-invo clean. Also, we consider the idealization $A \propto E$ of any A-module E and prove that $A \propto E$ is $g(x)$-invo clean ring if and only if A is an invo-clean and $2E = 0$. In Section 5, some classes of $g(x)$-invo clean rings are discussed.

2. $g(x)$-invo clean rings

In this section, we firstly define $g(x)$-invo clean elements and $g(x)$-invo clean rings. We study some of the basic properties of $g(x)$-invo clean rings. Moreover, we give some necessarily examples.

Definition. Let R be a ring and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called $g(x)$-invo clean if $r = v + s$ where $g(s) = 0$ and v is an involution of R. We say that R is $g(x)$-invo clean if every element in R is $g(x)$-invo clean.

Obviously, $g(x)$-invo clean rings are $g(x)$-clean. In contrast, $\mathbb{Z}_7$ is clean but is not invo-clean. Since $(x^2 - x)$-invo clean rings are precisely the invo-clean rings, we can say that for $g(x) = x^2 - x$, the ring $\mathbb{Z}_7$ is $g(x)$-clean, but it is not $g(x)$-invo clean.

In the other hand, invo-clean rings are exactly $(x^2 - x)$-invo clean. However, there are $g(x)$-invo clean rings which are not invo-clean and vice versa:

Example 2.1. Let $R = \mathbb{Z}_5$ and $g(x) = x^5 + 4x \in C(R)[x]$. Then:

(1) R is not invo-clean (In fact, the ring R has involutions $\{1, 4\}$, idempotents $\{0, 1\}$). Since the element 3 of R cannot be expressed as sum of an idempotent and an involution, then R is not invo-clean.

(2) R is $g(x)$-invo clean.

Example 2.2. Let R be a Boolean ring with the number of elements $|R| > 2$ and $c \in R$ with $c \in R\backslash\{0,1\}$. Define $g(x) = (x+1)(x+c)$. Then:

(1) R is invo-clean.

(2) R is not $g(x)$-invo clean.

Proof. (1) Since $e = (2e-1) + (1-e)$ with $(2e-1)^2 = 1$ and $(1-e)^2 = 1-e$, then any idempotent is an invo-clean element. Thus, R is invo-clean.

(2) Because if $c = v + s$ where $v \in Inv(R)$ and $g(s) = 0$, then it must be that $v = 1$ and $s = c - v$. But, clearly, $g(c-1) \neq 0$. Hence, R is not $g(x)$-invo clean. □

However, for some type of polynomials, invo-cleanness and $g(x)$-invo cleanness are equivalent.

Theorem 2.3. *Let R be a ring and $g(x) \in (x-a)(x-b)C(R)[x]$ where $a, b \in C(R)$. Then the following hold*:

(1) *R is invo-clean and $(b-a) \in Inv(R)$ if and only if R is $(x-a)(x-b)$-invo clean.*

(2) *If R is invo-clean and $(b-a) \in Inv(R)$, then R is $g(x)$-invo clean.*

Proof. (1) Suppose $r \in R$. Since R is $g(x)$-invo clean, there exist an involution v_1 and a root s_1 of $g(x)$ such that $b = v_1 + s_1$. Since $g(s_1) = (s_1-a)(s_1-b) = 0$, we have $s_1 = a$. This implies that $b - a$ is involution. Again by hypothesis, there exist an involution v_2 and a root s_2 of $g(x)$ such that $(b-a)r+a = v_2+s_2$. Set $e = (b-a)(s_2-a)$, i.e., $s_2 = (b-a)e+a$. Then we get $r = e+(b-a)v_2$. Note that $g(s_2) = (s_2-a)(s_2-b) = (b-a)e[(b-a)e+a-b] = (b-a)^2(e^2-e) = 0$ since $b-a \in C(R)$. Since $(b-a) \in Inv(R)$, we have $e^2 = e$, as required.

Conversely, for any $r \in R$, by hypothesis we may write $(b-a)(r-a) = e+v$ where $e^2 = e \in R$ and $v \in Inv(R)$. Thus, we have $r = [(b-a)e+a]+(b-a)v$. Note that $(b-a)v$ is an involution since $(b-a) \in Inv(R)$. Now we have $g((b-a)e+a) = (b-a)e[(b-a)e+a-b] = (b-a)^2e(e-1) = 0$, and so $(b-a)e+a$ is a root of $g(x)$. This completes the proof.

(2) This follows from (1). □

In fact, the condition $a, b \in C(R)$ in Theorem 2.3 can be replaced by $(b-a) \in C(R)$.

