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REVERSIBILITY OVER UPPER NILRADICALS

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Abstract. The studies of reversible and NI rings have done important roles in noncommutative ring theory. A ring R shall be called *QRUR* if $ab = 0$ for $a, b \in R$ implies that ba is contained in the upper nilradical of R, which is a generalization of the NI ring property. In this article we investigate the structure of QRUR rings and examine the QRUR property of several kinds of ring extensions including matrix rings and polynomial rings. We also show that if there exists a weakly semicommutative ring but not QRUR, then Köthe's conjecture does not hold.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. $N_*(R)$, $N^*(R)$, and $N(R)$ denote the lower nilradical (i.e., prime radical), the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R, respectively. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate x over a ring R is denoted by $R[x]$. Let $C_{f(x)}$ denote the set of all coefficients of given a polynomial $f(x)$. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0.

Following Cohn [7], a ring R (possibly without identity) is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson and Camillo [2], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible. Following Narbonne [21], a ring R (possibly without identity) is called *semicommutative* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. One can prove through simple computation that reduced rings are reversible and reversible rings are semicommutative. We will use these facts

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freely. A ring (possibly without identity) is called *Abelian* if every idempotent is central. Semicommutative rings are easily shown to be Abelian.

Following Birkenmeier et al. [5], a ring R (possibly without identity) is called *2-primal* if $N_*(R) = N(R)$. Note that a ring R is reduced if and only if R is both semiprime and 2-primal. Semicommutative rings are 2-primal through a simple computation, but the converse need not hold as can be seen by $U_2(D)$ over a 2-primal ring D, noting that $U_2(D)$ is 2-primal but non-Abelian.

Following [14], a ring R (possibly without identity) is called *QRPR* if $ab = 0$ for $a, b \in R$ implies $ba \in N_*(R)$. 2-primal rings are QRPR by [14, Lemma 1.1], but the converse need not hold by [14, Example 1.5].

Following Marks [20], a ring R (possibly without identity) is called *NI* if $N^*(R) = N(R)$. Note that a ring R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. 2-primal rings are clearly NI, but not conversely by [5, Example 3.3], [12, Example 1.2], or [20, Example 2.2].

Lemma 1.1 ([14, Lemma 1.2])**.** *For a ring R the following conditions are equivalent*:

(1) *R is NI*;
(2) *$a^2 \in N^*(R)$ for $a \in R$ implies $a \in N^*(R)$*;
(3) *$abc \in N^*(R)$ for $a, b, c \in R$ implies $acb \in N^*(R)$*;
(4) *$ab \in N^*(R)$ for $a, b \in R$ implies $ba \in N^*(R)$*;
(5) *$ab \in N^*(R)$ for $a, b \in R$ implies $aRb \subseteq N^*(R)$*;
(6) *$R/N^*(R)$ is 2-primal*;
(7) *$r_1 r_2 \cdots r_n \in N^*(R)$ for $r_i \in R$ implies $Rr_{\sigma(1)}Rr_{\sigma(2)}R \cdots Rr_{\sigma(n)}R \subseteq N^*(R)$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 2$.*

We start our study by the following that is induced from Lemma 1.1.

Definition 1.2. A ring R (possibly without identity) is called *quasi-reversible-over-upper-nilradical* (simply, *QRUR*) if $ab = 0$ for $a, b \in R$ implies $ba \in N^*(R)$.

The following is an immediate consequence of the definition.

Lemma 1.3. *For a ring R the following conditions are equivalent*:

(1) *R is QRUR*;
(2) *$ab = 0$ for $a, b \in R$ implies $baR \subseteq N^*(R)$*;
(3) *$ab = 0$ for $a, b \in R$ implies $Rba \subseteq N^*(R)$*;
(4) *$ab = 0$ for $a, b \in R$ implies $RbaR \subseteq N^*(R)$.*

We will use Lemma 1.3 freely. NI rings are QRUR by Lemma 1.1 and QRPR rings clearly are QRUR. But the converses need not hold by the following. For a given ring R and $k \geq 1$, we use $N_k(R) = \{a \in R \mid a^k = 0\}$.

Example 1.4. (1) There exists a QRPR (hence QRUR) ring which is not NI. We use the ring and argument in [10, Example 1]. Let F be a field and $A = F\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over F. Let $R = A/(x^2)^2$, where (x^2) is the ideal of A generated by x^2. We identity x, y with their images in R. Note that $N_*(R) = Rx^2R = N_2(R)$ and

$N(R) = xRx + Rx^2R + Fx$. Moreover $N_*(R) = N^*(R)$ by the computation in [13, Example 1.2 (2)]. This yields $N^*(R) \subsetneq N(R)$, and so R is not NI.

