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MORE EXPANSION FORMULAS FOR q,ω-APOSTOL BERNOULLI AND EULER POLYNOMIALS

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Abstract. The purpose of this article is to continue the study of q,ω-special functions in the spirit of Wolfgang Hahn from the previous papers by Annaby et al. and Varma et al., with emphasis on q,ω-Apostol Bernoulli and Euler polynomials, Ward-ω numbers and multiple q,ω-power sums. Like before, the q,ω-module for the alphabet of q,ω-real numbers plays a crucial role, as well as the q,ω-rational numbers and the Ward-ω numbers. There are many more formulas of this type, and the deep symmetric structure of these formulas is described in detail.

1. Introduction

Based on our previous papers on pure q-calculus [3] and [4], this paper is part of a series of five papers on q,ω-calculus. In each paper we start with many similar definitions, since the subject is quite new. Let $\omega \in \mathbb{R},\ 0 < \omega < 1$. Put $\omega_0 \equiv \frac{\omega}{1-q}$, $0 < q < 1$. We introduce a new calculus, which will be very similar to the well-known q-calculus, where many functions and operators appear again, with a similar name. The reason is that the q,ω-Appell polynomials form a ring, which is proved in one of these papers [7]. The convergence region in ω will always be a small interval above 0, depending on q. The subtle properties of absolute maximum for the two q,ω-additions are exemplified in [6].

The paper is organized as follows: In Section 1 we present preliminary definitions and theorems for q,ω-analogues, like the q,ω-difference operator. In Section 2 we define the four q,ω-additions, natural generalizations of the four q-additions, and point out that they obey identical laws. It is an intriguing fact that the q,ω-difference operator of the q,ω-addition is identical with the corresponding q-analogue, a formula that seems to be new. The q,ω-addition formulas and q,ω-differences for the q,ω-exponential functions are also identical. In Section 3 we investigate general q,ω-Appell polynomials in the spirit if [5].

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In Section 4 the two multiple alternating q,ω-power sums are defined together with some special cases. In Subsection 4.1 formulas containing q,ω-power sums in one dimension, q,ω-analogues of Wang and Wang, [15] are proved. Then in Subsection 4.2, mixed formulas of the same kind are proved.

Definition. The automorphism ϵ on the vector space of polynomials is defined by

$$\epsilon f(x) \equiv f(qx+\omega). \tag{1}$$

This automorphism is a generalization of the operator with the same name in q-calculus [2]. In [1, p. 136] it is proved that

$$\epsilon^k f(x) = f(q^k x + \omega\{k\}_q). \tag{2}$$

Definition. A q,ω-analogue of the mathematical object G is a mathematical function $F(q,\omega)$, with the property $\lim_{\omega\to 0} F(q,\omega) = G_q$, the q-analogue of G. Both F and G can depend on more, common variables. They can also be operators.

Definition. Let φ be a continuous real function of x. Then we define the q,ω-difference operator $\mathrm{D}_{q,\omega}$ as follows:

$$\mathrm{D}_{q,\omega}\left(\varphi\right)(x) \equiv \begin{cases} \frac{\varphi(qx+\omega)-\varphi(x)}{(q-1)x+\omega}, & \text{if } x \neq \omega_0; \\ \frac{d\varphi}{dx}(x) & \text{if } x = \omega_0. \end{cases} \tag{3}$$

We say that a function $f(x)$ is n times q,ω-differentiable if $\mathrm{D}^n_{q,\omega} f(x)$ exists. If we want to point out that this operator operates on the variable x, we write $\mathrm{D}_{q,\omega,x}$ for the operator. Furthermore, $\mathrm{D}_{q,\omega}(K) = 0$, like for the derivative.

Furthermore, we need the following chain rule:

Definition.

$$\mathrm{D}_{q,\omega}\left(\epsilon^k\varphi\right)(x) \equiv q^k \frac{\epsilon^{k+1}\varphi(x) - \epsilon^k\varphi(x)}{(q-1)x+\omega}. \tag{4}$$

The motivation for formula (4) is that it is identical with the q-calculus case and enables smooth proofs of the following formulas, like the Leibniz formula. It also follows from the chain rule (11).

Theorem 1.1. *The q,ω-difference operator is linear*

$$\mathrm{D}_{q,\omega}\sum_{k=0}^{\infty} a_k f_k(x) = \sum_{k=0}^{\infty} a_k \mathrm{D}_{q,\omega} f_k(x). \tag{5}$$

Theorem 1.2 ([1, (16), p. 137])**.** *The q,ω-difference operator for a product of functions.*

$$\mathrm{D}_{q,\omega}(fg)(x) = \mathrm{D}_{q,\omega}(f(x))g(x) + f(qx+\omega)\mathrm{D}_{q,\omega}(g(x)). \tag{6}$$

We now introduce two basic sequences, which generalize the Ciglerian polynomials in [2, 5.5].

Definition.

$$(x)_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1} (x - \omega\{m\}_q). \quad [14, (16)] \tag{7}$$

$$[x]_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1} (q^m x + \omega\{m\}_q). \quad [14, (15)] \tag{8}$$

The following names will be used for the ensuing q, ω-trigonometric and hyperbolic functions [6].

Definition. A function f of two variables x, ω is called x, ω-even if $f(-x, -\omega) = f(x, \omega)$. A function f of two variables x, ω is called x, ω-odd if $f(-x, -\omega) = -f(x, \omega)$.

Lemma 1.3. *Products and sums of any number of x, ω-even functions are x, ω-even. The product and quotient of an x, ω-even function and an x, ω-odd function are x, ω-odd.*

Lemma 1.4. *The two functions $(x)_{q,\omega}^{2k}$ and $[x]_{q,\omega}^{2k}$ are x, ω-even. The two functions $(x)_{q,\omega}^{2k+1}$ and $[x]_{q,\omega}^{2k+1}$ are x, ω-odd.*

The two following formulas correspond to the formula $\mathrm{D}x^n = nx^{n-1}$:

$$\mathrm{D}_{q,\omega}(x)_{q,\omega}^{n} = \{n\}_q (x)_{q,\omega}^{n-1}. \quad [9, 2.5], [14, (17)] \tag{9}$$

$$\mathrm{D}_{q,\omega}[x]_{q,\omega}^{n} = \{n\}_q [qx + \omega]_{q,\omega}^{n-1}. \quad [14, (18)] \tag{10}$$

Theorem 1.5. *The chain rule for the q, ω-difference operator.*

$$\mathrm{D}_{q,\omega}\left((ax)_{q,a\omega}^{n}\right) = a\{n\}_q (ax)_{q,a\omega}^{n-1}. \tag{11}$$

$$\mathrm{D}_{q,\omega}\left([ax]_{q,a\omega}^{n}\right) = a\{n\}_q [aqx + a\omega]_{q,a\omega}^{n-1}. \tag{12}$$

Proof. We prove (11) by induction. The formula (11) is true for $n = 1, 2$. Assume that it is true for $n - 1$. Then we have

$$\begin{aligned} & \mathrm{D}_{q,\omega}\left[(ax)_{q,a\omega}^{n-1}(ax - \{n-1\}_q a\omega)\right] \\ \overset{\text{by}(6)}{=} \; & a(ax)_{q,a\omega}^{n-1} + a^2\left[qx + \omega - \{n-1\}_q\right]\{n-1\}_q (ax)_{q,a\omega}^{n-2} \\ = \; & a(ax)_{q,a\omega}^{n-1}\left[1 + q\{n-1\}_q\right] = \text{RHS}. \end{aligned} \tag{13}$$

Formula (12) is proved in a similar style. □

We next introduce two q, ω-analogues of the exponential function:

Definition. The q, ω-exponential function $\mathrm{E}_{q,\omega}(z)$ [14, (21)] is defined by

$$\mathrm{E}_{q,\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{(z)_{q,\omega}^{k}}{\{k\}_q!}, \; |(1-q)z - \omega| < 1. \tag{14}$$

The complementary q,ω-exponential function $\mathrm{E}_{\frac{1}{q},\omega}(z)$ [14, (26)] is defined by

$$\mathrm{E}_{\frac{1}{q},\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{[z]_{q,\omega}^{k}}{\{k\}_{q}!},\ |\omega| < 1. \tag{15}$$

We have changed the name to $\mathrm{E}_{\frac{1}{q},\omega}(z)$ since $\mathrm{E}_{\frac{1}{q},0}(z) = \mathrm{E}_{\frac{1}{q}}(z)$ [2].

Theorem 1.6 ([14, (19)]). *The q,ω-exponential function is the unique solution of the first order initial value problem*

$$\mathrm{D}_{q,\omega} f(z) = f(z),\ f(0) = 1. \quad [14, (24)] \tag{16}$$

The complementary q,ω-exponential function is the unique solution of the first order initial value problem

$$\mathrm{D}_{q,\omega} f(z) = f(qz + \omega),\ f(0) = 1. \tag{17}$$

Theorem 1.7 ([14, (21)]). *The meromorphic continuation of the q,ω-exponential function $\mathrm{E}_{q,\omega}(z)$ is given by*

$$\mathrm{E}_{q,\omega}(z) = \frac{(-\omega; q)_{\infty}}{((1-q)z - \omega; q)_{\infty}}. \quad [14, (26)] \tag{18}$$

The meromorphic continuation of the complementary q,ω-exponential function $\mathrm{E}_{\frac{1}{q},\omega}(z)$ is given by

$$\mathrm{E}_{\frac{1}{q},\omega}(z) = \frac{((q-1)z + \omega; q)_{\infty}}{(\omega; q)_{\infty}}. \tag{19}$$

2. On the q,ω-addition with applications to q,ω-special functions

In order to use these functions, we need to generalize the q-addition. The ordinary q-addition is the special case $\omega = 0$. Just like for the q-addition, we use letters in an alphabet for the q,ω-additions. Equality between letters is denoted by $\sim$. In the following, beware of the fact that whenever we multiply the function argument x in $(x)_{q,\omega}^{\nu}$ or in $[x]_{q,\omega}^{\nu}$ by the constant a, we must also multiply ω by a.

