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AN ARTINIAN RING HAVING THE STRONG LEFSCHETZ PROPERTY AND REPRESENTATION THEORY

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Abstract. It is well-known that if $\operatorname{char}\Bbbk = 0$, then an Artinian monomial complete intersection quotient $\Bbbk[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n})$ has the strong Lefschetz property in the narrow sense, and it is decomposed by the direct sum of irreducible $\mathfrak{sl}_2$-modules. For an Artinian ring $A = \Bbbk[x_1,x_2,x_3]/(x_1^6,x_2^6,x_3^6)$, by the Schur-Weyl duality theorem, there exist 56 trivial representations, 70 standard representations, and 20 sign representations inside A. In this paper we find an explicit basis for A, which is compatible with the S_3-module structure.

1. Introduction

Let $R = \Bbbk[x_1,\dots,x_n] = \bigoplus_{i\geq 0} R_i$ be an n-variable polynomial ring over a field of characteristic 0, and let I be a homogeneous ideal of R, and $A := R/I$. The *Hilbert function of* A is a function, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, defined by

$$\mathbf{H}_A(t) := \dim_{\Bbbk} R_t - \dim_k I_t.$$

If I is a homogeneous ideal for which $\sqrt{I} = (x_1,\dots,x_n)$ and $m+1$ is the least positive integer such that $(x_1,\dots,x_n)^{m+1} \subseteq I$, then

$$A = \Bbbk \oplus A_1 \oplus \cdots \oplus A_m \quad \text{where} \quad A_m \neq 0.$$

In this case, we call *m the socle degree of A*.

We say that an Artinian $\Bbbk$-algebra $A = \oplus_{i\geq 0} A_i$ has the *weak Lefschetz property* (WLP) if there is a linear form $\ell \in A_1$ such that the linear map $\times\ell : A_i \to A_{i+1}$ has maximal rank for all $i \geq 0$. In addition, we say that A has the *strong Lefschetz property* (SLP) if the map $\times\ell^d : A_i \to A_{i+d}$ has maximal rank for every $i \geq 0$ and $d \geq 1$ ([4,5,7–11]). In these cases, ℓ is called a *weak or strong Lefschetz element* of A. If the Hilbert function of an Artinian algebra A having the SLP is symmetric and unimodal, then we say that A has the *SLP in the narrow sense* (see [4]).

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The WLP and SLP are strongly connected to many topics in algebraic geometry, commutative algebra, combinatorics, and representation theory. The manuscript [4] gives an overview of the Lefschetz properties from a different prospective focusing on representation theory and combinatorial connections and provides a wonderfully comprehensive exploration of the Lefschetz properties. R. Stanley [10] and J. Watanabe [11] proved that an Artinian monomial complete intersection quotient $A := \mathbb{k}[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$ has the SLP in the narrow sense.

Moreover, A has the SLP in the narrow sense if and only if A can be decomposed by

$$A \cong \bigoplus_{i=0}^{\lfloor \frac{m}{2} \rfloor} V(m-2i)^{\oplus a_i}, \tag{1.1}$$

where $a_0 = 1$, $a_i = \dim_{\mathbb{k}} A_i - \dim_{\mathbb{k}} A_{i-1}$ for $1 \leq i \leq m$, and $V(m-2i)$ is an $(m-2i+1)$-dimensional irreducible $\mathfrak{sl}_2$-module for such i (see [4, 8, 11] for the details of $\mathfrak{sl}_2$-representation theory).

Let S_n be the symmetric group on n-letters. For $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in \mathbb{k}[x_1, \ldots, x_n]$, S_n acts on $\mathbb{k}[x_1, \ldots, x_n]$ by

$$\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

Note that an ideal $I = (x_1^d, \ldots, x_n^d)$ is invariant under the action of the group S_n, and so an algebra R/I is isomorphic to the tensor product $V^{\otimes n}$ of an n-dimensional vector space $V = \mathbb{k}[x]/(x^d)$, where the tensor product $V^{\otimes n}$ is the space *Schur-Weyl* duality (see [3, 4, 11]). The general linear group $GL_d(\mathbb{k}) := GL_d$ acts on the space $V^{\otimes n}$, i.e.,