Corollary 2.4. *Let R be a ring. Then R is invo-clean if and only if R is (x^2+x)-invo clean.*

Proof. This follows from Theorem 2.3 when $a = 0$ and $b = -1$. □

Remark 2.5. The equivalence of (x^2+x)-invo clean and invo-clean is a global property. That is, it holds for a ring R but it may fail for a single element. For example, $1+1 = 2 \in \mathbb{Z}$ is invo-clean but it is not (x^2+x)-invo clean in $\mathbb{Z}$ since $\mathbb{Z}$ has only two involutions 1 and -1.

In [6, Proposition 10], Camillo and Yu showed that if $2 \in U(R)$, then R is clean if and only if every element of R is the sum of a unit and a square root of 1. Here we have a similar result for invo-clean rings.

Corollary 2.6. *A ring R is invo-clean and $2 \in Inv(R)$ if and only if every element of R is the sum of an involution and a square root of* 1.

Proof. Let $g(x) = (x+1)(x-1) = x^2 - 1$. Note that the condition that every element of R is the sum of an involution and a square root of 1 is equivalent to R being $g(x)$-invo clean. Hence by Theorem 2.3, the proof is immediate. □

Theorem 2.7. *Let R be a ring, $n \in \mathbb{N}$ and $a, b \in R$. Then R is $(ax^{2n} - bx)$-invo clean if and only if R is $(ax^{2n} + bx)$-invo clean.*

Proof. Suppose R is $(ax^{2n} - bx)$-invo clean. Then for any $r \in R$, $-r = v + s$ where $(as^{2n} - bs) = 0$ and $v \in Inv(R)$. So $r = (-v) + (-s)$ where $(-v) \in Inv(R)$ and $a(-s)^{2n} + b(-s) = 0$. Hence, r is $(ax^{2n} + bx)$-invo clean. Therefore, R is $(ax^{2n} + bx)$-invo clean. Now suppose R is $(ax^{2n} + bx)$-invo clean. Let $r \in R$. Then there exist s and v such that $-r = v + s$, $(as^{2n} + bs) = 0$ and $v \in Inv(R)$. So $r = (-v) + (-s)$ and $as^{2n} - bs = 0$ is satisfied. Hence, R is $(ax^{2n} - bx)$-invo clean. □

For example, we conclude that $(x^2 + x)$-invo clean rings and $(x^2 - x)$-invo clean rings are equivalent to invo-clean rings.

Remark 2.8. The equivalence in Theorem 2.7 does not hold for odd powers. For example, the ring $\mathbb{Z}_3$ is clearly a $(x^3 - x)$-invo clean which is not $(x^3 + x)$-invo clean.

Lemma 2.9. *Let R be a ring and $e \in Id(R)$. Then $Inv(eRe) = (eRe) \cap (\bar{e} + Inv(R))$, where $\bar{e} = 1 - e$.*

Proof. ($\subseteq$) If $v \in Inv(eRe)$, then $v^2 = e$. Since the product of v with $\bar{e}$ is zero, $(v - \bar{e})^2 = e + \bar{e} = 1$, and so $(v - \bar{e}) \in Inv(R)$. Then $v \in \bar{e} + Inv(R)$.

($\supseteq$) If $a = \bar{e} + v \in eRe$ with $v \in Inv(R)$, then $a - \bar{e} = v$, and hence $(a - \bar{e})^2 = 1$. Thus, $(ea - e\bar{e})^2 = e$, and so $ea^2 = e$. Therefore $a^2 = e$, and then $a \in Inv(eRe)$. □

For invo-clean rings, the author in [10, Theorem 2.2] proved that if R is an invo-clean ring and $e^2 = e$, then the corner ring eRe is an invo-clean ring. For $g(x)$-invo clean rings, we have the following result:

Theorem 2.10. *Let R be an $(x-a)(x-b)$-invo clean ring with $a, b \in C(R)$. Then for any $e^2 = e \in R$, eRe is $(x - ea)(x - eb)$-invo clean. In particular, if $g(x) \in (x - ea)(x - eb) \in C(R)[x]$ and R is $(x-a)(x-b)$-invo clean with $a, b \in C(R)$, then eRe is $g(x)$-invo clean.*

Proof. By Theorem 2.3 R is $(x-a)(x-b)$-invo clean if and only if R is invo-clean and $(b-a) \in Inv(R)$. If R is invo-clean, then eRe is invo-clean by [10, Theorem

2.2]. Again by Theorem 2.3 and Lemma 2.9, eRe is $(x-ea)(x-eb)$-invo clean. $\square$

Let R be a ring and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called $g(x)$-nil clean if $r = b + s$ where $g(s) = 0$ and b is a nilpotent of R. Then R is called $g(x)$-nil clean if every element in R is $g(x)$-nil clean [15]. Thus, we have the following Proposition.