Definition. Let $\{f_{\nu}\}_{\nu=0}^{\infty}$ denote an arbitrary sequence of real numbers. The q,ω-addition for the sequences $(x)_{q,\omega}^{k}$ is defined by

$$(f \oplus_{q} (x)_{q,\omega})^{\nu} \equiv \sum_{k=0}^{\nu} \binom{\nu}{k}_{q} f_{\nu-k} (x)_{q,\omega}^{k}. \tag{20}$$

The NWA q,ω-addition is defined as follows:

$$(x \oplus_{q,\omega} y)^{n} \equiv \sum_{k=0}^{n} \binom{n}{k}_{q} (x)_{q,\omega}^{n-k} (y)_{q,\omega}^{k}. \tag{21}$$

The NWA q,ω-subtraction is defined as follows:

$$(x \ominus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (-y)_{q,-\omega}^{k}. \tag{22}$$

The JHC q,ω-addition is defined as follows:

$$(x \boxplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [y]_{q,\omega}^{k}. \tag{23}$$

The JHC q,ω-subtraction is defined as follows:

$$(x \boxminus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [-y]_{q,-\omega}^{k}. \tag{24}$$

Theorem 2.1. *The NWA q,ω-addition is commutative and associative.*

Proof. Similar to the proof for NWA q-addition. ☐

Corollary 2.2. *Four q,ω-additions for the q,ω-exponential function.*

$$\mathrm{E}_{q,\omega}(x \oplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,\omega}(y). \tag{25}$$

$$\mathrm{E}_{q,\omega}(x \ominus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,-\omega}(-y). \tag{26}$$

$$\mathrm{E}_{q,\omega}(x \boxplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},\omega}(y). \tag{27}$$

$$\mathrm{E}_{q,\omega}(x \boxminus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},-\omega}(-y). \tag{28}$$

Definition. A q,ω-groupoid $(G_{q,\omega}, \oplus_{q,\omega}, \sim)$ is a set of letters with an associative and commutative mapping $\oplus_{q,\omega} : G_{q,\omega} \times G_{q,\omega} \mapsto G_{q,\omega}$. The associativity can be expressed as follows:

$$(a \oplus_{q,\omega} b) \oplus_{q,\omega} c \sim a \oplus_{q,\omega} (b \oplus_{q,\omega} c), a, b, c \in G_{q,\omega}. \tag{29}$$

Theorem 2.3 (Compare with [13, p. 39])**.** *In a q,ω-groupoid, all composite operations represent the same element. It is denoted by $\oplus_{q,\omega,l=0}^{j-1} a_l$, $\{a_l\}_{l=0}^{j-1} \in G_{q,\omega}$.*

Definition. A q,ω-module is a generalization of the vector space over a field, where the corresponding scalars belong to $\mathbb{R}$. In a q,ω-module we can multiply letters $\alpha \in \mathbb{Q}_{\oplus_{q,\omega}} \vee \mathbb{R}_{q,\omega}$ with scalars $b \in \mathbb{R}$ to form the letters $\gamma \in \mathbb{R}_{q,\omega}$:

$$\gamma \sim b\alpha. \tag{30}$$

This operation is distributive over the q,ω-addition:

$$b(\alpha \oplus_{q,\omega} \beta) \sim b\alpha \oplus_{q,\omega} b\beta,\ b \in \mathbb{R},\ \alpha,\ \beta \in \mathbb{R}_{q,\omega}. \tag{31}$$

The operations (30) and (31), as well as similar formulas for q,ω-modules are used in the rest of the article without further explanation.

Theorem 2.4. *The* q,ω*-differences for the* q,ω*-exponential functions are*:

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{q,a\omega}(ax) = a\ \mathrm{E}_{q,a\omega}(ax), \tag{32}$$

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{\frac{1}{q},a\omega}(ax) = a\ \mathrm{E}_{\frac{1}{q},a\omega}(aqx + a\omega). \tag{33}$$

Proof. This follows from the chain rule (11) and (12). ☐

In our second book [5] we introduced several new q-deformed number systems, semiring, biring etc., each with an extra index q. We can extend these number systems by adding another index ω. The proofs will be very similar, and we just state the definitions and corresponding theorems.

Definition. The Ward-ω number $\overline{n}_{q,\omega}$ is defined by

$$\overline{n}_{q,\omega} \sim 1 \oplus_{q,\omega} 1 \oplus_{q,\omega} \cdots \oplus_{q,\omega} 1, \tag{34}$$

where the number of 1 on the RHS is n.

Definition (An extension of [2, 4.70])**.**

$$(\overline{n}_{q,\omega})^k \equiv \sum_{m_1+\cdots+m_n=k} \binom{k}{m_1,\ldots,m_n}_q \prod_{i=1}^{n} (1)_{q,\omega}^{m_i}\ , \tag{35}$$

where each partition of k is multiplied with its number of permutations. We have the following special cases:

$$(\overline{0}_{q,\omega})^k = \delta_{k,0};\ (\overline{n}_{q,\omega})^0 = 1;\ (\overline{n}_{q,\omega})^1 = n. \tag{36}$$

Let $(\mathbb{N}_{\oplus_{q,\omega}}, \oplus_{q,\omega}, \odot_{q,\omega})$ denote the semiring of Ward-ω numbers $\overline{k}_{q,\omega},\ k \geq 0$ together with two binary operations: $\oplus_{q,\omega}$ is the usual q,ω-addition. The multiplication $\odot_{q,\omega}$ is defined as follows:

$$\overline{n}_{q,\omega} \odot_{q,\omega} \overline{m}_{q,\omega} \sim \overline{nm}_{q,\omega}, \tag{37}$$

where $\sim$ denotes the equivalence in the alphabet. In long formulas, the q,ω-multiplication is abbreviated by juxtaposition.

Theorem 2.5. *Functional equations for Ward-*ω *numbers operating on the* q*, omega-exponential function. First assume that the letters* $\overline{m}_{q,\omega}$ *and* $\overline{n}_{q,\omega}$ *are independent, i.e., come from two different functions, when operating with the functional. Furthermore,* $mnt < \frac{1+\omega}{1-q}$*. Then we have*

$$\mathrm{E}_{q,\omega}(\overline{m}_{q,\omega}\overline{n}_{q,\omega}t) = \mathrm{E}_{q,\omega}(\overline{mn}_{q,\omega}t). \tag{38}$$

Furthermore,

$$\mathrm{E}_{q,\omega}(\overline{jm}_{q,\omega}) = \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega})^m = \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega})^j = \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega} \odot_{q,\omega} \overline{m}_{q,\omega}). \tag{39}$$

Definition. Let the q,ω-rational numbers $\mathbb{Q}_{q,\omega}$ be defined as follows:

$$\mathbb{Q}_{q,\omega} \equiv \left\{ \frac{\overline{m}_{q,\omega}}{\overline{n}_{q,\omega}},\ m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N},\ m \neq n,\ \frac{\overline{0}_{q,\omega}}{\overline{n}_{q,\omega}} \sim \theta,\ \frac{\overline{n}_{q,\omega}}{\overline{n}_{q,\omega}} \sim 1 \right\}, \tag{40}$$

together with a linear functional

$$v,\ \mathbb{R}[x] \times \mathbb{Q}_{\oplus_{q,\omega}} \to \mathbb{R}, \tag{41}$$

called the evaluation. If $v(x) = \sum_{k=0}^{n} a_k x^k$, then

$$v\left(\frac{\overline{m}_{q,\omega}}{\overline{n}_{q,\omega}}\right) \equiv \sum_{k=0}^{n} a_k \frac{(\overline{m}_{q,\omega})^k}{(\overline{n}_{q,\omega})^k}. \tag{42}$$

3. q, ω-Appell polynomials

The most general form of polynomial in this article is the Hahn–Appell polynomial, which we will now define.

Definition. Let $\mathcal{A}_{q,\omega}$ denote the set of real sequences $\{u_{\nu,q}\}_{\nu=0}^{\infty}$ such that

$$\sum_{\nu=0}^{\infty} |u_{\nu,q}| \frac{r^\nu}{\{\nu\}_q!} < \infty \tag{43}$$

for some q, ω-dependent convergence radius $r = r(q) > 0$, where $0 < q < 1$.

The q, ω-Appell number sequence is denoted by $\{\Phi^{(n)}_{\nu,q,\omega}\}_{\nu=0}^{\infty}$.

Definition. Assume that $h(t, q, \omega),\ h(t, q, \omega)^{-1} \in \mathbb{R}[[t]]$. For $f_n(t, q, \omega) = h(t, q, \omega)^n$, the multiplicative q, ω-Appell numbers of degree ν and order n $\Phi_{\nu,q,\omega} \in \mathcal{A}_{q,\omega}$ are given by the generating function

$$f_n(t, q, \omega) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\nu,q,\omega},\ \Phi^{(n)}_{0,q,\omega} = 1. \tag{44}$$

It will be convenient to fix the value for $n = 0$ and $n = 1$:

$$\Phi^{(0)}_{\nu,q,\omega} \equiv \delta_{0,\nu},\ \Phi^{(1)}_{\nu,q,\omega} \equiv \Phi_{\nu,q,\omega}. \tag{45}$$

Definition (Compare with [14, (31)])**.** For $f_n(t, q, \omega) \in \mathbb{R}[[t]]$ as above, the multiplicative q, ω-Appell polynomial sequence $\{\Phi^{(n)}_{\nu;q,\omega}(x)\}_{\nu=0}^{\infty}$ of degree ν and order n is defined by the generating function

$$f_n(t, q, \omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\nu;q,\omega}(x). \tag{46}$$

It will be convenient to fix the value for $n = 0$ and $n = 1$:

Theorem 3.1.

$$\Phi^{(0)}_{\nu,q,\omega}(x) = (x)^\nu_{q,\omega},\ \Phi^{(1)}_{\nu,q,\omega}(x) \equiv \Phi_{\nu,q,\omega}(x). \tag{47}$$

We prove the first equation.

Proof. By using the linearity of $\mathrm{D}_{q,\omega}$, (5), and the q, ω Taylor formula [7], it would suffice to prove that

$$\mathrm{D}^k_{q,\omega,x}(xt)^k_{q,\omega t} = t^k \{k\}_q!.$$

But this is obvious by the chain rule (11). ☐

Definition. For $f_n(t,q,\omega) \in \mathbb{R}[[t]]$ as above, the complementary, multiplicative q,ω-Appell polynomial sequence $\{\Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x)\}_{\nu=0}^{\infty}$ of degree ν and order n is defined by the generating function

$$f_n(t,q,\omega)\mathrm{E}_{\frac{1}{q},\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x). \tag{48}$$

It will be convenient to fix the value for $n=0$ and $n=1$:

Definition.

$$\Phi^{(0)}_{\nu,\frac{1}{q},\omega}(x) \equiv [x]^{\nu}_{q,\omega},\ \Phi^{(1)}_{\nu,\frac{1}{q},\omega}(x) \equiv \Phi_{\nu,\frac{1}{q},\omega}(x). \tag{49}$$

We next present generalizations of the three formulas [2, 4.107, 4.108, 4.111].

Theorem 3.2.

$$\mathrm{D}_{q,\omega}\Phi_{\nu;q,\omega}(x) = \{\nu\}_q\Phi_{\nu-1;q,\omega}(x). \tag{50}$$

[14, (30)] *in umbral form*:

$$\Phi_{\nu;q,\omega}(x) \ddot{=} (\Phi_{q,\omega} \oplus_{q,\omega} x)^{\nu}. \tag{51}$$

Theorem 3.3. *Assume that M and K are the x-order and y-order, respectively. Then we have*:

$$\Phi^{(M+K)}_{\nu,q,\omega}(x \oplus_{q,\omega} y) = \sum_{k=0}^{\nu}\binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y). \tag{52}$$

Proof. We show that both sides of (52) have the same generating function.

$$\begin{aligned}
f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x \oplus_{q,\omega} y)t) &\overset{\text{by}(25)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt),\\
f_K(t,q,\omega)\mathrm{E}_{q,\omega t}(yt) &\overset{\text{by}(46)}{=} \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x)\sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;q,\omega}(y)\\
&= \sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\sum_{k=0}^{\nu}\binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y).
\end{aligned} \tag{53}$$

□

Remark 3.4. Formula (52) defines $\Phi^{(M+K)}_{\nu,q,\omega}(x \oplus_{q,\omega} y)$ as the right hand side of the formula. There is no other definition of this function.