$$g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n,$$

for $g \in GL_d$ and $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. It is clear that two actions *commute* with each other, i.e., $g \circ \sigma = \sigma \circ g$. Hence the space $V^{\otimes n}$ is given a structure of a bimodule for the product group $S_n \times GL_d$. By the Schur-Weyl duality theorem, the tensor product $V^{\otimes n}$ is isomorphic to

$$A \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} S^\lambda \otimes V(\lambda)$$

as an $S_n \times GL_d$-module, where λ is a partition of n with length $\ell(\lambda) \leq d$ and $V(\lambda)$ is an irreducible $\mathfrak{sl}_2$-module associated with a partition λ of n (see [4,6,8] for details).

In this article, we find an explicit basis for A, which is compatible with the S_n-module structure for $n = 3$ and $a_1 = a_2 = a_3 = 6$. Moreover, if we find a highest weight vector (representation) in each irreducible $\mathfrak{sl}_2$-module component of A (see [4,8]), then we can find the rest of representations (vectors) in the basis for A applying $\times \ell := x_1 + x_2 + x_3$ as many times as we need. Thus we introduce only highest and lowest weight vectors in each irreducible $\mathfrak{sl}_2$-module component of A with the three representations, i.e., trivial, standard, and sign representations.

We linked full calculations for Section 3 to Arxiv to make this paper shortened (see Lie-algebra-fulltext.pdf).

2. $\mathfrak{sl}_2$-representation theory and Schur-Weyl duality

In this section, we first introduce the definition of a *Lie algebra* and $\mathfrak{sl}_2$-representation theory. As we mentioned in the introduction, an Artinian ring $A := \mathbb{k}[x_1, x_2, x_3]/(x_1^d, x_2^d, x_3^d)$ can be decomposed by irreducible $\mathfrak{sl}_2$-modules (see Equation (1.1)). Moreover, we shall introduce how to find a representation in each irreducible $\mathfrak{sl}_2$-module component of A among the three representations, i.e., trivial, standard, and sign representation having a highest weight inside A with $d = 6$, and show the details how to find and calculate them in each degree (in each irreducible $\mathfrak{sl}_2$-module component of A) in the next section.

Definition 2.1. Let $\mathfrak{g}$ be a vector space over a field $\mathbb{k}$. $\mathfrak{g}$ is a *Lie algebra* if there exists a bilinear product $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(a) $[x, y] = -[y, x]$;
(b) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Let $\mathfrak{sl}_2$ be the set of all 2×2 matrices having trace 0. Define

$$[x, y] = xy - yx$$

for $x, y \in \mathfrak{sl}_2$. Set

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \tag{2.1}$$

Thus $\mathfrak{sl}_2$ is a Lie algebra generated by e, f, h with defining relations (2.1), i.e.,

$$\mathfrak{sl}_2 = \mathbb{k}e \oplus \mathbb{k}h \oplus \mathbb{k}f.$$

For each $m \in \mathbb{Z}_{\geq 0}$, there exists a unique (up to isomorphism) $(m + 1)$-dimensional irreducible $\mathfrak{sl}_2$-module $V(m)$ with a basis $\{u, fu, \ldots, f^m u\}$ [6], where the $\mathfrak{sl}_2$-action is given by

$$\begin{aligned} e \cdot (f^k u) &= k(m - k + 1) f^{k-1} u, \\ f \cdot (f^k u) &= f^{k+1} u, \quad \text{and} \\ h \cdot (f^k u) &= (m - 2k) f^k u. \end{aligned} \tag{2.2}$$

For a finite-dimensional $\mathfrak{sl}_2$-module V, $v \in V$ is called a *highest weight vector* if $e \cdot v = 0$, and $w \in V$ is called a *lowest weight vector* if $f \cdot w = 0$. We say that v has *weight* k if $h \cdot v = kv$ (see [4, 6, 8]).