Proposition 2.11. *Let R be a ring and $g(x) \in C(R)[x]$. If R is a $g(x)$-invo clean ring with $2 \in Nil(R)$, then R is $g(1-x)$-nil clean with bounded index of nilpotence.*

Proof. Given $r \in R$, we write $r = v + s$, where $v^2 = 1$ and $g(s) = 0$. But $(1+v)^2 = 2+2v = 2(1+v)$, and hence $(1+v)^3 = 2(1+v)^2 = 2^2(1+v)$, etc. By induction we derive that $(1+v)^{n+1} = 2^n(1+v)$ for all $n \in \mathbb{N}$. Thus $(1+v)^t = 0$ for some appropriate natural t (since $2 \in Nil(R)$), that is, $(1+v) \in Nil(R)$. Furthermore, one may write that $r = (v+1) - (1-s)$, whence R is $g(1-x)$-nil clean, as claimed. $\square$

Corollary 2.12. *If R is an invo-clean ring with $2 \in Nil(R)$, then R is nil clean with bounded index of nilpotence.*

Proof. Since invo-clean (resp. nil clean) is (x^2-x)-invo clean (resp. (x^2-x)-nil clean). $\square$

3. General properties of $g(x)$-invo clean rings

Let R and S be two rings. Consider the ring homomorphism $\psi : C(R) \to C(S)$ with $\psi(1_R) = 1_S$. Then ψ induces a map ψ' from $C(R)[x]$ to $C(S)[x]$ such that for $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, $g_\psi(x) := \psi'(g(x)) = \sum_{i=0}^{n} \psi(a_i)x^i \in C(S)[x]$. We should note that if $n \in \mathbb{Z}$, then $\psi(n) = \psi(1+\cdots+1) = n\psi(1) = n$. So, if $g(x) \in \mathbb{Z}[x]$, then $g_\psi(x) = g(x)$.

Next, we give some properties of the class of $g(x)$-invo clean rings. We start by a simple result.

Proposition 3.1. *Let R and S be two rings, $\psi : R \to S$ be a ring epimorphism and $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$. If R is $g(x)$-invo clean, then S is $g_\psi(x)$-invo clean.*

Proof. Let $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ and consider $g_\psi(x) := \sum_{i=0}^{n} \psi(a_i)x^i \in C(S)[x]$. For every $\alpha \in S$, there exists $r \in R$ such that $\psi(r) = \alpha$. Since R is $g(x)$-invo clean, there exist $s \in R$ and $v \in Inv(R)$ such that $r = v + s$ and $g(s) = 0$. So $\alpha = \psi(r) = \psi(v+s) = \psi(v) + \psi(s)$ with $\psi(v) \in Inv(S)$ and $g_\psi(\psi(s)) = \sum_{i=0}^{n} \psi(a_i)(\psi(s))^i = \sum_{i=0}^{n} \psi(a_i)\psi(s^i) = \sum_{i=0}^{n} \psi(a_i s^i) = \psi(\sum_{i=0}^{n} a_i s^i) = \psi(g(s)) = \psi(0) = 0$. Therefore, S is $g_\psi(x)$-invo clean. $\square$

Now by Proposition 3.1, the following holds:

Corollary 3.2. *If R is $g(x)$-invo clean, then for any ideal I of R, R/I is $\overline{g(x)}$-invo clean where $\overline{g(x)} \in C(R/I)[x]$.*

Proof. Let $\psi : R \to R/I$ be the canonical epimorphism. Note that if $a \in C(R)$, then $\bar{a} \in C(R/I)$, and so the result follows from Proposition 3.1. $\square$

Proposition 3.3. *Let $R_1, R_2, \ldots, R_n$ be rings and $g(x) \in \mathbb{Z}[x]$. Then $R := \prod_{i=1}^{n} R_i$ is $g(x)$-invo clean if and only if R_i is $g(x)$-invo clean for all $i \in \{1, 2, \ldots, n\}$.*

Proof. $\Rightarrow$): Let R be $g(x)$-invo clean. Define $\pi_j : \prod_{i=1}^{n} R_i \to R_j$ by $\pi_j((a_i)_i) = a_j$. Since for all $i \in \{1, 2, \ldots, n\}$, π_j is a ring epimorphism, so by Corollary 3.2, for every $i \in \{1, 2, \ldots, n\}$, R_i is $g(x)$-invo clean.