Theorem 3.5. *Assume that M and K are the x-order and y-order, respectively. Then we have*:

$$\Phi^{(M+K)}_{\nu,q,\omega}(x \boxplus_{q,\omega} y) = \sum_{k=0}^{\nu}\binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y). \tag{54}$$

Proof. We show that both sides of (54) have the same generating function.

$$\begin{aligned} f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x \boxplus_{q,\omega} y)t) &\overset{\text{by}(27)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) f_K(t,\tfrac{1}{q},\omega),\\ \mathrm{E}_{\frac{1}{q},\omega t}(yt) &\overset{\text{by}(46),(48)}{=} \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x)\sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;\frac{1}{q},\omega}(y)\\ &= \sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\sum_{k=0}^{\nu}\binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y). \end{aligned} \tag{55}$$

☐

Theorem 3.6 (A q,ω-analogue of [12, p. 125])**.**

$$(\mathrm{E}_{q,\omega}(t)-1)f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\triangle_{\mathrm{NWA},q,\omega}\Phi^{(n)}_{\nu,q,\omega}(x). \tag{56}$$

Proof. Operate on (46) with $\triangle_{\mathrm{NWA},q,\omega}$. ☐

Theorem 3.7 (A q,ω-analogue of [12, p. 125])**.**

$$\frac{(\mathrm{E}_{q,\omega}(t)+1)}{2}f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\nabla_{\mathrm{NWA},q,\omega}\Phi^{(n)}_{\nu,q,\omega}(x). \tag{57}$$

Proof. Operate on (46) with $\nabla_{\mathrm{NWA},q,\omega t}$. ☐

Theorem 3.8 (A q,ω-analogue of [12, p. 125])**.**

$$(\mathrm{E}_{\frac{1}{q},\omega}(t)-1)f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\triangle_{\mathrm{JHC},q,\omega}\Phi^{(n)}_{\nu,q,\omega}(x). \tag{58}$$

Proof. Operate on (46) with $\triangle_{\mathrm{JHC},q,\omega t}$. ☐

Theorem 3.9 (A q,ω-analogue of [12, p. 125])**.**

$$\frac{(\mathrm{E}_{\frac{1}{q},\omega}(t)+1)}{2}f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\nabla_{\mathrm{JHC},q,\omega}\Phi^{(n)}_{\nu,q,\omega}(x). \tag{59}$$

Proof. Operate on (46) with $\nabla_{\mathrm{JHC},q,\omega t}$. ☐

4. Multiple q,ω-power sums

Definition (A q,ω-analogue of [11, (20) p. 381])**.** The multiple q,ω-power sum is defined by

$$s^{(l)}_{\mathrm{NWA},\lambda,m,q,\omega}(n) \equiv \sum_{|\vec{j}|=l}\binom{l}{\vec{j}}\lambda^k\left(\overline{k}_{q,\omega}\right)^m, \tag{60}$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1},\ \forall j_i \geq 0$.

Definition (A q,ω-analogue of [11, (46) p. 386])**.** The multiple alternating q,ω-power sum is defined by

$$\sigma^{(l)}_{\mathrm{NWA},\lambda,m,q,\omega}(n) \equiv (-1)^l \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} (-\lambda)^k \left(\overline{k}_{q,\omega}\right)^m, \tag{61}$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$, $\forall j_i \geq 0$.

For $l = 1$, formulas (60) and (61) reduce to single sums. In order to keep the same notation as in [2], we make a slight change from [15, p. 309]. The following definitions are special cases of the q,ω-power sums in [8].

Definition (Almost a q,ω-analogue of [15, p. 309])**.** The q,ω-power sum and the alternate q,ω-power sum (with respect to λ) are defined by

$$s_{\mathrm{NWA},\lambda,m,q,\omega}(n) \equiv \sum_{k=0}^{n-1} \lambda^k (\overline{k}_{q,\omega})^m, \tag{62}$$

$$\sigma_{\mathrm{NWA},\lambda,m,q,\omega}(n) \equiv \sum_{k=0}^{n-1} (-1)^k \lambda^k (\overline{k}_{q,\omega})^m. \tag{63}$$

Their respective generating functions are

$$\sum_{m=0}^{\infty} s_{\mathrm{NWA},\lambda,m,q,\omega}(n) \frac{t^m}{\{m\}_q!} = \frac{\lambda^n \mathrm{E}_{q,\omega}(\overline{n}_{q,\omega} t) - 1}{\lambda \mathrm{E}_{q,\omega}(t) - 1} \tag{64}$$

and

$$\sum_{m=0}^{\infty} \sigma_{\mathrm{NWA},\lambda,m,q,\omega}(n) \frac{t^m}{\{m\}_q!} = \frac{(-1)^{n+1} \lambda^n \mathrm{E}_{q,\omega}(\overline{n}_{q,\omega} t) + 1}{\lambda \mathrm{E}_{q,\omega}(t) + 1}. \tag{65}$$

Proof. Let us prove (64). We have

$$\begin{aligned} \sum_{m=0}^{\infty} s_{\mathrm{NWA},\lambda,m,q,\omega}(n) \frac{t^m}{\{m\}_q!} &= \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \lambda^k \frac{(\overline{k}_{q,\omega} t)^m}{\{m\}_q!} \\ &= \sum_{k=0}^{n-1} \lambda^k (\mathrm{E}_{q,\omega}(t))^k = \mathrm{RHS}. \end{aligned} \tag{66}$$

□

We have the following special cases:

$$s_{\mathrm{NWA},\lambda,m,q,\omega}(1) = \sigma_{\mathrm{NWA},\lambda,m,q,\omega}(1) = \delta_{0,m}, \tag{67}$$

$$s_{\mathrm{NWA},\lambda,m,q,\omega}(2) = \delta_{0,m} + \lambda,\ \sigma_{\mathrm{NWA},\lambda,m,q,\omega}(2) = \delta_{0,m} - \lambda. \tag{68}$$

4.1. Single formulas for q,ω-power sums

Theorem 4.1 (A q,ω-analogue of [15, p. 310], and extensions of [2, pp. 121, 131]).

$$s_{\mathrm{NWA},\lambda,m,q,\omega}(n)=\frac{\lambda^{n}\mathcal{B}_{\mathrm{NWA},\lambda,m+1,q,\omega}(\overline{n}_{q,\omega})-\mathcal{B}_{\mathrm{NWA},\lambda,m+1,q,\omega}}{\{m+1\}_{q}}. \tag{69}$$

$$\sigma_{\mathrm{NWA},\lambda,m,q,\omega}(n)=\frac{(-1)^{n+1}\lambda^{n}\mathcal{F}_{\mathrm{NWA},\lambda,m,q,\omega}(\overline{n}_{q,\omega})-\mathcal{F}_{\mathrm{NWA},\lambda,m,q,\omega}}{2}. \tag{70}$$

Theorem 4.2 (A q,ω-analogue of [15, (18), p. 311]).

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}_{q}\frac{(\overline{i}_{q,\omega})^{k}}{i}(\overline{j}_{q,\omega})^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(\overline{j}_{q,\omega}x\right)s_{\mathrm{NWA},\lambda^{j},n-k,q,\omega}(i)\\
&=\sum_{k=0}^{n}\binom{n}{k}_{q}\frac{(\overline{j}_{q,\omega})^{k}}{j}(\overline{i}_{q,\omega})^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^{j},k,q,\omega}\left(\overline{i}_{q,\omega}x\right)s_{\mathrm{NWA},\lambda^{i},n-k,q,\omega}(j)\\
&=\frac{(\overline{i}_{q,\omega})^{n}}{i}\sum_{m=0}^{i-1}\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\\
&=\frac{(\overline{j}_{q,\omega})^{n}}{j}\sum_{m=0}^{j-1}\lambda^{im}\mathcal{B}_{\mathrm{NWA},\lambda^{j},n,q,\omega}\left(\overline{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\overline{j}_{q,\omega}}\right).
\end{aligned} \tag{71}$$

Proof. The following function is symmetric in i and j.

$$\begin{aligned}
f_{q,\omega}(t)&\equiv\frac{t\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)(\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1)}{(\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1)(\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1)}\\
&=\left(\frac{(\overline{i}_{q,\omega}t)^{1}\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)}{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1}\right)\left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1}\right)\frac{1}{i}.
\end{aligned} \tag{72}$$

We can expand $f_{q,\omega}(t)$ in two ways by using the formula for a geometric sequence.

$$\begin{aligned}
&f_{q,\omega}(t)\\
&=\left(\sum_{\nu=0}^{\infty}\mathcal{B}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega}\left(\overline{j}_{q,\omega}x\right)\frac{(\overline{i}_{q,\omega}t)^{\nu}}{\{\nu\}_{q}!}\right)\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^{j},m,q,\omega}(i)\frac{(\overline{j}_{q,\omega}t)^{m}}{\{m\}_{q}!}\right)\frac{1}{i}\\
&=\frac{\left(\overline{i}_{q,\omega}\right)^{1}t}{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1}\sum_{m=0}^{i-1}\lambda^{jm}\left(\mathrm{E}_{q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\overline{i}_{q,\omega}t\right)\frac{1}{i}\\
&=\sum_{\nu=0}^{\infty}\left(\frac{(\overline{i}_{q,\omega})^{\nu}}{i}\sum_{m=0}^{i-1}\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\right)\frac{t^{\nu}}{\{\nu\}_{q}!}.
\end{aligned} \tag{73}$$

Finally, equate the coefficients of $\frac{t^{\nu}}{\{\nu\}_{q}!}$ and use the symmetry in i and j of $f_{q,\omega}(t)$. ☐

Corollary 4.3 (A q,ω-analogue of [15, (19), p. 311]).

$$\begin{aligned}\mathcal{B}_{\mathrm{NWA},\lambda,n,q,\omega}\left(\overline{i}_{q,\omega}x\right)&=\sum_{k=0}^{n}\binom{n}{k}_{q}\frac{(\overline{i}_{q,\omega})^{k}}{i}\mathcal{B}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(x\right)s_{\mathrm{NWA},\lambda,n-k,q,\omega}(i)\\&=\frac{(\overline{i}_{q,\omega})^{n}}{i}\sum_{m=0}^{i-1}\lambda^{m}\mathcal{B}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(x\oplus_{q,\omega}\frac{\overline{m}_{q,\omega}}{\overline{i}_{q,\omega}}\right).\end{aligned}\tag{74}$$

Proof. Put $j=1$ in (71) and use (68). ☐

Corollary 4.4 (A q,ω-analogue of [15, (20), p. 311]).

$$\begin{aligned}&\sum_{m=0}^{1}\lambda^{im}\mathcal{B}_{\mathrm{NWA},\lambda^{2},n,q,\omega}\left(\overline{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\overline{2}_{q,\omega}}\right)\\&=\frac{2}{(\overline{2}_{q,\omega})^{n}}\sum_{k=0}^{n}\binom{n}{k}_{q}\frac{(\overline{i}_{q,\omega})^{k}}{i}(\overline{2}_{q,\omega})^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(\overline{2}_{q,\omega}x\right)s_{\mathrm{NWA},\lambda^{2},n-k,q,\omega}(i)\\&=\frac{2}{(\overline{2}_{q,\omega})^{n}}\frac{(\overline{i}_{q,\omega})^{n}}{i}\sum_{m=0}^{i-1}\lambda^{2m}\mathcal{B}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(\overline{2}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{2m}_{q,\omega}}{\overline{i}_{q,\omega}}\right).\end{aligned}\tag{75}$$

Proof. Put $j=2$ in (71) and multiply by $\frac{2}{(\overline{2}_{q,\omega})^{n}}$. ☐