Definition-Example 2.2 ([8, Example 2.2])**.** There are 3 irreducible representations of S_3 corresponding to the partitions $\lambda = (3)$, $(2,1)$, and $(1,1,1)$ of 3. The standard tableaux of shape λ are given below (see [1, 2, 8]).

$$\lambda = (3), \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}\,,$$

$$\lambda = (2,1), \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,,$$

$$\lambda = (1,1,1), \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}\,.$$

Hence $\dim_{\Bbbk} S^{(3)} = \dim_{\Bbbk} S^{(1,1,1)} = 1$ and $\dim_{\Bbbk} S^{(2,1)} = 2$. The 1-dimensional representations $S^{(3)}$ and $S^{(1,1,1)}$ are called the *trivial representation* and *sign representation*, respectively. We will call the 2-dimensional representation $S^{(2,1)}$ the *standard representation.*

Let $A := \Bbbk[x_1, x_2, x_3]/(x_1^6, x_2^6, x_3^6)$. Then the Hilbert function of A is

$$\mathbf{H}_A \; : \; 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 25 \quad 27 \quad 27 \quad 25 \quad 21 \quad 15 \quad 10 \quad 6 \quad 3 \quad 1.$$

Since A has the SLP in the narrow sense, we see that the $\mathfrak{sl}_2$-module decomposition of A is

$$A \cong V(15) \oplus V(13)^{\oplus 2} \oplus V(11)^{\oplus 3} \oplus V(9)^{\oplus 4} \oplus V(7)^{\oplus 5} \oplus V(5)^{\oplus 6} \oplus V(3)^{\oplus 4} \oplus V(1)^{\oplus 2}.$$

The Schur-Weyl duality implies

$$A \cong V((3)) \otimes S^{(3)} \oplus V((2,1)) \otimes S^{(2,1)} \oplus V((1,1,1)) \otimes S^{(1,1,1)}.$$

By counting the number of semi-standard tableaux with entries in $1, 2, \dots, 6$ (see [1, 2]), we obtain

$$\dim_{\Bbbk} V((3)) = 56, \quad \dim_{\Bbbk} V((2,1)) = 70, \quad \text{and} \quad \dim_{\Bbbk} V((1,1,1)) = 20.$$

It follows that there are 56 copies of trivial representations, 70 copies of standard representations, and 20 copies of sign representations in the S_3-module decomposition of A (see Figure 1). It is not hard to find where each representation exists in each irreducible $\mathfrak{sl}_2$-module component of A since it is enough to find a highest weight vector in each irreducible $\mathfrak{sl}_2$-module component of A (see the bold **1**'s in the following diagram). We can also obtain all representations after we apply the multiplication map by $\ell = x_1 + x_2 + x_3$ to a highest weight vector as many times as we need.

While the sum of each column in Figure 1 indicates the Hilbert function, the sum of each row specifies the dimension of an irreducible $\mathfrak{sl}_2$-module component of A. Since a degree 0 highest weight vector of an irreducible $\mathfrak{sl}_2$-module $V(15)$ in Figure 1 is $1 \in A_0$, we see that 1 generates a trivial representation.

A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	16 trivial representations
	1	1	1	1	1	1	1	1	1	1	1	1	1	1		14 standard representations
	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
		1	1	1	1	1	1	1	1	1	1	1	1			12 trivial representations
		1	1	1	1	1	1	1	1	1	1	1	1			12 standard representations
		1	1	1	1	1	1	1	1	1	1	1	1			
			1	1	1	1	1	1	1	1	1	1				10 trivial representations
			1	1	1	1	1	1	1	1	1	1				10 sign representations
			1	1	1	1	1	1	1	1	1	1				10 standard representations
			1	1	1	1	1	1	1	1	1	1				
				1	1	1	1	1	1	1	1					8 trivial representations
				1	1	1	1	1	1	1	1					8 standard representations
				1	1	1	1	1	1	1	1					
				1	1	1	1	1	1	1	1					8 standard representations
				1	1	1	1	1	1	1	1					
					1	1	1	1	1	1						6 trivial representations
					1	1	1	1	1	1						6 sign representations
					1	1	1	1	1	1						6 standard representations
					1	1	1	1	1	1						
					1	1	1	1	1	1						6 standard representations
					1	1	1	1	1	1						
						1	1	1	1							4 trivial representations
						1	1	1	1							4 sign representations
						1	1	1	1							4 standard representations
						1	1	1	1							
							1	1								2 standard representations
							1	1								