$\Leftarrow$): Let $(x_1, x_2, \ldots, x_n) \in \prod_{i=1}^{n} R_i$. For each i, write $x_i = v_i + s_i$ where $v_i \in Inv(R_i)$, $g(s_i) = 0$. Let $v = (v_1, v_2, \ldots, v_n)$ and $s = (s_1, s_2, \ldots, s_n)$. Then it is clear that $v \in R$ and $g(s) = 0$. Therefore, R is $g(x)$-invo clean. $\square$

Let R be a ring with an identity and S be a ring (not necessary unitary) which is an (R, R)-bimodule such that $(s_1 s_2)a = s_1(s_2 a)$, $a(s_1 s_2) = (a s_1)s_2$ and $(s_1 a)s_2 = s_1(a s_2)$ for all $a \in R$, $s_1, s_2 \in S$. The ideal-extension $I(R, S)$ of R by S is defined as the additive abelian group $I(R, S) = R \oplus S$ with multiplication $(a_1, s_1)(a_2, s_2) = (a_1 a_2, a_1 s_2 + s_1 a_2 + s_1 s_2)$. If $g(x) = (a_0, s_0) + (a_1, s_1)x + \cdots + (a_n, s_n)x^n \in C(I(R, S))[x]$, then clearly $g_R(x) = a_0 + a_1 x + \cdots + a_n x^n \in C(R)[x]$.

Proposition 3.4. *Let R and S be as above. If $I(R, S)$ is $g(x)$-invo clean, then R is $g_R(x)$-invo clean.*

Proof. If we define $\mu_R : I(R, S) \to R$ by $\mu_R(r, s) = r$, then μ_R is a ring epimorphism. The result follows by Corollary 3.2. $\square$

Let R be a ring and $\alpha : R \to R$ be a ring endomorphism. By $R[[x, \alpha]]$ we denote the ring of skew formal power series over R, that is all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R. The skew polynomial ring $R[x, \alpha]$ can be defined in an analogous way. One can prove that $R[[x, \alpha]] \simeq I(R, \langle x \rangle)$ where $\langle x \rangle$ is the ideal generated by x.

Corollary 3.5. *Let R be a ring and $\alpha : R \to R$ be a ring endomorphism. If $R[[x, \alpha]]$ (or in particular $R[[x]]$) is $g(x)$-invo clean, then R is $g_\mu(x)$-invo clean where $\mu : R[[x, \alpha]] \to R$ is defined by $\mu(f) = f(0)$.*

In general, the ring of polynomials $R[x]$ over a ring R is not $g(x)$-clean. This is also true for commutative $g(x)$-invo clean rings.

Lemma 3.6. *Let R be a commutative ring and $f = \sum_{i=0}^{n} a_i x^i \in R[x]$ be an involution element. Then a_0 is an involution and a_i is nilpotent for each i.*

Proof. Since f is involution, $f^2 = 1$. So $a_0^2 = 1$. Therefore, a_0 is an involution. Now, to end the proof, it is enough to show that for each prime ideal P of R; every $a_i \in P$. Since P is prime, thus $(R/P)[x]$ is an integral domain. Define $\varphi : R[x] \to (R/P)[x]$ by $\varphi(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n}(a_i + P)x^i$. Clearly, φ is an epimorphism. But $\varphi(f)\varphi(f) = \varphi(1)$, and so $\deg(\varphi(f)\varphi(f)) = \deg(\varphi(1))$. So, $\deg(\varphi(f)) = 0$. Thus, $a_1 + P = a_2 + P = \cdots = a_n + P = P$, as required. □

Theorem 3.7. *If R is a commutative ring, then $R[x]$ is not invo-clean (hence not $(x^2 - x)$-invo clean).*

Proof. We show that x is not invo-clean in $R[x]$. Suppose that $x = v + e$, where $v \in Inv(R[x])$ and $e \in Id(R[x])$. Since $Id(R) = Id(R[x])$ and $x = v + e$, so $x - e$ is an involution. Hence, by Lemma 3.6, 1 should be nilpotent, which is a contradiction. □

A Morita context (A, B, V, W, ψ, ϕ) consists of two rings A, B, two bimodules ${}_AV_B$, ${}_BW_A$ and a pair of bimodule homomorphisms $\psi : V \otimes_B W \to A$ and $\phi : W \otimes_A V \to B$, such that $\psi(v \otimes w)v' = v\phi(w \otimes v')$, $\phi(w \otimes v)w' = w\psi(v \otimes w')$. With such a Morita context we associate the ring $T = \left[\begin{smallmatrix} A & V \\ W & B \end{smallmatrix}\right] = \{\left[\begin{smallmatrix} a & v \\ w & b \end{smallmatrix}\right] : a \in A, b \in B, v \in V, w \in W\}$ under the usual matrix addition and multiplication defined as:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' + \psi(v \otimes w') & av' + vb' \\ wa' + bw' & \phi(w \otimes v') + bb' \end{bmatrix}.$$

We call T a Morita context ring. If $g(x) = \left[\begin{smallmatrix} a_0 & v_0 \\ w_0 & b_0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} a_1 & v_1 \\ w_1 & b_1 \end{smallmatrix}\right] x + \cdots + \left[\begin{smallmatrix} a_n & v_n \\ w_n & b_n \end{smallmatrix}\right] x^n \in C(T)[x]$, then clearly $g_A(x) = a_0 + a_1 x + \cdots + a_n x^n \in C(A)[x]$ and $g_B(x) = b_0 + b_1 x + \cdots + b_n x^n \in C(B)[x]$.