Now suppose that $ab = 0$ for $a, b \in R$. Then $ba \in N_2(R)$. This yields $ba \in N_2(R) = N_*(R) = N^*(R)$ by the computation in [6, Example 2.2]. Thus R is QRPR.

(2) There exists a QRUR ring that is not QRPR. We use the ring and argument in [12, Example 1.2]. Let S be a 2-primal ring, n be a positive integer and R_n be the 2^n by 2^n upper triangular matrix ring over S, i.e., $R_n = U_{2^n}(S)$. Each R_n is a 2-primal (hence NI) ring by [5, Proposition 2.5]. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Next set $R = \cup_{i=1}^{\infty} R_n$. Then R is an NI (but not 2-primal) ring with $N_*(R) = 0$, by the argument in [12, Example 1.2]. Furthermore R is QRUR by Lemma 1.1. Now let $a = e_{23}$ and $b = e_{12}$ in $R_2 \subset R$. Then $ab = 0$, but $ba = e_{13} \notin N_*(R)$ since $N_*(R) = 0$. So R is not QRPR.

Following Liang et al. [19], a ring R is called *weakly semicommutative* if $ab = 0$ for $a, b \in R$ implies $aRb \subseteq N(R)$. This notion is a proper generalization of semicommutative rings. Liang et al. showed that $U_2(R)$ is weakly semicommutative over a semicommutative ring R.

Proposition 1.5. (1) *QRUR rings are weakly semicommutative.*

(2) *The class of weakly semicommutative rings contains both NI rings and QRPR rings.*

Proof. We apply the proof of [14, Proposition 1.6]. Let R be a QRUR ring and suppose that $ab = 0$ for $a, b \in R$. Then $(aRbaRba)(baRb) = 0$. Since R is QRUR, we get $(barb)(arbarba) \in N^*(R)$ and this yields

$$(arb)^5 = ar[(barb)(arbarba)]rb \in N^*(R)$$

for all $r \in R$. Thus $arb \in N(R)$, and so R is weakly semicommutative.

(2) is immediately obtained from (1). □

Köthe [17] conjectured in 1930 that if a ring has no nonzero nil ideals, then it has no nonzero nil one-sided ideals. There are many assertions equivalent to this conjecture. For example, Köthe conjecture holds if and only if the sum of two nil right ideals is nil if and only if the upper nilradical contains all nil right ideals [18]. We do not know any example of a weakly semicommutative ring that is not QRUR. The following shows that the converse of Proposition 1.5(1) holds for rings satisfying Köthe's conjecture.

Proposition 1.6. (1) *Let R be a ring which satisfies Köthe's conjecture. If R is weakly semicommutative, then R is QRUR.*

(2) *Suppose that there exists a weakly semicommutative ring but not QRUR. Then Köthe's conjecture does not hold.*

Proof. (1) Let R be a weakly semicommutative ring that satisfies Köthe's conjecture. Let $ab = 0$ for $a, b \in R$. Since R is weakly semicommutative, $aRb \subseteq N(R)$. It then follows that Rba and baR are contained in $N(R)$. Since R satisfies Köthe's conjecture, Rba and baR are contained in $N^*(R)$. This implies that R is QRUR.

(2) is an immediate consequence of (1). □

2. Properties and examples

In this section we investigate properties of QRUR rings and extend the class of QRUR rings by applying them. The following provides us a method by which we can examine the QRUR property of given rings. Let R be a ring. Following [23, Definition 2.6.5], an ideal P of R is called *strongly prime* if P is prime and R/P has no nonzero nil ideals. Maximal ideals are clearly strongly prime, but there exist many strongly prime ideals which are not maximal (e.g., zero ideals of non-simple domains). An ideal P of R is called *minimal strongly prime* if P is minimal in the space of strongly prime ideals in R. $N^*(R) = \bigcap\{P \mid P \text{ is a strongly prime ideal of } R\} = \bigcap\{P \mid P \text{ is a minimal strongly prime ideal of } R\}$ by help of [23, Proposition 2.6.7].

Theorem 2.1. *Let R be a ring and I be a proper ideal of R such that R/I is QRUR. If I is NI as a ring without identity, then R is QRUR.*

Proof. We apply the proof of [14, Theorem 1.8]. Let $ab = 0$ for $a, b \in R$. Then bIa is a nil subset of I, and $\bar{b}\bar{a} \in N^*(R/I)$ since R/I is QRUR.