FIGURE 1. $\mathfrak{sl}_2$-decompositions (the bold **1**'s are the locations of highest weight vectors)

Now consider a degree 1 highest weight vectors of the two irreducible $\mathfrak{sl}_2$-module components $V(13)^{\oplus 2}$ of A in Figure 1. Recall that $\mathbf{H}_A(1) = 3$ and we already have a trivial representation ℓ in degree 1. Furthermore, notice that we have a 2-dimensional standard representation

$$\Bbbk(x_1 - x_2) \oplus \Bbbk(x_1 - x_3)$$

in degree 1 from the partition

$$\lambda = (2,1), \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Hence we find all kinds of representations having a highest weight in degree 1.

Now we look at the degree 2 (the three irreducible $\mathfrak{sl}_2$-module components $V(11)^{\oplus 3}$ of A). We still don't have a sign representation in degree 2, and so we have to decide 3-dimensional representations in degree 2 having a highest weight with trivial and standard representations, which are one trivial representation and one 2-dimensional standard representation. Indeed, they are from the previous cases for $3 \leq d \leq 5$ in [8].

As we mentioned before, we have a highest weight sign representation

$$\Bbbk\big((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\big)$$

in degree 3 (in the four irreducible $\mathfrak{sl}_2$-module components $V(9)^{\oplus 4}$ of A) from the partition

$$\lambda = (1,1,1), \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

We also assign one trivial and one 2-dimensional standard representations having a highest weight in degree 3 as in the previous cases for $3 \le d \le 5$, recursively (see [8]). After we apply multiplication map by ℓ, we obtain 9 more trivial, standard, and sign representations, respectively.

By an analogous argument with the previous cases for $3 \le d \le 5$ in [8], we find trivial, standard, and sign representations in degrees $4, 5, 6$, and 7 (in irreducible $\mathfrak{sl}_2$-module components $V(7)^{\oplus 5}$, $V(5)^{\oplus 6}$, $V(3)^{\oplus 4}$, and $V(1)^{\oplus 2}$ of A) in Figure 1, respectively.

3. The $(S_3 \times GL_6)$-module structure of $\Bbbk[x_1, x_2, x_3]/(x_1^6, x_2^6, x_3^6)$

In this section, we find an explicit basis for $A := \Bbbk[x_1, x_2, x_3]/(x_1^6, x_2^6, x_3^6)$ in Theorem 3.1, which is compatible with the S_3-module structure based on Schur-Weyl duality with trivial, standard, and sign representations.

As we mentioned in the introduction, we linked full calculations to Arxiv to make this paper shortened (see Lie-algebra-fulltext.pdf).

3.1. 56 trivial representations

We start with trivial representations inside A. In Section 2, we mention that there exist 56 trivial representations inside A with the location of a highest weight vector in each irreducible $\mathfrak{sl}_2$-module component of A in Figure 1. We now find them in each degree.

First, a highest weight vector $1 \in A_0$ in degree 0 generates the trivial representation in degree 0, and so we obtain 15 more trivial representations.

Recall that we don't have any trivial representation in degree 1 (in the two irreducible $\mathfrak{sl}_2$-module components $V(13)^{\oplus 2}$ of A).

Since the polynomials of degree 2, $x_1^2 + x_2^2 + x_3^2$ and $x_1x_2 + x_1x_3 + x_2x_3$, are invariant under S_3-action, we see that

$$P = a(x_1^2 + x_2^2 + x_3^2) + b(x_1x_2 + x_1x_3 + x_2x_3)$$

is a candidate polynomial for a generator of the degree 2 trivial representation. Moreover, since we expect P is a highest weight vector of $V(11)$, we need to impose the condition $F^{12}(P) = 0$, which gives $4a + 5b = 0$. We may take

$$P = 5(x_1^2 + x_2^2 + x_3^2) - 4(x_1x_2 + x_1x_3 + x_2x_3)$$

and P generates 11 more trivial representations.