Proposition 3.8. *Let $T = \left[\begin{smallmatrix} A & V \\ W & B \end{smallmatrix}\right]$ be a Morita context with ψ, $\phi = 0$. If T is $g(x)$-invo clean, then A is $g_A(x)$-invo clean and B is $g_B(x)$-invo clean.*

Proof. Assume that T is $g(x)$-invo clean with ψ, $\phi = 0$. Let $I = \left[\begin{smallmatrix} 0 & V \\ W & B \end{smallmatrix}\right]$ and $J = \left[\begin{smallmatrix} A & V \\ W & B \end{smallmatrix}\right]$. Then clearly I and J are ideals of T and moreover, $T/I \cong A$ and $T/J \cong B$. It follows by Corollary 3.2 that A is $g_A(x)$-invo clean and B is $g_B(x)$-invo clean. □

Corollary 3.9. *Let A, B be two rings and M be an (A, B)-bimodule. Let $T = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ be the formal triangular matrix ring. If T is $g(x)$-invo clean, then A is $g_A(x)$-invo clean and B is $g_B(x)$-invo clean.*

In the following proposition, we consider a particular case of formal triangular matrix rings. Let R be a commutative ring and M an R-module. The trivial extension of R by M is the (commutative) ring:

$$R(M) = \left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} : r \in R, m \in M \right\}$$

with the usual matrix addition and multiplication. We note that if $\left[\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right] \in Inv(R(M))$, then clearly $r \in Inv(R)$. We recall that R naturally embeds into $R(M)$ via $r \to \left[\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right]$. Thus any polynomial $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ can be written as $g(x) = \sum_{i=0}^{n} \left[\begin{smallmatrix} r_i & 0 \\ 0 & r_i \end{smallmatrix}\right] x^i \in R(M)[x]$ and conversely.

Proposition 3.10. *Let R be a commutative ring, M an R-module and $2M = 0$. Then the idealization $R(M)$ of R and M is $g(x)$-invo clean if and only if R is $g(x)$-invo clean.*

Proof. ($\Rightarrow$) Note that $R \simeq R(M)/\widetilde{M}$ where $\widetilde{M} = \{\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] : m \in M\}$. Hence R is $g(x)$-invo clean by Corollary 3.2.

$\Leftarrow$) Let $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $r \in R$. Since R is $g(x)$-invo clean, we have $r = v + s$, where $v \in Inv(R)$ and $g(s) = 0$. Then for $m \in M$, $\left[\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right] = \left[\begin{smallmatrix} v & m \\ 0 & v \end{smallmatrix}\right] + \left[\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} v & m \\ 0 & v \end{smallmatrix}\right] \in Inv(R(M)$ (since $2M = 0$). Moreover, we have:

$$\begin{aligned} g(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}) &= a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + a_2 \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix} + \cdots + a_n \begin{bmatrix} s^n & 0 \\ 0 & s^n \end{bmatrix} \\ &= \begin{bmatrix} a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n & 0 \\ 0 & a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $R(M)$ is $g(x)$-invo clean. □

4. $(x^2 - x)$-invo clean rings

Let A and B be two commutatives rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. The amalgamation of A with B along J with respect to f is defined as $A \bowtie^f J = \{(a, f(a)+j) \mid a \in A, j \in J\}$. It is easy to check that $A \bowtie^f J$ is a subring of $A \times B$ (with the usual componentwise operations). For more properties of $A \bowtie^f J$, one can see [11, 12]. In the following theorem, we investigate the invo-cleanness (hence the $(x^2 - x)$-invo cleanness) of $A \bowtie^f J$. Recall that a ring R is called invo-clean if every $r \in R$ can be written as $r = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$. If, in addition, the existing idempotent e is unique, then R is called uniquely invo-clean.

Theorem 4.1. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B.*

1. *If $A \bowtie^f J$ is an invo-clean (resp., a uniquely invo-clean) ring, then A is an invo-clean (resp., a uniquely invo-clean) ring and $f(A) + J$ is an invo-clean ring.*
2. *Assume that $\frac{f(A)+J}{J}$ is uniquely invo-clean. Then $A \bowtie^f J$ is an invo-clean ring if and only if A and $f(A) + J$ are invo-clean rings.*

Proof. (1) If $A \bowtie^f J$ is an invo-clean, we know by [11, Prop. 5.1] that A and $f(A) + J$ are homomorphic images of $A \bowtie^f J$, and so by [9, Lemma 2.1], A and

$f(A)+J$ are invo-clean . Assume now that $A \bowtie^f J$ is uniquely invo-clean and consider $v+e = v'+e'$ where $v, v' \in Inv(A)$ and $e, e' \in Id(A)$. Then $(v, f(v))+(e, f(e)) = (v', f(v'))+(e', f(e'))$ and clearly $(v, f(v))$,$(v', f(v')) \in Inv(A \bowtie^f J)$ and $(e, f(e))$, $(e', f(e')) \in Id(A \bowtie^f J)$. Thus, $(v, f(v)) = (v', f(v'))$ and $(e, f(e)) = (e', f(e'))$. Hence $v = v'$ and $e = e'$. Consequently, A is uniquely invo-clean.