Suppose that I is NI as a ring without identity. Then $I/N^*(I)$ is a reduced ring, entailing that $N(I) = N^*(I)$ is a nil ideal of R by help of Andrunakievic [8, Lemma 61] (i.e., $(RN^*(I)R)^3 \subseteq IRN^*(I)RI = IN^*(I)I \subseteq N^*(I)$). This yields $bIa \subseteq N(I) = N^*(I) \subseteq N^*(R)$, so

$$(baI)(baI)\cdots(baI) = b(aIbaIb\cdots aIb)aI \subseteq bIaI \subseteq N^*(I).$$

Since $I/N^*(I)$ is a reduced ring, we get $baRI = baI \subseteq N^*(I) \subseteq N^*(R)$. Then $baI \subseteq P$ for any minimal strongly prime ideal P of R. But P is prime, so $ba \in P$ or $I \subseteq P$. Here assume $ba \notin P$. Then $I \subseteq P$, and so $\bar{b}\bar{a} \in N^*(R/I) \subseteq P/I$. This yields $ba \in P$, a contradiction. Consequently $ba \in P$, and thus $ba \in N^*(R)$ because P runs over all minimal strongly prime ideals of R. This concludes that R is QRUR. □

As an application of Theorem 2.1, consider $E = U_n(R)$ for $n \geq 2$ over an NI ring R. Then

$$N(E) = \{(a_{ij}) \in U_n(R) \mid a_{ii} \in N^*(R) \text{ for all } i\} = N^*(E),$$

entailing $\frac{E}{N^*(E)} \cong \underbrace{\frac{R}{N^*(R)} \oplus \cdots \oplus \frac{R}{N^*(R)}}_{n\text{-times}}$. Since $\frac{R}{N^*(R)}$ is a reduced ring, $\frac{E}{N^*(E)}$ is QRUR by Proposition 2.2 to follow. But $N^*(E)$ is NI, so E is QRUR by

Theorem 2.1, letting $I = N^*(E)$. Theorem 2.1 is also applicable to the case of setting $I = \{(a_{ij}) \in U_n(R) \mid a_{ii} = 0 \text{ for all } i\}$, noting that $\frac{R}{I} \cong \underbrace{R \oplus \cdots \oplus R}_{n\text{-times}}$.

$\prod$ (resp., $\oplus$) denotes the direct product (resp., direct sum) of rings.

Proposition 2.2. *The class of QRUR rings is closed under subrings and direct sums.*

Proof. We apply the proof of [14, Proposition 1.9]. Let R be a QRUR ring and S be a subring of R. Take $ab = 0$, for $a, b \in S$. Then $ba \in N^*(R)$. Since $N^*(R) \cap S \subseteq N^*(S)$, we have $ba \in N^*(S)$.

Suppose that R_i is a QRUR ring for each i in a nonempty index set I, and let D be the direct sum of R_i's. Let $a = (a_i), b = (b_i) \in D$ with $ab = 0$. Then $a_i b_i = 0$ for each $i \in I$ and so $b_i a_i \in N^*(R_i)$. Notice that $N^*(\oplus_{i\in I} R_i) = \oplus_{i\in I} N^*(R_i)$. So we have $ba \in N^*(\oplus_{i\in I} R_i)$, entailing D is QRUR. $\square$

As in the proof of Proposition 2.2, $N^*(R) \cap S \subseteq N^*(S)$ holds for any ring R. But the converse inclusion need not hold as can be seen by the ring R in Example 1.4(2). In fact, consider the subring $R_1 = U_2(A)$ over a NI ring A. Then $N^*(R_1) = \left(\begin{smallmatrix} N^*(A) & A \\ 0 & N^*(A) \end{smallmatrix}\right) \neq 0$ and $N^*(R) = 0$.

For a given ring R we usually write $D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$, $UN_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$ and $V_n(R) = \{(m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\}$. It is easily checked that $V_n(R) \cong R[x]/x^n R[x]$.