Let us move onto the degree 3 cases. By an analogous argument, since the polynomials of degree 3, $x_1^3 + x_2^3 + x_3^3$, $x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2$, and $x_1x_2x_3$, are invariant under S_3-action, we see that

$$P = a(x_1^3+x_2^3+x_3^3)+b(x_1^2x_2+x_1^2x_3+x_2^2x_3+x_1x_2^2+x_1x_3^2+x_2x_3^2)+c(x_1x_2x_3)$$

can be a candidate polynomial for a generator of the degree 3 trivial representation. Since we expect P is a highest weight vector of $V(9)$, we need to have $F^{10}(P) = 0$, which yields

$$126a + 380b + 75c = 0 \quad \text{and} \quad 27a + 94b + 20c = 0.$$

Taking $a = 50$, $b = -45$, and $c = 144$, we get

$$P = 50(x_1^3+x_2^3+x_3^3)-45(x_1^2x_2+x_1^2x_3+x_2^2x_3+x_1x_2^2+x_1x_3^2+x_2x_3^2)+144(x_1x_2x_3).$$

As usual, apply F repeatedly to get 9 more trivial representations.

By the same argument as above, for the degree 4 candidate, let

$$\begin{aligned} P = {} & a(x_1^4 + x_2^4 + x_4^4) + b(x_1^3x_2 + x_1^3x_3 + x_2^3x_3 + x_1x_2^3 + x_1x_3^3 + x_2x_3^3) \\ & + c(x_1^2x_2^2 + x_1^2x_3^3 + x_2^2x_3^2) + d(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2). \end{aligned}$$

Imposing the condition $F^8(P) = 0$, we obtain the following equations

$$8a+28b+16c+15d = 0, \quad 8a+37b+24c+30d = 0, \quad \text{and} \quad a+8b+6c+8d = 0.$$

Then we get $a = 10t$, $b = -8t$, $c = 9t$, and $d = 0$ for some $t \in \mathbb{N}$. Hence we have

$$\begin{aligned} P = {} & 10(x_1^4 + x_2^4 + x_3^4) - 8(x_1^3x_2 + x_1^3x_3 + x_2^3x_3 + x_1x_2^3 + x_1x_3^3 + x_2x_3^3) \\ & + 9(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2). \end{aligned}$$

For the degree 5 candidate, let

$$\begin{aligned} P = {} & a(x_1^5 + x_2^5 + x_3^5) + b(x_1^4x_2 + x_1^4x_3 + x_2^4x_3 + x_1x_2^4 + x_1x_3^4 + x_2x_3^4) \\ & + c(x_1^3x_2^2 + x_1^3x_3^2 + x_2^3x_3^2 + x_1^2x_2^3 + x_1^2x_3^2 + x_2^2x_3^3) \\ & + d(x_1^3x_2x_3 + x_1x_2^3x_3 + x_1x_2x_3^3) \\ & + e(x_1^2x_2^2x_3 + x_1x_2^2x_3^2 + x_1^2x_2x_3^2). \end{aligned}$$

Imposing the condition $F^6(P) = 0$, we obtain the following equations

$$\begin{aligned} 6a + 36b + 60c + 15d + 10e &= 0, \\ 15a + 111b + 171c + 90d + 95e &= 0, \\ 10a + 60b + 86c + 60d + 75e &= 0, \quad \text{and} \\ b + 3c + 2d + 3e &= 0. \end{aligned}$$

Taking $a = 150$, $b = -75$, $c = 15$, $d = 96$, and $e = -54$,

$$\begin{aligned} P = {} & 150(x_1^5 + x_2^5 + x_3^5) - 75(x_1^4x_2 + x_1^4x_3 + x_2^4x_3 + x_1x_2^4 + x_1x_3^4 + x_2x_3^4) \\ & + 15(x_1^3x_2^2 + x_1^3x_3^2 + x_2^3x_3^2 + x_1^2x_2^3 + x_1^2x_3^2 + x_2^2x_3^3) \\ & + 96(x_1^3x_2x_3 + x_1x_2^3x_3 + x_1x_2x_3^3) - 54(x_1^2x_2^2x_3 + x_1x_2^2x_3^2 + x_1^2x_2x_3^2). \end{aligned}$$

So, we have 6 more trivial representations.