Moreover, we have

$$\mathcal{B}_{\mathrm{NWA},\lambda,n,q,\omega}\left(x\right)=\frac{(\overline{2}_{q,\omega})^{n}}{2}\sum_{m=0}^{1}\lambda^{m}\mathcal{B}_{\mathrm{NWA},\lambda^{2},n,q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\oplus_{q,\omega}\frac{\overline{m}_{q,\omega}}{\overline{2}_{q,\omega}}\right).\tag{76}$$

Proof. Put $i=2$ in (74) and replace x by $\frac{x}{\overline{2}_{q,\omega}}$. ☐

Theorem 4.5 (A q,ω-analogue of [15, (22) p. 312]). *For i and j either both odd, or both even, we have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}_{q}(\overline{i}_{q,\omega})^{k}(\overline{j}_{q,\omega})^{n-k}\mathcal{F}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(\overline{j}_{q,\omega}x\right)\sigma_{\mathrm{NWA},\lambda^{j},n-k,q,\omega}(i)\\&=\sum_{k=0}^{n}\binom{n}{k}_{q}(\overline{j}_{q,\omega})^{k}(\overline{i}_{q,\omega})^{n-k}\mathcal{F}_{\mathrm{NWA},\lambda^{j},k,q,\omega}\left(\overline{i}_{q,\omega}x\right)\sigma_{\mathrm{NWA},\lambda^{i},n-k,q,\omega}(j)\\&=(\overline{i}_{q,\omega})^{n}\sum_{m=0}^{i-1}\lambda^{jm}(-1)^{m}\mathcal{F}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\\&=(\overline{j}_{q,\omega})^{n}\sum_{m=0}^{j-1}\lambda^{im}(-1)^{m}\mathcal{F}_{\mathrm{NWA},\lambda^{j},n,q,\omega}\left(\overline{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\overline{j}_{q,\omega}}\right).\end{aligned}\tag{77}$$

Proof. Let us define the following symmetric function

$$
\begin{aligned}
f_{q,\omega}(t) &\equiv \frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)((-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1)}{(\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)+1)(\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)+1)} \\
&= \frac{1}{2}\left(\frac{2\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)}{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)+1}\right)\left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1}{\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)+1}\right).
\end{aligned}
\tag{78}
$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$
\begin{aligned}
&f_{q,\omega}(t) \\
&= \frac{1}{2}\left(\sum_{\nu=0}^{\infty}\mathcal{F}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega}\left(\overline{j}_{q,\omega}x\right)\frac{(\overline{i}_{q,\omega}t)^{\nu}}{\{\nu\}_{q}!}\right)\left(\sum_{m=0}^{\infty}\sigma_{\mathrm{NWA},\lambda^{j},m,q,\omega}(i)\frac{(\overline{j}_{q,\omega}t)^{m}}{\{m\}_{q}!}\right) \\
&= \frac{1}{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)+1}\sum_{m=0}^{i-1}(-1)^{m}\lambda^{jm}\mathrm{E}_{q,\omega t}\left(\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\overline{i}_{q,\omega}t\right) \\
&= \frac{1}{2}\sum_{\nu=0}^{\infty}\left((\overline{i}_{q,\omega})^{\nu}\sum_{m=0}^{i-1}(-1)^{m}\lambda^{jm}\mathcal{F}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\right)\frac{t^{\nu}}{\{\nu\}_{q}!}.
\end{aligned}
\tag{79}
$$

The theorem follows by equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_{q}!}$ and using the symmetry in i and j of $f_{q,\omega}(t)$. ☐

Theorem 4.6 (A q,ω-analogue of [15, (24) p. 313])**.** *For i odd we have*

$$
\begin{aligned}
\mathcal{F}_{\mathrm{NWA},\lambda,n,q,\omega}\left(\overline{i}_{q,\omega}x\right) &= \sum_{k=0}^{n}\binom{n}{k}_{q}(\overline{i}_{q,\omega})^{k}\mathcal{F}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(x\right)\sigma_{\mathrm{NWA},\lambda,n-k,q,\omega}(i) \\
&= (\overline{i}_{q,\omega})^{n}\sum_{m=0}^{i-1}(-\lambda)^{m}\mathcal{F}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(x\oplus_{q,\omega}\frac{\overline{m}_{q,\omega}}{\overline{i}_{q,\omega}}\right)
\end{aligned}
\tag{80}
$$

(A q,ω-analogue of [15, (25) p. 313]). *For i even,*

$$
\begin{aligned}
&\sum_{m=0}^{1}\lambda^{im}(-1)^{m}\mathcal{F}_{\mathrm{NWA},\lambda^{2},n,q,\omega}\left(\overline{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\overline{2}_{q,\omega}}\right) \\
&= \frac{1}{(\overline{2}_{q,\omega})^{n}}\sum_{k=0}^{n}\binom{n}{k}_{q}(\overline{i}_{q,\omega})^{k}(\overline{2}_{q,\omega})^{n-k}\mathcal{F}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(\overline{2}_{q,\omega}x\right) \\
&\quad\times\sigma_{\mathrm{NWA},\lambda^{2},n-k,q,\omega}(i) \\
&= \frac{(\overline{i}_{q,\omega})^{n}}{(\overline{2}_{q,\omega})^{n}}\sum_{m=0}^{i-1}(-1)^{m}\lambda^{2m}\mathcal{F}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(\overline{2}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{2m}_{q,\omega}}{\overline{i}_{q,\omega}}\right).
\end{aligned}
\tag{81}
$$

Proof. Put $j=1$ or 2 in (77), and divide by $(\overline{2}_{q,\omega})^{n}$. ☐

4.2. q,ω-power sums, mixed formulas

We now turn to mixed formulas, which contain polynomials of both kinds.

Theorem 4.7 (A q,ω-analogue of [15, (26) p. 313])**.** *If i is even, then*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}_q \frac{(\bar{i}_{q,\omega})^k}{i}(\bar{j}_{q,\omega})^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^i,k,q,\omega}\left(\bar{j}_{q,\omega}x\right)\sigma_{\mathrm{NWA},\lambda^j,n-k,q,\omega}(i)\\
&= -\frac{\{n\}_q}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}_q(\bar{j}_{q,\omega})^k(\bar{i}_{q,\omega})^{n-k-1}\\
&\quad\times\mathcal{F}_{\mathrm{NWA},\lambda^j,k,q,\omega}\left(\bar{i}_{q,\omega}x\right)s_{\mathrm{NWA},\lambda^i,n-k-1,q,\omega}(j)\\
&= \frac{(\bar{i}_{q,\omega})^n}{i}\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^i,n,q,\omega}\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\\
&= -\frac{\{n\}_q}{2}(\bar{j}_{q,\omega})^{n-1}\sum_{m=0}^{j-1}\lambda^{im}\mathcal{F}_{\mathrm{NWA},\lambda^j,n-1,q,\omega}\left(\bar{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}}\right).
\end{aligned}\tag{82}$$

Proof. Let us define the following function

$$\begin{aligned}
f_{q,\omega}(t)&\equiv\frac{t\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)((-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1)}{(\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1)(\lambda^j\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1)}\\
&=\left(\frac{(\bar{i}_{q,\omega}t)^1\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)}{\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)\left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1}{\lambda^j\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1}\right)\frac{1}{i}.
\end{aligned}\tag{83}$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$\begin{aligned}
&f_{q,\omega}(t)\\
&=\left(\sum_{\nu=0}^{\infty}\mathcal{B}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\bar{j}_{q,\omega}x\right)\frac{(\bar{i}_{q,\omega}t)^\nu}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}\sigma_{\mathrm{NWA},\lambda^j,m,q,\omega}(i)\frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!}\right)\frac{1}{i}\\
&=\frac{\left(\bar{i}_{q,\omega}t\right)^1}{\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\mathrm{E}_{q,\omega t}\left(\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\bar{i}_{q,\omega}t\right)\frac{1}{i}\\
&=\sum_{\nu=0}^{\infty}\left(\frac{(\bar{i}_{q,\omega})^\nu}{i}\sum_{m=0}^{i-1}(-1)^m\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.
\end{aligned}\tag{84}$$

By equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$, we obtain rows 1 and 3 of formula (82).

On the other hand, we can rewrite $f_{q,\omega}(t)$ in the following way:

$$\begin{aligned} f_{q,\omega}(t) &= -\frac{t}{2}\frac{2\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)(\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1)}{(\lambda^{i}\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1)(\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1)} \\ &= -\frac{t}{2}\left(\frac{2\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)}{\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1}\right)\left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda^{i}\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right). \end{aligned} \tag{85}$$

By using the formula for a geometric sequence, we can expand (85) in two ways:

$$\begin{aligned} &f_{q,\omega}(t) \\ = &-\frac{t}{2}\left(\sum_{\nu=0}^{\infty}\mathcal{F}_{\mathrm{NWA},\lambda^{j},\nu,q,\omega}\left(\bar{i}_{q,\omega}x\right)\frac{(\bar{j}_{q,\omega}t)^{\nu}}{\{\nu\}_q!}\right)\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^{i},m,q,\omega}(j)\frac{(\bar{i}_{q,\omega}t)^{m}}{\{m\}_q!}\right) \\ = &-\frac{t}{2}\sum_{m=0}^{j-1}\lambda^{im}\frac{2}{\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1}\mathrm{E}_{q,\omega t}\left(\left(\bar{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}}\right)\bar{j}_{q,\omega}t\right) \\ = &-\frac{t}{2}\sum_{\nu=0}^{\infty}\left((\bar{j}_{q,\omega})^{\nu}\sum_{m=0}^{j-1}\lambda^{im}\mathcal{F}_{\mathrm{NWA},\lambda^{j},\nu,q,\omega}\left(\bar{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}}\right)\right)\frac{t^{\nu}}{\{\nu\}_q!}. \end{aligned} \tag{86}$$

By equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_q!}$, we obtain rows 2 and 4 of formula (82). ☐

Corollary 4.8 (A q,ω-analogue of [15, (28) p. 313]). *If i is even, then*

$$\begin{aligned} \mathcal{F}_{\mathrm{NWA},\lambda,n-1,q,\omega}\left(\bar{i}_{q,\omega}x\right) = &-\frac{2}{\{n\}_q}\sum_{k=0}^{n}\binom{n}{k}_q\frac{(\bar{i}_{q,\omega})^{k}}{i}\mathcal{B}_{\mathrm{NWA},\lambda^{i},k,q,\omega}(x) \\ &\times\sigma_{\mathrm{NWA},\lambda,n-k,q,\omega}(i) \\ = &-\frac{2(\bar{i}_{q,\omega})^{n}}{i\{n\}_q}\sum_{m=0}^{i-1}(-\lambda)^{m}\mathcal{B}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(x\oplus_{q,\omega}\frac{\overline{m}_{q,\omega}}{\bar{i}_{q,\omega}}\right). \end{aligned} \tag{87}$$

Proof. Put $j=1$ in formula (82) and multiply by $-\frac{2}{\{n\}_q}$. ☐

Corollary 4.9 (A q,ω-analogue of [15, (29) p. 313]).