(2) If $A \bowtie^f J$ is invo-clean, then so A and $f(A)+J$ by (1). Conversely, assume that A and $f(A)+J$ are invo-clean rings and consider $(a, j) \in A \times J$. Since A is invo-clean, we write that $a = v+e$ for some $v \in Inv(A)$ and $e \in Id(A)$. Furthermore, since $f(A)+J$ is invo-clean, $f(a)+j = (f(x)+k)+(f(y)+l)$ with $(f(x)+k)$ and $(f(y)+l)$ are respectively involution and idempotent element of $f(A)+J$. It is clear that $\overline{f(x)} = \overline{f(x)+k}$ (resp. $\overline{f(v)}$) and $\overline{f(y)} = \overline{f(y)+l}$ (resp. $\overline{f(e)}$) are respectively involution and idempotent element of $\frac{f(A)+J}{J}$, and we have $\overline{f(a)} = \overline{f(v)} + \overline{f(e)} = \overline{f(x)} + \overline{f(y)}$. Thus, $\overline{f(v)} = \overline{f(x)}$ and $\overline{f(e)} = \overline{f(y)}$ since $\frac{f(A)+J}{J}$ is uniquely invo-clean. Consider $k, l' \in J$ such that $f(x) = f(v)+k'$ and $f(y) = f(e)+l'$. We have, $(a, f(a)+j) = (v, f(v)+k'+k)+(e, f(e)+l'+l)$, and it is clear that $(v, f(v)+k'+k) \in Inv(A \bowtie^f J)$ and $(e, f(e)+l'+l) \in Id(A \bowtie^f J)$. Consequently, $A \bowtie^f J$ is invo-clean. □

Remark 4.2. Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B.

(1) If $B = J$, we have $A \bowtie^f J = A \times B$. Hence $A \bowtie^f J$ is invo-clean if and only if A and B are invo-clean (by [9, Proposition 2.13]).
(2) If $f^{-1}(J) = \{0\}$, we have $A \bowtie^f J \cong f(A)+J$ by [11, Proposition 5.1(3)]. Hence, $A \bowtie^f J$ is invo-clean if and only if $f(A)+J$ is invo-clean.

In a duplication ring, we obtain:

Corollary 4.3. *Let A be a ring and I an ideal such that A/I is an uniquely invo-clean. Then $A \bowtie I$ is invo-clean if and only if so is A.*

Proof. In this case, we have $f(A)+I = A+I = A$. Thus Theorem 4.1 completes the proof. □

Proposition 4.4. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B such that $J \subset Id(B)$. Then $A \bowtie^f J$ is invo-clean if and only if A is invo-clean.*

Proof. Let $(a, j) \in A \times J$. Hence there exist an idempotent e and an involution v such that $a = v+e$ (since A is invo-clean). Hence $(a, f(a)+j) = (v, f(v))+(e, f(e)+j)$, and then for all $j \in J$, we have $2j = 0$ and $j^2 = j$ (since $J \subset Id(B)$). Therefore, $(f(e)+j)^2 = (f(e))^2 + 2jf(e) + j^2 = (f(e)+j)$, and so $(a, f(a)+j)$ is an invo-clean element of $A \bowtie^f J$. Thus $A \bowtie^f J$ is invo-clean. The converse implication is clear. □

For more examples of invo-clean rings, we consider the method of idealization. Let A be a commutative ring and E an A-module. Nagata [16] introduced the idealization $A \propto E$ of A and E. The idealization of A and E (or trivial extension ring of A by E) is the ring $A \propto E$ with multiplication given by $(a_1, e_1)(a_2, e_2) = (a_1a_2, a_1e_2 + a_2e_1)$. This construction has been extensively studied and has many applications in different contexts, see [2, 3].