Note that so far we have found 52 trivial representations. We shall find 4 more trivial representations in degree 6. Let

$$\begin{aligned} P = {} & a(x_1^5x_2 + x_1^5x_3 + x_2^5x_3 + x_1x_2^5 + x_1x_3^5 + x_2x_3^5) \\ & + b(x_1^4x_2^2 + x_1^4x_3^2 + x_2^4x_3^2 + x_1^2x_2^4 + x_1^2x_3^4 + x_2^2x_3^4) \\ & + c(x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) + d(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) \\ & + e(x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^2x_2^3x_3 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3) \\ & + f(x_1^2x_2^2x_3^2). \end{aligned}$$

By applying the condition $F^4(P) = 0$, we obtain the following equations

$$\begin{array}{lcl} 2a + 8b + 6d & = & 0, \\ 5a + 16b + 5c + 12d + 10e & = & 0, \\ 10a + 16b + 12c + 6d + 23e + 4f & = & 0, \\ 7b + 4c + 6d + 16e + 3f & = & 0, \quad \text{and} \\ 8b + 6c + 9d + 44e + 12f & = & 0. \end{array}$$

Hence we have

$$a = 0,\ b = 15,\ c = -24,\ d = -20,\ e = 12,\ \text{and}\ f = -12,$$

i.e.,

$$\begin{aligned} P = {} & 15(x_1^4x_2^2 + x_1^4x_3^2 + x_2^4x_3^2 + x_1^2x_2^4 + x_1^2x_3^4 + x_2^2x_3^4) \\ & - 24(x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) - 20(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) \\ & + 12(x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^2x_2^3x_3 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3) \\ & - 12(x_1^2x_2^2x_3^2), \end{aligned}$$

and thus we have 4 more trivial representations. Hence we have constructed the basis of all 56 trivial representations inside A in Figure 1 according to highest weight vectors in degrees $0, 2, 3, 4, 5$, and 6.

3.2. 70 standard representations

Now we work on the standard representations inside A. As we mentioned in Section 2, we know the two polynomials

$$P_1 = x_1 - x_2 \quad \text{and} \quad Q_1 = x_1 - x_3$$

generate an standard representation in degree 1, and 13 more in higher degree by multiplying F repeatedly.

Consider an standard representation in degree 2 having a highest weight. Since two polynomials

$$P_1 = x_1 - x_2 \quad \text{and} \quad Q_1 = x_1 - x_3$$

generate an standard representation, we can put

$$P_2 = (x_1 - x_2)(a(x_1 + x_2) + bx_3) = ax_1^2 - ax_2^2 + bx_1x_3 - bx_2x_3$$

and impose the condition $F^{12}(P_2) = 0$. Then we get an equation $8a + 5b = 0$ and obtain

$$P_2 = 5x_1^2 - 5x_2^2 - 8x_1x_3 + 8x_2x_3, \quad \text{and}$$
$$Q_2 = 5x_1^2 - 5x_3^2 - 8x_1x_2 + 8x_2x_3.$$

It is obvious that P_2 and Q_2 are linearly independent. Then 11 more standard representations generated by P_2 and Q_2.

For the degree 3 candidate, we begin with

$$P_3 = ax_1^3 - ax_2^3 + bx_1^2x_2 - bx_1x_2^2 + cx_1^2x_3 - cx_2^2x_3 + dx_1x_3^2 - dx_2x_3^2$$

and impose the condition $F^8(P_3) = 0$. Then we get

$$\begin{aligned} 9a + 5b + 8c + 3d &= 0, \quad \text{and} \\ 27a + 15b + 35c + 20d &= 0. \end{aligned}$$

If we take $a = 5$, $b = -9$, and $c = d = 0$, then we obtain

$$P_3 = 5x_1^3 - 5x_2^3 - 9x_1^2x_2 + 9x_1x_2^2,$$
$$Q_3 = 5x_1^3 - 5x_3^2 - 9x_1^2x_3 + 9x_1x_3^2.$$

Now we get 10 more standard representations.