$$\begin{aligned} &\mathcal{F}_{\mathrm{NWA},\lambda,n-1,q,\omega}(x) \\ &= -\frac{2}{\{n\}_q}\sum_{k=0}^{n}\binom{n}{k}_q\frac{(\overline{2}_{q,\omega})^{k}}{2}\mathcal{B}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right)\sigma_{\mathrm{NWA},\lambda,n-k,q,\omega}(2) \\ &= -\frac{(\overline{2}_{q,\omega})^{n}}{\{n\}_q}\sum_{m=0}^{1}(-\lambda)^{m}\mathcal{B}_{\mathrm{NWA},\lambda^{2},n,q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\oplus_{q,\omega}\frac{\overline{m}_{q,\omega}}{\overline{2}_{q,\omega}}\right). \end{aligned} \tag{88}$$

Proof. Put $i=2$ in formula (87), and replace x by $\frac{x}{\overline{2}_{q,\omega}}$. ☐

Corollary 4.10 (A q,ω-analogue of [15, (31) p. 314]). *If i is even, then*

$$\begin{aligned}\sum_{m=0}^{1} \lambda^{im} \mathcal{F}_{\mathrm{NWA},\lambda^2,n-1,q,\omega}\left(\overline{i}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{im}_{q,\omega}}{\overline{2}_{q,\omega}}\right) \\ = -\frac{2}{\{n\}_q (\overline{2}_{q,\omega})^{n-1}} \sum_{k=0}^{n} \binom{n}{k}_q \frac{(\overline{i}_{q,\omega})^k}{i} (\overline{2}_{q,\omega})^{n-k} \\ \times \mathcal{B}_{\mathrm{NWA},\lambda^i,k,q,\omega}\left(\overline{2}_{q,\omega} x\right) \sigma_{\mathrm{NWA},\lambda^2,n-k,q,\omega}(i) \\ = \frac{1}{(\overline{2}_{q,\omega})^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (\overline{2}_{q,\omega})^k (\overline{i}_{q,\omega})^{n-k-1} \\ \times \mathcal{F}_{\mathrm{NWA},\lambda^2,k,q,\omega}\left(\overline{i}_{q,\omega} x\right) s_{\mathrm{NWA},\lambda^i,n-k-1,q,\omega}(2) \\ = -\frac{2}{\{n\}_q (\overline{2}_{q,\omega})^{n-1}} \frac{(\overline{i}_{q,\omega})^n}{i} \sum_{m=0}^{i-1} (-1)^m \lambda^{2m} \\ \times \mathcal{B}_{\mathrm{NWA},\lambda^i,n,q,\omega}\left(\overline{2}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{2m}_{q,\omega}}{\overline{i}_{q,\omega}}\right).\end{aligned} \tag{89}$$

Proof. Put $j = 2$ in formula (82) and multiply by $-\frac{2}{\{n\}_q (\overline{2}_{q,\omega})^{n-1}}$. ☐

Corollary 4.11 (A q,ω-analogue of [15, (32) p. 314]).

$$\begin{aligned}\sum_{m=0}^{1} (-1)^{m+1} \lambda^m \mathcal{B}_{\mathrm{NWA},\lambda,n,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{2m}_{q,\omega}}{\overline{2}_{q,\omega}}\right) \\ = \frac{\{n\}_q (\overline{2}_{q,\omega})^{n-1}}{(\overline{2}_{q,\omega})^n} \sum_{m=0}^{1} \lambda^m \mathcal{F}_{\mathrm{NWA},\lambda,n-1,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{2m}_{q,\omega}}{\overline{2}_{q,\omega}}\right).\end{aligned} \tag{90}$$

Proof. Put $i = 2$ in formula (89), replace x and λ^2 by $\frac{x}{\overline{2}_{q,\omega}}$ and λ, and multiply by $\frac{\{n\}_q (\overline{2}_{q,\omega})^{n-1}}{(\overline{2}_{q,\omega})^n}$. ☐

Corollary 4.12 (A q,ω-analogue of [15, (33) p. 314]).

$$\begin{aligned}\sum_{m=0}^{1} (-1)^m \lambda^{jm} \mathcal{B}_{\mathrm{NWA},\lambda^2,n,q,\omega}\left(\overline{j}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\overline{2}_{q,\omega}}\right) \\ = -\frac{\{n\}_q}{(\overline{2}_{q,\omega})^n} \sum_{k=0}^{n-1} \binom{n-1}{k}_q \\ \times (\overline{j}_{q,\omega})^k (\overline{2}_{q,\omega})^{n-k-1} \mathcal{F}_{\mathrm{NWA},\lambda^j,k,q,\omega}\left(\overline{2}_{q,\omega} x\right) s_{\mathrm{NWA},\lambda^2,n-k-1,q,\omega}(j) \\ = -\frac{\{n\}_q}{(\overline{2}_{q,\omega})^n} (\overline{j}_{q,\omega})^{n-1} \sum_{m=0}^{j-1} \lambda^{2m} \mathcal{F}_{\mathrm{NWA},\lambda^j,n-1,q,\omega}\left(\overline{2}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{2m}_q}{\overline{j}_{q,\omega}}\right).\end{aligned} \tag{91}$$

Proof. Put $i = 2$ in formula (82) and multiply by $\frac{2}{(\overline{2}_{q,\omega})^n}$. ☐

Corollary 4.13 (A q,ω-analogue of [15, (26) p. 315]). *If i is odd, then*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}_{q}(\overline{i}_{q,\omega})^{k}(\overline{j}_{q,\omega})^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(\overline{j}_{q,\omega}x\right)\sigma_{\mathrm{NWA},\lambda^{j},n-k,q,\omega}(i)\\
&=\sum_{k=0}^{n}\binom{n}{k}_{q}(\overline{i}_{q,\omega})^{k}(\overline{j}_{q,\omega})^{n-k}\mathcal{B}_{\mathrm{NWA},\lambda^{i},k,q,\omega}\left(\overline{j}_{q,\omega}x\right)\sigma_{\mathrm{NWA},\lambda^{j},n-k,q,\omega}(i)\\
&=(\overline{i}_{q,\omega})^{n}\sum_{m=0}^{i-1}(-1)^{m}\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\\
&=(\overline{i}_{q,\omega})^{n-1}\sum_{m=0}^{i-1}\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^{i},n,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)(-1)^{m}.
\end{aligned}\tag{92}$$

Proof. We can rewrite $f_{q,\omega}(t)$ in the following way:

$$\begin{aligned}
f_{q,\omega}(t)&=\frac{(\overline{i}_{q,\omega}t)^{1}\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)(\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1)}{i(\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1)(\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)+1)}\\
&=\frac{t}{2}\left(\frac{2\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}xt)}{\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)+1}\right)\left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1}{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1}\right).
\end{aligned}\tag{93}$$

By using the formula for a geometric sequence, we can expand (93) in two ways:

$$\begin{aligned}
&f_{q,\omega}(t)\\
&=\left(\sum_{\nu=0}^{\infty}\mathcal{B}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega}\left(\overline{j}_{q,\omega}x\right)\frac{(\overline{i}_{q,\omega}t)^{\nu}}{\{\nu\}_{q}!}\right)\left(\sum_{m=0}^{\infty}\sigma_{\mathrm{NWA},\lambda^{j},m,q,\omega}(i)\frac{(\overline{j}_{q,\omega}t)^{m}}{\{m\}_{q}!}\right)\frac{1}{i}\\
&=\sum_{m=0}^{i-1}\lambda^{jm}\frac{(-1)^{m}(\overline{i}_{q,\omega}t)^{1}}{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1}\mathrm{E}_{q,\omega}\left(\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\overline{j}_{q,\omega}t\right)\frac{1}{i}\\
&=\sum_{\nu=0}^{\infty}\left((\overline{i}_{q,\omega})^{\nu}\sum_{m=0}^{i-1}\lambda^{jm}\mathcal{B}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\right)\frac{t^{\nu}}{\{\nu\}_{q}!}\frac{(-1)^{m}}{i}.
\end{aligned}\tag{94}$$

By equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_{q}!}$, we obtain rows 2 and 4 of formula (82). ☐

5. More expansion formulas

Theorem 5.1. *A triple sum of NWA q,ω-Apostol-Euler polynomials is equal to a double sum of NWA q,ω-Apostol-Euler polynomials.*

$$\begin{aligned}
&\sum_{|\nu|=n}\binom{n}{\vec{\nu}}_{q}(\overline{i}_{q,\omega})^{\nu_1}(\overline{j}_{q,\omega})^{\nu_2}\mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^{i},\nu_1,q,\omega}\left(\overline{j}_{q,\omega}x\right)\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^{j},\nu_2,q,\omega}\left(\overline{i}_{q,\omega}y\right)\\
&\times\sigma_{\mathrm{NWA},\lambda^{j},\nu_3,q,\omega}(i)(\overline{j}_{q,\omega})^{\nu_3}
\end{aligned}\tag{95}$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{n} \binom{n}{\nu}_{q} (\bar{i}_{q,\omega})^{\nu} (\bar{j}_{q,\omega})^{n-\nu} \mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^{j},n-\nu,q,\omega} \left(\bar{i}_{q,\omega} y\right) \\
&\quad \times \sum_{m=0}^{i-1} \lambda^{jm} (-1)^{m} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right).
\end{aligned}
$$

Proof. Define the following function, note that $f_{q,\omega}(t)$ is symmetric when i, j have the same parity.

$$
\begin{aligned}
f_{q,\omega}(t) &\equiv \frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x \oplus_{q,\omega} y)t)((-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1)}{(\lambda^{i}\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)+1)^{k}(\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1)^{k}} \\
&= 2^{1-2k}\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x \oplus_{q,\omega} y)t) \left(\frac{2}{\lambda^{i}\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)+1} \right)^{k} \\
&\quad \times \left(\frac{2}{\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1} \right)^{k-1} \left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1}{\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1} \right).
\end{aligned}
\tag{96}
$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$
\begin{aligned}
f_{q,\omega}(t) &\overset{(65),\,[8]}{=} 2^{1-2k} \left(\sum_{\nu=0}^{\infty} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega} \left(\bar{j}_{q,\omega} x\right) \frac{(\bar{i}_{q,\omega}t)^{\nu}}{\{\nu\}_{q}!} \right) \\
&\left(\sum_{m=0}^{\infty} \sigma_{\mathrm{NWA},\lambda^{j},m,q,\omega}(i) \frac{(\bar{j}_{q,\omega}t)^{m}}{\{m\}_{q}!} \right) \left(\sum_{l=0}^{\infty} \mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^{j},l,q,\omega} \left(\bar{i}_{q,\omega} y\right) \frac{(\bar{j}_{q,\omega}t)^{l}}{\{l\}_{q}!} \right) \\
&= 2^{1-2k} \frac{2^{k}}{\left(\lambda^{i}\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)+1\right)^{k}} \frac{2^{k-1}}{\left(\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1\right)^{k-1}} \sum_{m=0}^{i-1} (-1)^{m} \lambda^{jm} \\
&\quad \times \mathrm{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \bar{j}_{q,\omega} y \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega} t \right) \\
&= 2^{1-2k} \sum_{m=0}^{i-1} (-1)^{m} \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^{l} t^{l}}{\{l\}_{q}!} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^{i},l,q,\omega} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
&\sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^{n} t^{n}}{\{n\}_{q}!} \mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^{j},n,q,\omega} (\bar{i}_{q,\omega} y).
\end{aligned}
\tag{97}
$$

The theorem follows by equating the coefficients of $\frac{t^{n}}{\{n\}_{q}!}$. ☐