Corollary 2.3. *For a given ring R and $n \geq 2$, the following conditions are equivalent*:

(1) *R is QRUR*;
(2) *$U_n(R)$ is QRUR*;
(3) *$D_n(R)$ is QRUR*;
(4) *$V_n(R)$ is QRUR*;
(5) *$R[x]/x^n R[x]$ is QRUR.*

Proof. (1) $\Rightarrow$ (2). Let R be QRUR and consider $U_n(R)$. Let $I = UN_n(R)$. Then I is a nil ideal of $U_n(R)$, hence I is NI as a ring without identity. Note $U_n(R)/I$ is isomorphic to $\prod_{i=1}^n R_i$, where $R_i = R$ for all i. Since R is QRUR, $\prod_{i=1}^n R_i$ is QRUR by Proposition 2.2, noting $\prod_{i=1}^n R_i = \oplus_{i=1}^n R_i$. So $U_n(R)/I$ is also QRUR. Thus $U_n(R)$ is QRUR by Theorem 2.1.

(Another proof) We apply the proof of [14, Proposition 1.10]. Let R be a QRUR ring, and suppose $AB = 0$ for $A = (a_{ij}), B = (b_{ij}) \in U_n(R)$. Then $a_{ii} b_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. Since R is QRUR, we have $b_{ii} a_{ii} \in N^*(R)$. Notice that

$$N^*(U_n(R)) = \begin{pmatrix} N^*(R) & R & \cdots & R \\ 0 & N^*(R) & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N^*(R) \end{pmatrix}.$$

Thus $BA \in N^*(U_n(R))$.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are proved by Proposition 2.2. (4) $\Leftrightarrow$ (5) is obvious because $V_n(R)$ and $R[x]/x^nR[x]$ are isomorphic. $\square$

From [11], given a ring R and a R-bimodule M, the trivial extension of R by M is the ring $T(R, M)$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$, and the usual matrix operations are used. The proof of the following is similar to one of Corollary 2.3.

Corollary 2.4. *Let R and M be a R-bimodule. Then R is QRUR if and only if $T(R, M)$ is QRUR.*

One may suspect that if a ring R is QRUR, then $\mathrm{Mat}_n(R)$ is QRUR (or weakly semicommutative) for $n \geq 2$. But $\mathrm{Mat}_n(R)$ cannot be weakly semicommutative (hence cannot be QRUR by Proposition 1.5(1)) by the argument in [14, Example 1.12].

Let A be an algebra (with or without identity) over a commutative ring S. Due to Dorroh [9], the *Dorroh extension* of A by S is the Abelian group $A \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. Use $A \times_{dor} S$ to denote this extension.

Corollary 2.5. *Let A be an NI algebra (with or without identity) over a commutative ring S. Then $A \times_{dor} S$ is QRUR.*

Proof. Let $R = A \times_{dor} S$ and $I = A \times_{dor} 0$. Then R/I is isomorphic to S, so commutative (hence QRUR). Moreover I is NI by hypothesis. Thus R is QRUR by Theorem 2.1. $\square$

Following Rege and Chhawchharia [22, Definition 1.1], a ring R is called *Armendariz* if whenever any polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, we have $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Every reduced ring is Armendariz by [4, Lemma 1]. Armendariz rings are Abelian by the proof of [1, Theorem 6] (or [16, Lemma 7]).

The concepts of Armendariz and QRUR are independent of each other by the following.

Example 2.6. (1) We use the construction and argument in [3, Theorem 4.7]. Let F be a field and $A = F\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over F. Let $R = A/(b^2)$, where (b^2) is the ideal of A. Then R is Armendariz by [3, Theorem 4.7]. We identify a, b with their images in R. R is not QRUR by Lemma 1.3 because $abb = 0$ but $baba \notin N(R)$.

(2) Let R be a QRUR ring. Then $U_2(R)$ is a QRUR ring by Proposition 2.3. But this ring is non-Abelian, and so $U_2(R)$ is not Armendariz since Armendariz rings are Abelian.

The QRUR property can go up to polynomial rings when given rings are Armendariz as we see in the following that is similar to [14, Proposition 1.15].

Proposition 2.7. *Let R be an Armendariz ring. If R is QRUR, then $R[x]$ is QRUR.*

Proof. Let R be a QRUR ring. Assume $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $fg = 0$. Then since R is Armendariz, $a_i b_j = 0$ for all i and j. Since R is QRUR, we have $b_j a_i \in N^*(R)$ for all i, j.

Since R is Armendariz, $N_*(R) = N^*(R)$ by [15, Lemma 2.3 (5)] and $R[x]$ is also Armendariz by [1, Theorem 2]. Thus $N^*(R)[x] = N_*(R)[x] = N_*(R[x]) = N^*(R[x])$ by [18, Theorem 10.19]. Then we obtain $gf \in N^*(R[x])$ from the fact that $b_j a_i \in N^*(R)$ for all i, j. This implies that $R[x]$ is QRUR. $\square$

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