Let us work on the degree 4 case. Let

$$\begin{aligned} P_4 = {} & ax_1^4 - ax_2^4 + bx_1^3x_2 - bx_1x_2^3 + cx_1^3x_3 - cx_2^3x_3 + dx_1^2x_3^2 - dx_2^2x_3^2 \\ & + ex_1x_3^3 - ex_2x_3^3 + fx_1^2x_2x_3 - fx_1x_2^2x_3 \end{aligned}$$

be a candidate for a degree 4 highest weight vector of $V(8)$. Then the condition $F^8(P) = 0$ yields a system of linear equations

$$\begin{aligned} 224a + 280b + 252c + 112d + 14e + 140f &= 0, \\ 280a + 350b + 504c + 392d + 112e + 280f &= 0, \\ 168a + 168b + 420c + 504d + 252e + 210f &= 0, \quad \text{and} \\ 56a + 112b + 210c + 280d + 140e + 140f &= 0. \end{aligned}$$

If we take $d = e = 0$, then we have $a = 25$, $b = c = -20$, and $f = 36$, and thus we obtain two dimensional standard representations.

$$\begin{cases} P_4 = 25x_1^4 - 25x_2^4 - 20x_1^3x_2 + 20x_1x_2^3 - 20x_1^3x_3 + 20x_2^3x_3 \\ \qquad + 36x_1^2x_2x_3 - 36x_1x_2^2x_3, \\ Q_4 = 25x_1^4 - 25x_3^4 - 20x_1^3x_3 + 20x_1x_3^3 - 20x_1^3x_2 + 20x_2x_3^3 \\ \qquad + 36x_1^2x_2x_3 - 36x_1x_2x_3^2. \end{cases}$$

Now we have 7 more 2-dimensional standard representations.

On the other hand, if we take $b = f = 0$, then we have $a = 5$, $c = e = -8$, and $d = 9$. So we have another 2-dimensional standard representations given below:

$$\begin{cases} P_4' = 5x_1^4 - 5x_2^4 - 8x_1^3x_3 + 8x_2^3x_3 + 9x_1^2x_3^2 - 9x_2^2x_3^2 - 8x_1x_3^3 + 8x_2x_3^3, \\ Q_4' = 5x_1^4 - 5x_3^4 - 8x_1^3x_2 + 8x_2x_3^3 + 9x_1^2x_2^2 - 9x_2^2x_3^2 - 8x_1x_2^3 + 8x_2^3x_3. \end{cases}$$

Now we have another 7 more 2-dimensional standard representations. Note that the pairs $(F^i(P_4), F^i(Q_4))$ and $(F^i(P_4'), F^i(Q_4'))$ generate two distinct (linearly independent) standard representations in degree 4 for each $i = 0, 1, \ldots, 7$.

We now move on to the degree 5. Let

$$\begin{aligned} P_5 = {} & ax_1^5 - ax_2^5 + bx_1^4x_2 - bx_1x_2^4 + cx_1^4x_3 - cx_2^4x_3 + dx_1^3x_2^2 - dx_1^2x_2^3 \\ & + ex_1^3x_3^2 - ex_2^3x_3^2 + fx_1^2x_3^3 - fx_2^2x_3^3 + gx_1x_3^4 - gx_2x_3^4 \\ & + hx_1^3x_2x_3 - hx_1x_2^3x_3 + ix_1^2x_2x_3^2 - ix_1x_2^2x_3^2 \end{aligned}$$

be a candidate for a degree 5 highest weight vector of $V(6)$. Then the condition $F^6(P) = 0$ yields a system of linear equations

$$\begin{array}{lcl} 15a + 45b + 24c + 30d + 9e + 30h + 5i & = & 0, \\ 20a + 60b + 60c + 40d + 54e + 14f + 75h + 30i & = & 0, \\ 15a + 30b + 60c + 15d + 90e + 54f + 9g + 60h + 45i & = & 0, \\ 15b + 20c + 15d + 45e + 30f + 5g + 40h + 30i & = & 0, \\ 6a + 6b + 30c + 60e + 60f + 24g + 15h + 20i & = & 0, \quad \text{and} \\ 6b + 15c + 6d + 60e + 75f + 30g + 30h + 40i & = & 0. \end{array}$$