Theorem 5.2 (Almost a q,ω-analogue of [10, p. 3351])**.** *Assume that i and j are either both odd, or both even. Then we have*

$$
\sum_{\nu=0}^{n} \binom{n}{\nu}_{q} (\bar{j}_{q,\omega})^{\nu} (\bar{i}_{q,\omega})^{n-\nu} \mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^{i},n-\nu,q,\omega} \left(\bar{j}_{q,\omega} y\right)
\tag{98}
$$

$$\begin{aligned}&\sum_{m=0}^{j-1}\lambda^{im}(-1)^m\mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^j,\nu,q,\omega}\left(\overline{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\overline{j}_{q,\omega}}\right)\\&=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\overline{i}_{q,\omega})^{\nu}(\overline{j}_{q,\omega})^{n-\nu}\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q,\omega}\left(\overline{i}_{q,\omega}y\right)\\&\times\sum_{m=0}^{i-1}\lambda^{jm}(-1)^m\mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right).\end{aligned}$$

Proof. This follows from the previous proof, and then using the symmetry for i and j. ☐

Theorem 5.3. *A triple sum of NWA q,ω-Apostol-Bernoulli polynomials is equal to a double sum of NWA q,ω-Apostol-Bernoulli polynomials.*

$$\begin{aligned}&\sum_{|\nu|=n}\binom{n}{\nu}_q(\overline{i}_{q,\omega})^{\nu_1}(\overline{j}_{q,\omega})^{\nu_2}(\overline{j}_{q,\omega})^{\nu_3}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega}\left(\overline{j}_{q,\omega}x\right)\\&\times\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega}\left(\overline{i}_{q,\omega}y\right)s_{\mathrm{NWA},\lambda^j,\nu_3,q,\omega}(i)\\&=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\overline{i}_{q,\omega})^{\nu}(\overline{j}_{q,\omega})^{n-\nu}\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q,\omega}\left(\overline{i}_{q,\omega}y\right)\\&\sum_{m=0}^{i-1}\lambda^{jm}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\overline{i}_{q,\omega}}\right).\end{aligned}\tag{99}$$

Proof. Define the following symmetric function

$$\begin{aligned}\phi_{q,\omega}(t)&\equiv\frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega}y)t)(\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1)}{(\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1)^k(\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1)^k}t^k\\&=\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega}y)t)\left(\frac{\overline{i}_{q,\omega}t}{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1}\right)^k\left(\frac{\overline{j}_{q,\omega}t}{\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1}\right)^{k-1}\\&\times\left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda^{j}\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1}\right)\frac{t^{1-2k}}{(\overline{i}_{q,\omega})^k(\overline{j}_{q,\omega})^{k-1}}.\end{aligned}\tag{100}$$

By using the formula for a geometric sequence, we can expand $\phi_{q,\omega}(t)$ in two ways:

$$\begin{aligned}\phi_{q,\omega}(t)&\overset{\text{by}(64)}{=}\left(\sum_{\nu=0}^{\infty}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\overline{j}_{q,\omega}x\right)\frac{(\overline{i}_{q,\omega}t)^{\nu}}{\{\nu\}_q!}\right)\\&\times\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^j,m,q,\omega}(i)\frac{(\overline{j}_{q,\omega}t)^{m}}{\{m\}_q!}\right)\end{aligned}\tag{101}$$

$$\begin{aligned}
&\times \left(\sum_{l=0}^{\infty} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q,\omega}\left(\bar{i}_{q,\omega}y\right) \frac{(\bar{j}_{q,\omega}t)^l}{\{l\}_q!} \right) \frac{t^{1-2k}}{(\bar{i}_{q,\omega})^k(\bar{j}_{q,\omega})^{k-1}} \\
&= \frac{(\bar{i}_{q,\omega}t)^k}{\left(\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1\right)^k} \frac{(\bar{j}_{q,\omega}t)^{k-1}}{\left(\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1\right)^{k-1}} \\
&\times \sum_{m=0}^{i-1} \lambda^{jm} \mathrm{E}_{q,\omega t}\left(\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \bar{j}_{q,\omega}y \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega}t \right) \\
&\times \frac{t^{1-2k}}{(\bar{i}_{q,\omega})^k(\bar{j}_{q,\omega})^{k-1}} \\
&= \frac{t^{1-2k}}{(\bar{i}_{q,\omega})^k(\bar{j}_{q,\omega})^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^l t^l}{\{l\}_q!} \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,l,q,\omega} \\
&\times \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^n t^n}{\{n\}_q!} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n,q,\omega}(\bar{i}_{q,\omega}y).
\end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. ☐

Theorem 5.4 (A q,ω-analogue of [17, p. 2994], [16, p. 551])**.**

$$\begin{aligned}
&\sum_{|\nu|=n} \binom{n}{\vec{\nu}}_q (\bar{i}_{q,\omega})^{\nu_1}(\bar{j}_{q,\omega})^{\nu_2}(\bar{j}_{q,\omega})^{\nu_3} \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega}\left(\bar{j}_{q,\omega}x\right) \\
&\times \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega}\left(\bar{i}_{q,\omega}y\right) s_{\mathrm{NWA},\lambda^j,\nu_3,q,\omega}(i) \\
= &\sum_{|\nu|=n} \binom{n}{\vec{\nu}}_q (\bar{j}_{q,\omega})^{\nu_1}(\bar{i}_{q,\omega})^{\nu_2}(\bar{i}_{q,\omega})^{\nu_3} \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^j,\nu_1,q,\omega}\left(\bar{i}_{q,\omega}x\right) \\
&\times \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,\nu_2,q,\omega}\left(\bar{j}_{q,\omega}y\right) s_{\mathrm{NWA},\lambda^i,\nu_3,q,\omega}(j).
\end{aligned} \tag{102}$$

Proof. Use the symmetry in $\phi_{q,\omega}(t)$. ☐

Theorem 5.5 (A q,ω-analogue of [17, p. 2996])**.** *We have*

$$\begin{aligned}
&\sum_{\nu=0}^{n} \binom{n}{\nu}_q \sum_{l=0}^{i-1}\sum_{m=0}^{j-1} \lambda^{l+m} (\bar{i}_{q,\omega})^{\nu}(\bar{j}_{q,\omega})^{n-\nu} \\
&\times \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,\nu,q,\omega}\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\overline{jl}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,n-\nu,q,\omega}\left(\bar{i}_{q,\omega}y \oplus_{q,\omega} \frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}} \right) \\
= &\sum_{\nu=0}^{n} \binom{n}{\nu}_q \sum_{l=0}^{j-1}\sum_{m=0}^{i-1} \lambda^{l+m} (\bar{j}_{q,\omega})^{\nu}(\bar{i}_{q,\omega})^{n-\nu} \\
&\times \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,\nu,q,\omega}\left(\bar{i}_{q,\omega}x \oplus_{q,\omega} \frac{\overline{il}_{q,\omega}}{\bar{j}_{q,\omega}} \right) \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,n-\nu,q,\omega}\left(\bar{j}_{q,\omega}y \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right).
\end{aligned} \tag{103}$$

Proof. We can expand the following symmetric function $\phi'_{q,\omega}(t)$ by using the formula for a geometric sequence:

(104)

$$\begin{aligned}
&\phi'_{q,\omega}(t)\\
&\equiv \frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega} y)t)(\lambda^{i}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1)(\lambda^{j}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1)}{(\lambda\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1)^{k}(\lambda\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1)^{k}}t^{2k-2}\\
&\quad\times \mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega} y)t)\frac{1}{(\bar{i}_{q,\omega})^{k-1}(\bar{j}_{q,\omega})^{k-1}}\\
&\quad\times\left(\frac{\bar{i}_{q,\omega}t}{\lambda\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)^{k-1}\left(\frac{\bar{j}_{q,\omega}t}{\lambda\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1}\right)^{k-1}\\
&\quad\times\left(\frac{\lambda^{i}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1}\right)\left(\frac{\lambda^{j}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)\\
&=\frac{1}{(\bar{i}_{q,\omega})^{k-1}(\bar{j}_{q,\omega})^{k-1}}\sum_{l=0}^{i-1}\sum_{m=0}^{j-1}\lambda^{l+m}\left(\frac{\bar{i}_{q,\omega}t}{\lambda\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)^{k-1}\left(\frac{\bar{j}_{q,\omega}t}{\lambda\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1}\right)^{k-1}\\
&\quad\times \mathrm{E}_{q,\omega t}\left(\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jl}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\bar{i}_{q,\omega}t\right)\mathrm{E}_{q,\omega t}\left(\left(\bar{i}_{q,\omega}y\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}}\right)\bar{j}_{q,\omega}t\right)\\
&=\frac{1}{(\bar{i}_{q,\omega})^{k-1}(\bar{j}_{q,\omega})^{k-1}}\left(\sum_{l=0}^{i-1}\lambda^{l}\sum_{\nu_1=0}^{\infty}\frac{(\bar{i}_{q,\omega})^{\nu_1}t^{\nu_1}}{\{\nu_1\}_q!}\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda,\nu_1,q,\omega}\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jl}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\right)\\
&\quad\times\left(\sum_{m=0}^{j-1}\lambda^{m}\sum_{\nu_2=0}^{\infty}\frac{(\bar{j}_{q,\omega})^{\nu_2}t^{\nu_2}}{\{\nu_2\}_q!}\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda,\nu_2,q,\omega}\left(\bar{i}_{q,\omega}y\oplus_{q,\omega}\frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}}\right)\right).
\end{aligned}$$

The theorem follows by using the symmetry in $\phi'_{q,\omega}(t)$ and changing $k-1$ to k. ☐

Theorem 5.6 (A q,ω-analogue of [17, p. 2997])**.** *We have*

$$\begin{aligned}
&\sum_{\nu=0}^{n}\binom{n}{\nu}_{q}\sum_{l=0}^{i-1}(\bar{i}_{q,\omega}\bar{j}_{q,\omega})^{n-\nu}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,n-\nu,q,\omega}\left(\bar{i}_{q,\omega}y\right)\\
&\quad\times\sum_{m=0}^{j-1}\lambda^{l+m}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,\nu,q,\omega}\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jl}_{q,\omega}}{\bar{i}_{q,\omega}}\oplus_{q,\omega}\overline{m}_{q,\omega}\right)\\
&=\sum_{\nu=0}^{n}\binom{n}{\nu}_{q}\sum_{l=0}^{j-1}(\bar{j}_{q,\omega})^{\nu}(\bar{i}_{q,\omega})^{n-\nu}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,n-\nu,q,\omega}\left(\bar{j}_{q,\omega}y\right)\\
&\quad\times\sum_{m=0}^{i-1}\lambda^{l+m}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda,\nu,q,\omega}\left(\bar{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{il}_{q,\omega}}{\bar{j}_{q,\omega}}\oplus_{q,\omega}\overline{m}_{q,\omega}\right).
\end{aligned}\tag{105}$$