Lemma 4.5. *If A is an invo-clean ring, then any $q \in Nil(A)$ satisfies the equation $q^2 + 2q = 0$.*

Proof. If $q \in Nil(A)$, write $q = v + e$ where $v \in Inv(A)$ and $e \in Id(A)$. Thus $(-v) = (-q) + e$, where $(-v) \in Inv(A)$ and $(-q) \in Nil(A)$. Then by [9, Corollary 2.6], we conclude that $e = 1$. Therefore $q = v + 1$, and hence $q^2 + 2q = 0$. □

Proposition 4.6. *Let A be a commutative ring, E an A-module and $R := A \propto E$ the trivial extension ring of A by E. Then R is invo-clean if and only if A is invo-clean and $2E = 0$.*

Proof. (⇒) If $A \propto E$ is invo-clean, then $A \cong (A \propto E)/(0 \propto E)$ is invo-clean by [9, Lemma 2.1]. On the other hand, let $x \in E$. Then by Lemma 4.5 $(0, x)^2 + 2(0, x) = (0, 0)$ (since $(0, x) \in Nil(A \propto E)$ and $A \propto E$ is invo-clean), which shows that $2x = 0$. Hence $2E = 0$.

(⇐) Let $(a, x) \in A \propto E$ and write $a = v + e$, where $v \in Inv(A)$ and $e \in Id(A)$. Thus $(a, x) = (v, x) + (e, 0)$, and it is clear that $(v, x) \in Inv(A \propto E)$ and $(e, 0) \in Id(A \propto E)$. Consequently, $A \propto E$ is invo-clean. □

Clearly, invo-clean rings are clean rings. But in general, clean rings may not be invo-clean. Then to enrich the literature with new example of clean ring but not invo-clean, we propose the next example.

Example 4.7. Let $A := \mathbb{Z}_5$ and let $R := A \propto A$ be the trivial ring extension of A by A. Then:

(1) By [8, Corollary 2.12], R is a clean ring since A is a clean ring.
(2) Since A is not invo-clean, R is not invo-clean by Proposition 4.6.

If G is a group and R is a ring, we denote the group ring over R by RG. If RG is invo-clean, then R is invo-clean by [9, Lemma 2.1]. But it seems to be difficult to characterize R and G for which RG is invo-clean in general. In the following we will give some rings and groups such that RG is invo-clean.

Proposition 4.8. *Let R be a ring where $2 \in U(R)$ and $G = \{1, g\}$ be a group with two elements. Then RG is invo-clean if and only if R is invo-clean.*

Proof. One direction is trivial.

Conversely, if R is invo-clean, since 2 is invertible, by [14, Proposition 3] $RG \cong R \times R$. Hence, RG is invo-clean by [9, Proposition 2.13]. □

In the next proposition, we determine conditions under which the group ring RG is invo-clean where $G = C_n$ the cyclic group of order n.

Proposition 4.9. *Let R be a ring and $2 \in U(R)$. Then, RC_4 is invo-clean if and only if R is invo-clean.*

Proof. As $2 \in U(R)$, $RC_4 \cong R \times R \times R[x]/\langle x^2+1\rangle$ by Yi and Zhou [24, Lemma 3.3]. But as $2 \in U(R)$, we have $R[x]/\langle x^2+1\rangle \cong RC_2 \cong R \times R$. Therefore, the claim follows. □

Proposition 4.10. *If R is an invo-clean ring with $2 \in U(R)$, then RC_{2^k} is invo-clean for all $k \geq 0$.*

Proof. We know that $RC_{2^k} \cong (RC_k)C_2$. So it suffices to show that if RC_2 is invo-clean. But RC_2 is invo-clean by Proposition 4.8, as required. □

5. Unitly invo-clean rings

In this section, we explore and discuss the original notion of unitly invo-clean rings stated in Problem 3 of [9].

Definition ([9])**.** A ring R is called unitly invo-clean if $U(R) = Inv(R)+Id(R)$, i.e., for each $a \in U(R)$, there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $a = v + e$.

Remark 5.1. Although homomorphic images of units, idempotents and involutions are again units, idempotents and involutions, respectively, it follows in general that even an epimorphic image of a unitly invo-clean ring need not be unitly invo-clean. For instance, the ring $\mathbb{Z}$ is unitly invo-clean, while $\mathbb{Z}_5$ is not.

However, the following is valid:

Proposition 5.2. *Suppose that R is a ring with $I \subseteq J(R)$. Then R/I is a unitly invo-clean ring provided that R is a unitly invo-clean ring.*

Proof. We have here $I \subseteq J(R)$, which implies that $U(R) \to U(R/I)$ is surjective. Hence if $w = u + I \in U(R/I)$, then $u \in U(R) = Inv(R) + Id(R)$, so that $u = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$. Thus $w = u + I = (v + I) + (e + I) \in Inv(R/I) + Id(R/I)$, as needed. □

Corollary 5.3. *Let R be a ring. If R is a unitly invo-clean, then $R[[x]]/(x^n)$ ($n \in \mathbb{N}$) is a unitly invo-clean.*

Proof. Clearly, $R[[x]]/(x^n) = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} | \ a_0, \ldots, a_{n-1}\}$. Let $\alpha : R[[x]]/(x^n) \longrightarrow R$ be a morphism such that $\alpha(f) = f(0)$. It is easy to check that α is an R-epimorphism and $ker\alpha$ is a nil ideal of R, and therefore the result follows from Proposition 5.2. □

The nil property of the Jacobson radical can be strengthened by the following observation.