Proof. Similar to above. ☐

Theorem 5.7 (A q,ω-analogue of [16, p. 552])**.** *We have*

$$\begin{aligned}
&\frac{1}{(\overline{i}_{q,\omega})^k(\overline{j}_{q,\omega})^{k-1}}\sum_{m=0}^{n}\binom{n}{m}_q(\overline{i}_{q,\omega})^m(\overline{j}_{q,\omega})^{n-m}\\
&\times\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-m,q,\omega}\left(\overline{i}_{q,\omega}y\right)\sum_{l=0}^{i-1}\lambda^{jl}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,m,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jl}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\\
=&\frac{1}{(\overline{j}_{q,\omega})^k(\overline{i}_{q,\omega})^{k-1}}\sum_{m=0}^{n}\binom{n}{m}_q(\overline{j}_{q,\omega})^m(\overline{i}_{q,\omega})^{n-m}\\
&\times\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,n-m,q,\omega}\left(\overline{j}_{q,\omega}y\right)\sum_{l=0}^{j-1}\lambda^{il}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^j,m,q,\omega}\left(\overline{i}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{il}_{q,\omega}}{\overline{j}_{q,\omega}}\right).
\end{aligned}\tag{106}$$

Proof. We can expand the following symmetric function $\psi_{q,\omega}(t)$ by using the formula for a geometric sequence:

$$\begin{aligned}
\psi_{q,\omega}(t)&\equiv\frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega}y)t)(\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1)}{(\lambda^i\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1)^k(\lambda^j\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1)^k}t^{2k-1}\\
&=\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega}y)t)\frac{1}{(\overline{i}_{q,\omega})^k(\overline{j}_{q,\omega})^{k-1}}\left(\frac{\overline{i}_{q,\omega}t}{\lambda^i\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1}\right)^k\\
&\quad\times\left(\frac{\overline{j}_{q,\omega}t}{\lambda^j\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1}\right)^{k-1}\left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda^j\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1}\right)\\
&=\frac{1}{(\overline{i}_{q,\omega})^k(\overline{j}_{q,\omega})^{k-1}}\left(\frac{\overline{i}_{q,\omega}t}{\lambda^i\mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)-1}\right)^k\left(\frac{\overline{j}_{q,\omega}t}{\lambda^j\mathrm{E}_{q,\omega}(\overline{j}_{q,\omega}t)-1}\right)^{k-1}\\
&\quad\times\sum_{l=0}^{i-1}\lambda^{lj}\mathrm{E}_{q,\omega t}\left(\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jl}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\overline{i}_{q,\omega}t\right)\mathrm{E}_{q,\omega t}\left(\left(\overline{i}_{q,\omega}y\right)\overline{j}_{q,\omega}t\right)\\
&=\frac{1}{(\overline{i}_{q,\omega})^k}\frac{1}{(\overline{j}_{q,\omega})^{k-1}}\\
&\quad\times\left(\sum_{l=0}^{i-1}\lambda^{jl}\sum_{\nu_1=0}^{\infty}\frac{(\overline{i}_{q,\omega})^{\nu_1}t^{\nu_1}}{\{\nu_1\}_q!}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jl}_{q,\omega}}{\overline{i}_{q,\omega}}\right)\right)\\
&\quad\times\sum_{\nu_2=0}^{\infty}\frac{(\overline{j}_{q,\omega})^{\nu_2}t^{\nu_2}}{\{\nu_2\}_q!}\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega}\left(\overline{i}_{q,\omega}y\right)\\
&=\frac{1}{(\overline{i}_{q,\omega})^k(\overline{j}_{q,\omega})^{k-1}}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}_q\right.\\
&\quad\times\sum_{l=0}^{i-1}\lambda^{jl}(\overline{i}_{q,\omega})^m(\overline{j}_{q,\omega})^{n-m}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,m,q,\omega}\left(\overline{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jl}_{q,\omega}}{\overline{i}_{q,\omega}}\right)
\end{aligned}\tag{107}$$

$$\times\, \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-m,q,\omega}\left(\bar{i}_{q,\omega}y\right)\Big)\frac{t^n}{\{n\}_q!}.$$

The theorem follows by using the symmetry in $\psi_{q,\omega}(t)$. □

6. Mixed formulas

Theorem 6.1 (A q,ω-analogue of [10, (3.9) p. 3356])**.** *A triple sum of mixed q,ω-Apostol polynomials is equal to a double sum of mixed q,ω-Apostol polynomials.*

$$\begin{aligned}
&\sum_{|\nu|=n}\binom{n}{\vec{\nu}}_q(\bar{i}_{q,\omega})^{\nu_1}(\bar{j}_{q,\omega})^{\nu_2}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega}\left(\bar{j}_{q,\omega}x\right)\\
&\times\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega}\left(\bar{i}_{q,\omega}y\right)\sigma_{\mathrm{NWA},\lambda^j,\nu_3,q,\omega}(i)(\bar{j}_{q,\omega})^{\nu_3}\\
&=\sum_{\nu=0}^{n}\binom{n}{\nu}_q(\bar{i}_{q,\omega})^{\nu}(\bar{j}_{q,\omega})^{n-\nu}\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q,\omega}\left(\bar{i}_{q,\omega}y\right)\sum_{m=0}^{i-1}\lambda^{jm}(-1)^m\\
&\times\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right).
\end{aligned}\tag{108}$$

Proof. Define the following function

$$\begin{aligned}
g_{q,\omega}(t)&\equiv\frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega}y)t)((-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1)}{(\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1)^k(\lambda^j\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1)^k}\\
&=\frac{2^{1-k}}{(\bar{i}_{q,\omega}t)^k}\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega}y)t)\left(\frac{\bar{i}_{q,\omega}t}{\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)^k\\
&\times\left(\frac{2}{\lambda^j\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1}\right)^{k-1}\left(\frac{(-1)^{i+1}\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)+1}{\lambda^j\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1}\right).
\end{aligned}\tag{109}$$

By using the formula for a geometric sequence, we can expand $g_{q,\omega}(t)$ in two ways:

$$\begin{aligned}
g_{q,\omega}(t)&\overset{\text{by}(65)}{=}\frac{2^{1-k}}{(\bar{i}_{q,\omega}t)^k}\left(\sum_{\nu=0}^{\infty}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\bar{j}_{q,\omega}x\right)\frac{(\bar{i}_{q,\omega}t)^{\nu}}{\{\nu\}_q!}\right)\\
&\times\left(\sum_{m=0}^{\infty}\sigma_{\mathrm{NWA},\lambda^j,m,q,\omega}(i)\frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!}\right)\\
&\times\left(\sum_{l=0}^{\infty}\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q,\omega}\left(\bar{i}_{q,\omega}y\right)\frac{(\bar{j}_{q,\omega}t)^l}{\{l\}_q!}\right)\\
&=\frac{2^{1-k}}{(\bar{i}_{q,\omega}t)^k}\left(\frac{\bar{i}_{q,\omega}t}{\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)^k\frac{2^{k-1}}{\left(\lambda^j\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1\right)^{k-1}}
\end{aligned}\tag{110}$$

$$
\begin{aligned}
&\times \sum_{m=0}^{i-1}(-1)^m \lambda^{jm}\mathrm{E}_{q,\omega t}\left(\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \bar{j}_{q,\omega}y \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\bar{i}_{q,\omega}t\right)\\
=\ &\frac{2^{1-k}}{(\bar{i}_{q,\omega}t)^k}\sum_{m=0}^{i-1}(-1)^m\\
&\times \lambda^{jm}\sum_{l=0}^{\infty}\frac{(\bar{i}_{q,\omega})^l t^l}{\{l\}_q!}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,l,q,\omega}\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\\
&\times \sum_{n=0}^{\infty}\frac{(\bar{j}_{q,\omega})^n t^n}{\{n\}_q!}\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n,q,\omega}(\bar{i}_{q,\omega}y).
\end{aligned}
$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. ☐

Theorem 6.2 (A q,ω-analogue of [10, p. 3353])**.** *Under the assumption that i is even, a triple sum of mixed q,ω-Apostol polynomials is equal to another triple sum of mixed q,ω-Apostol polynomials.*

(111)

$$
\begin{aligned}
&\sum_{|\nu|=n}\binom{n}{\vec{\nu}}_q(\bar{i}_{q,\omega})^{\nu_1}(\bar{j}_{q,\omega})^{\nu_2}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega}\left(\bar{j}_{q,\omega}x\right)\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega}\left(\bar{i}_{q,\omega}y\right)\\
&\times s_{\mathrm{NWA},\lambda^j,\nu_3,q,\omega}(i)(\bar{j}_{q,\omega})^{\nu_3}\\
=\ &-\frac{\{n\}_q(\bar{i}_{q,\omega})^k}{2(\bar{i}_{q,\omega})^{k-1}}\sum_{|\nu|=n-1}\binom{n-1}{\vec{\nu}}_q(\bar{i}_{q,\omega})^{\nu_1}(\bar{j}_{q,\omega})^{\nu_2}\\
&\times(\bar{j}_{q,\omega})^{\nu_3}\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega}\left(\bar{j}_{q,\omega}y\right)\mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega}\left(\bar{i}_{q,\omega}x\right)s_{\mathrm{NWA},\lambda^i,\nu_3,q,\omega}(j).
\end{aligned}
$$

Proof. We can write $g_{q,\omega}(t)$ as follows:

$$
\begin{aligned}
g_{q,\omega}(t)\overset{\text{by}(64),(109)}{=}\ &\frac{2^{1-k}}{(\bar{i}_{q,\omega}t)^k}\left(\sum_{\nu=0}^{\infty}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega}\left(\bar{j}_{q,\omega}x\right)\frac{(\bar{i}_{q,\omega}t)^\nu}{\{\nu\}_q!}\right)\\
&\times\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^j,m,q,\omega}(i)\frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!}\right)\\
&\times\left(\sum_{l=0}^{\infty}\mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q,\omega}\left(\bar{i}_{q,\omega}y\right)\frac{(\bar{j}_{q,\omega}t)^l}{\{l\}_q!}\right)\qquad(112)\\
=\ &-\frac{2^{-k}}{(\bar{i}_{q,\omega}t)^{k-1}}\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x\oplus_{q,\omega}y)t)\left(\frac{\bar{i}_{q,\omega}t}{\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)^{k-1}\\
&\times\left(\frac{2}{\lambda^j\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)+1}\right)^k\left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda^i\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)-1}\right)\\
\overset{\text{by}(64)}{=}\ &-\frac{2^{-k}}{(\bar{i}_{q,\omega}t)^{k-1}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\sum_{\nu=0}^{\infty} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^j,\nu,q,\omega} \left(\bar{i}_{q,\omega} x \right) \frac{(\bar{j}_{q,\omega} t)^{\nu}}{\{\nu\}_q!} \right) \\
&\times \left(\sum_{m=0}^{\infty} s_{\mathrm{NWA},\lambda^i,m,q,\omega}(j) \frac{(\bar{j}_{q,\omega} t)^{m}}{\{m\}_q!} \right) \\
&\times \left(\sum_{l=0}^{\infty} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,l,q,\omega} \left(\bar{j}_{q,\omega} y \right) \frac{(\bar{i}_{q,\omega} t)^{l}}{\{l\}_q!} \right).
\end{aligned}
$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

Theorem 6.3 (A q,ω-analogue of [10, p. 3353])**.** *Under the assumption that i is even, a double sum of mixed q,ω-Apostol polynomials is equal to another double sum of mixed q,ω-Apostol polynomials.*

(113)

$$
\begin{aligned}
&\sum_{\nu=0}^{n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu} (\bar{j}_{q,\omega})^{n-\nu} \mathcal{F}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q,\omega} \left(\bar{i}_{q,\omega} y \right) \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \\
&\times \mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
= {}& - \frac{\{n\}_q (\bar{i}_{q,\omega})^k}{2(\bar{i}_{q,\omega})^{k-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (\bar{i}_{q,\omega})^{n-k-1} (\bar{j}_{q,\omega})^k \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,n-k-1,q,\omega} \left(\bar{j}_{q,\omega} y \right) \\
&\times \sum_{m=0}^{j-1} \lambda^{im} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^j,k,q,\omega} \left(\bar{i}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}}) \right).
\end{aligned}
$$

Proof. We can expand $g_{q,\omega}(t)$ as follows:

$$
\begin{aligned}
g_{q,\omega}(t) \overset{\mathrm{by}(109)}{=} {}& - \frac{2^{-k}}{(\bar{i}_{q,\omega} t)^{k-1}} \mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x \oplus_{q,\omega} y)t) \\
&\times \left(\frac{\bar{i}_{q,\omega} t}{\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega} t) - 1} \right)^{k-1} \left(\frac{2}{\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega} t) + 1} \right)^{k} \qquad (114) \\
&\times \left(\frac{\lambda^{ij} \mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega} t) - 1}{\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega} t) - 1} \right) \\
= {}& - \frac{2^{-k}}{(\bar{i}_{q,\omega} t)^{k-1}} \left(\frac{\bar{i}_{q,\omega} t}{\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega} t) - 1} \right)^{k-1} \frac{2^k}{\left(\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega} t) + 1 \right)^k} \\
&\times \sum_{m=0}^{j-1} \lambda^{im} \times \mathrm{E}_{q,\omega t} \left(\left(\bar{i}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}} \right) \bar{j}_{q,\omega} t \right) E_{q,\omega t}(\overline{ij}_{q,\omega} y t) \\
= {}& - \frac{2^{-k}}{(\bar{i}_{q,\omega} t)^{k-1}} \sum_{m=0}^{j-1} \lambda^{im} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^l t^l}{\{l\}_q!} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^i,l,q,\omega} \left(\bar{j}_{q,\omega} y \right)
\end{aligned}
$$

$$\times \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^n t^n}{\{n\}_q!} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^j,n,q,\omega}(\bar{i}_{q,\omega}x \oplus_{q,\omega} \frac{\overline{im}_{q,\omega}}{\bar{j}_{q,\omega}}).$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. ☐

Theorem 6.4. *Assume that $\vec{\nu}$ on the left hand side is a vector with three components and with length n.*

$$\begin{aligned}
&\sum_{|\nu|=n} \binom{n}{\vec{\nu}}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} (\bar{j}_{q,\omega})^{\nu_3} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega} \left(\bar{j}_{q,\omega}x\right) \\
&\times \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega} \left(\bar{i}_{q,\omega}y\right) s_{\mathrm{NWA},\lambda^j,\nu_3,q,\omega}(i) \\
=& \sum_{\nu=0}^{n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu} (\bar{j}_{q,\omega})^{n-\nu} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q,\omega} \left(\bar{i}_{q,\omega}y\right) \\
&\times \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right).
\end{aligned} \tag{115}$$

Proof. Define the following function

$$\begin{aligned}
\Psi_{q,\omega}(t) \equiv& \frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x \oplus_{q,\omega} y)t)(\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t) - 1)}{(\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1)^k (\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1)^k} t^{k-1} \\
=& \mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x \oplus_{q,\omega} y)t) \left(\frac{2}{\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)+1}\right)^k \left(\frac{\bar{j}_{q,\omega}t}{\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1}\right)^{k-1} \\
&\times \left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t) - 1}{\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1}\right) \frac{2^{-k}}{(\bar{j}_{q,\omega})^{k-1}}.
\end{aligned} \tag{116}$$

By using the formula for a geometric sequence, we can expand $\Psi_{q,\omega}(t)$ in two ways:

$$\begin{aligned}
\Psi_{q,\omega}(t) \overset{\text{by}(64)}{=}& \left(\sum_{\nu=0}^{\infty} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega}x\right) \frac{(\bar{i}_{q,\omega}t)^{\nu}}{\{\nu\}_q!}\right) \\
&\times \left(\sum_{m=0}^{\infty} s_{\mathrm{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!}\right) \\
&\times \left(\sum_{l=0}^{\infty} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,l,q,\omega} \left(\bar{i}_{q,\omega}y\right) \frac{(\bar{j}_{q,\omega}t)^l}{\{l\}_q!}\right) \frac{2^{-k}}{(\bar{j}_{q,\omega})^{k-1}} \\
=& \frac{2^k}{\left(\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1\right)^k} \frac{(\bar{j}_{q,\omega}t)^{k-1}}{\left(\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1\right)^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \\
&\times \mathrm{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \bar{j}_{q,\omega}y \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right) \bar{i}_{q,\omega}t\right) \frac{2^{-k}}{(\bar{j}_{q,\omega})^{k-1}}
\end{aligned} \tag{117}$$

$$
\begin{aligned}
&= \sum_{m=0}^{i-1} \frac{2^{-k}\lambda^{jm}}{(\bar{j}_{q,\omega})^{k-1}} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^l t^l}{\{l\}_q!} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^i,l,q,\omega} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
&\quad \times \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^n t^n}{\{n\}_q!} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n,q,\omega}(\bar{i}_{q,\omega} y).
\end{aligned}
$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. ☐

Similar formulas with $\mathcal{H}$ polynomials can easily be constructed.

Theorem 6.5. *Assume that $\vec{\nu}$ on the left hand side is a vector with three components and length n.*

$$
\begin{aligned}
&\sum_{|\nu|=n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} (\bar{j}_{q,\omega})^{\nu_3} \mathcal{H}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega} \left(\bar{j}_{q,\omega} x\right) \\
&\quad \times \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega} \left(\bar{i}_{q,\omega} y\right) s_{\mathrm{NWA},\lambda^j,\nu_3,q,\omega}(i) \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu} (\bar{j}_{q,\omega})^{n-\nu} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q,\omega} \left(\bar{i}_{q,\omega} y\right) \\
&\quad \times \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{H}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right).
\end{aligned} \tag{118}
$$

Proof. Use $\Psi_{q,\omega}(t)$ again. ☐

Theorem 6.6 (A q,ω-analogue of [10, (3.11) p. 3356])**.** *A triple sum of mixed q,ω-Apostol polynomials is equal to a double sum of mixed q,ω-Apostol polynomials.*

$$
\begin{aligned}
&\sum_{|\nu|=n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} (\bar{j}_{q,\omega})^{\nu_3} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu_1,q,\omega} \left(\bar{j}_{q,\omega} x\right) \\
&\quad \times \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,\nu_2,q,\omega} \left(\bar{i}_{q,\omega} y\right) n_{\mathrm{NWA},\lambda^j,\nu_3,q,\omega}(i) \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu} (\bar{j}_{q,\omega})^{n-\nu} \mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^j,n-\nu,q,\omega} \left(\bar{i}_{q,\omega} y\right) \\
&\quad \times \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}} \right).
\end{aligned} \tag{119}
$$

Proof. Define the following function

$$
f_{q,\omega}(t) \equiv \frac{\mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x \oplus_{q,\omega} y)t)(\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t) - 1)}{(\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1)^k (\lambda^j \mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1)^k} t^k
$$

$$
= \mathrm{E}_{q,\omega t}(\overline{ij}_{q,\omega}(x \oplus_{q,\omega} y)t) \left(\frac{2}{\lambda^i \mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1} \right)^k \tag{120}
$$

$$\times\left(\frac{\bar{j}_{q,\omega}t}{\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1}\right)^{k-1}\left(\frac{\lambda^{ij}\mathrm{E}_{q,\omega}(\overline{ij}_{q,\omega}t)-1}{\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1}\right)\frac{1}{2^{k}(\bar{j}_{q,\omega})^{k-1}}.$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$\begin{aligned}f_{q,\omega}(t)\overset{\text{by}(64)}{=}&\left(\sum_{\nu=0}^{\infty}\mathcal{B}^{(k)}_{\mathrm{NWA},\lambda^{i},\nu,q,\omega}\left(\bar{j}_{q,\omega}x\right)\frac{(\bar{i}_{q,\omega}t)^{\nu}}{\{\nu\}_{q}!}\right)\\ &\times\left(\sum_{m=0}^{\infty}s_{\mathrm{NWA},\lambda^{j},m,q,\omega}(i)\frac{(\bar{j}_{q,\omega}t)^{m}}{\{m\}_{q}!}\right)\\ &\times\left(\sum_{l=0}^{\infty}\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^{j},l,q,\omega}\left(\bar{i}_{q,\omega}y\right)\frac{(\bar{j}_{q,\omega}t)^{l}}{\{l\}_{q}!}\right)\frac{1}{2^{k}(\bar{j}_{q,\omega})^{k-1}}\\ =&\frac{2^{k}}{\left(\lambda^{i}\mathrm{E}_{q,\omega}(\bar{i}_{q,\omega}t)+1\right)^{k}}\frac{(\bar{j}_{q,\omega}t)^{k-1}}{\left(\lambda^{j}\mathrm{E}_{q,\omega}(\bar{j}_{q,\omega}t)-1\right)^{k-1}}\sum_{m=0}^{i-1}\lambda^{jm}\\ &\times\mathrm{E}_{q,\omega t}\left(\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\bar{j}_{q,\omega}y\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\bar{i}_{q,\omega}t\right)\frac{1}{2^{k}(\bar{j}_{q,\omega})^{k-1}}\\ =&\sum_{m=0}^{i-1}\frac{\lambda^{jm}}{2^{k}(\bar{j}_{q,\omega})^{k-1}}\sum_{l=0}^{\infty}\frac{(\bar{i}_{q,\omega})^{l}t^{l}}{\{l\}_{q}!}\mathcal{F}^{(k)}_{\mathrm{NWA},\lambda^{i},l,q,\omega}\left(\bar{j}_{q,\omega}x\oplus_{q,\omega}\frac{\overline{jm}_{q,\omega}}{\bar{i}_{q,\omega}}\right)\\ &\times\sum_{n=0}^{\infty}\frac{(\bar{j}_{q,\omega})^{n}t^{n}}{\{n\}_{q}!}\mathcal{B}^{(k-1)}_{\mathrm{NWA},\lambda^{j},n,q,\omega}(\bar{i}_{q,\omega}y).\end{aligned}\tag{121}$$

The theorem follows by equating the coefficients of $\frac{t^{n}}{\{n\}_{q}!}$. ☐

7. Discussion

Many of the proofs use the formula for a geometric sequence in q,ω-form and the generating function for the q,ω-Appell polynomials and the power sums. The integers i and j are crucial for the formulas; by the generating function, if λ^{i},ν appears as index in a polynomial, certainly the factor $(\bar{i}_{q,\omega})^{\nu}$ will also appear. If the orders of two polynomials in a formula are k and $k-1$, the last one with index λ^{j}, and argument $\bar{i}_{q,\omega}y$, a function $\sigma_{\mathrm{NWA},\lambda^{j},m,q,\omega}(i)$ or $s_{\mathrm{NWA},\lambda^{j},m,q,\omega}(i)$, together with $(\bar{j}_{q,\omega})^{m}$ will appear. If a polynomial has λ^{i},ν as index, it will have $(\bar{j}_{q,\omega})$ in the function argument, and vice versa. Most of the q-transformations can be generalized to the q,ω case, a general exception is when the q,ω-addition is expanded and coefficients of the dummy variable t are equated.

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