Proposition 5.4. *If R is a unitly invo-clean ring, then $J(R)$ is nil with index of nilpotence at most* 3.

Proof. Let $j \in J(R)$. We write $1+j = v+e$, where $v \in Inv(R)$ and $e \in Id(R)$. In both cases, since $J(R) + U(R) = U(R)$, we derive that $v - j = 1 - e \in U(R) \cap Id(R) = \{1\}$, and hence $e = 0$. Thus $j = v - 1$ implies that $j^2 = -2j$. Consequently, $j^3 = -2j^2$, then $j^3 = 4j$. Replacing j by $2j$ in the last equality, we obtain $8j^3 = 8j$ whence $8j(1 - j^2) = 0$. Since $1 - j^2 \in 1 + J(R) \subseteq U(R)$, it follows that $8j = 0$. On the other hand, substituting j by $2j$ in $j^2 = -2j$ and multiplying both sides of these two equalities by 4, we have $4j^2 = -4j = -8j$, i.e., $4j = 0$. Finally, $j^3 = 4j = 0$. □

We now arrange to prove the following.

Proposition 5.5. *If R is a unitly invo-clean ring with $2 \in U(R)$, then $Nil(R) = J(R) = \{0\}$.*

Proof. Since in view of Proposition 5.4 it must be that $J(R) \subseteq Nil(R)$, we need to consider only nilpotent elements. To that aim, suppose $q \in Nil(R)$. Then $1 + q \in U(R)$. Write $1 + q = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$ (since R is a unitly invo-clean ring). Thus $v = q + (1 - e)$. Appealing to [11, Corollary 2.6], we conclude that $e = 0$. Therefore $q = v - 1$, and hence $q^2 = 2-2v = -2(v-1) = -2q$. This leads to $q(q+2) = 0$. Since $q+2 \in U(R)$, we have $q = 0$, as expected. □

Proposition 5.6. *Let R be a unitly invo-clean ring and $4 = 0$. Then $Z(R)$ is a unitly invo-clean ring.*

Proof. For any $z \in U(Z(R)) \subseteq U(R)$, write $z = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$. It follows by squaring that $z^2 - 2ze = 1 - e$. Squaring again, we deduce that $z^4 = 1 - e$, so that $e = 1 - z^4 \in Z(R)$. We therefore infer that $v \in Z(R)$, and hence $z = v + e \in Inv(Z(R)) + Id(Z(R))$. □

Proposition 5.7. *Suppose that R is a nil-clean ring. Then R is unitly invo-clean if and only if any $q \in Nil(R)$ satisfies the equation $q^2 + 2q = 0$.*

Proof. (⇒) As in proof of Proposition 5.5, we derive that $q^2 = -2q$, and then $q^2 + 2q = 0$.

(⇐) Given $r \in U(R)$, we write $r = q + e$, where $q \in Nil(R)$ and $e \in Id(R)$ (since R is a nil-clean ring). Thus $r = q + e = (1 + q) - (1 - e)$. One checks that $(1 + q)^2 = q^2 + 2q + 1 = 1$ and $(1 - e)^2 = 1 - e$, as required. □

As an interesting consequence, we obtain the following one.

Corollary 5.8. *Let R be a nil-clean ring of characteristic* 2*. Then R is unitly invo-clean if and only if the index of nilpotence of R is* 2.

Remark 5.9. In regard to the above statement, it is worth noticing that $\mathbb{Z}_8$ is both unitly invo-clean and nil-clean containing the element 2 of nilpotence index 3. However, it is readily seen that 2 satisfies the equality $q^2 + 2q = 0$ because $2^2 + 2 \cdot 2 = 8 = 0$.

Likewise, $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$ is a nil-clean ring which is not necessarily unitly invo-clean (compare with Corollary 5.8). In fact, $\mathbb{Z}_{16}$ is indecomposable, that is, the only idempotents are 0 and 1 as well as all involutions are 1, 7, 9 and 15. So, the unit 5 cannot be represented as a sum of an involution and an idempotent, as expected.

Proposition 5.10. *If R is a unitly invo-clean ring with $3 \in U(R)$, then $24 = 0$. In particular, $6 \in Nil(R)$.*

Proof. Write $3 = v + e$, where v is an involution and e is an idempotent. Thus $(3 - v)^2 = 3 - v$ implies that $5v = 7$, whence $24 = 0$ by squaring both sides of the equality. In addition, $6^3 = 216 = 24 \cdot 9 = 0$, and hence $6 \in Nil(R)$, as asserted. □

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