If we take $h = i = 0$, then we get

$$a = 1, b = 1, c = -2, d = -1, e = 2, f = -2, \text{ and } g = 2.$$

Now we get 2-dimensional standard representation in degree 5 given below:

$$\begin{cases} P_5 = x_1^5 - x_2^5 + x_1^4x_2 - x_1x_2^4 - 2x_1^4x_3 + 2x_2^4x_3 - x_1^3x_2^2 + x_1^2x_2^3 \\ \qquad + 2x_1^3x_3^2 - 2x_2^3x_3^2 - 2x_1^2x_3^3 + 2x_2^2x_3^3 + 2x_1x_3^4 - 2x_2x_3^4, \\ Q_5 = x_1^5 - x_3^5 + x_1^4x_3 - x_1x_3^4 - 2x_1^4x_2 + 2x_2x_3^4 - x_1^3x_3^2 + x_1^2x_3^3 \\ \qquad + 2x_1^3x_2^2 - 2x_2^2x_3^3 - 2x_1^2x_2^3 + 2x_2^3x_3^2 + 2x_1x_2^4 - 2x_2^4x_3. \end{cases}$$

Taking $d = f = g = 0$, we get

$$a = 15, b = -5, c = -10, e = 5, h = 8, \text{ and } i = -9.$$

Hence we obtain another 2-dimensional standard representations.

$$\begin{cases} P_5' = 15x_1^5 - 15x_2^5 - 5x_1^4x_2 + 5x_1x_2^4 - 10x_1^4x_3 + 10x_2^4x_3 + 5x_1^3x_3^2 - 5x_2^3x_3^2 \\ \qquad + 8x_1^3x_2x_3 - 8x_1x_2^3x_3 - 9x_1^2x_2x_3^2 + 9x_1x_2^2x_3^2, \\ Q_5' = 15x_1^5 - 15x_3^5 - 5x_1^4x_3 + 5x_1x_3^4 - 10x_1^4x_2 + 10x_2x_3^4 + 5x_1^3x_2^2 - 5x_2^2x_3^3 \\ \qquad + 8x_1^3x_2x_3 - 8x_1x_2x_3^3 - 9x_1^2x_2^2x_3 + 9x_1x_2^2x_3^2. \end{cases}$$

Hence we have 12 standard representations in degree 5.

We now work on the degree 6 case. Let

$$\begin{aligned} P_6 = {} & (x_1 - x_2)(a(x_1^5 + x_2^5) + bx_3^5 + c(x_1^4x_2 + x_1x_2^4) + d(x_1^4x_3 + x_2^4x_3) \\ & + e(x_1^3x_2^2 + x_1^2x_2^3) + f(x_1^3x_3^2 + x_2^3x_3^2) + g(x_1^2x_3^3 + x_2^2x_3^3) + h(x_1x_3^4 + x_2x_3^4) \\ & + p(x_1^3x_2x_3 + x_1x_2^3x_3) + q(x_1^2x_2^2x_3) + r(x_1x_2x_3^3) + s(x_1^2x_2x_3^2 + x_1x_2^2x_3^2)) \end{aligned}$$

be a candidate for a degree 6 highest weight vector, which is annihilated by F^4. Then we obtain a system of linear equations.

$$\begin{array}{lcl}
4a+4c+2d-8e-p-2q &=& 0, \\
6a+6c+8d-12e+f-4p-8q-5s &=& 0, \\
4c+6d-4e+4f-g-6q-2r-8s &=& 0, \\
4a+6d-4e-3g-6p-6q-3r-12s &=& 0, \\
c+4d-e+6f-4g-5h-4q-8r-12s &=& 0, \\
a-c-6f-8g-4h-4p-4r-6s &=& 0, \\
b+d+4f+6g+4h &=& 0, \\
3b-d+6g+12h+p+6r+4s &=& 0, \quad \text{and} \\
2b-4f+8h-p+q+6r+4s &=& 0.
\end{array}$$

If we take $e=0$ and $s=-24$, then

$$\begin{aligned}
&a=15,\ b=40,\ c=-5,\ d=20,\ f=-20,\ g=20, \\
&h=-25,\ p=32,\ q=24,\ \text{and } r=24.
\end{aligned}$$

We thus have a 2-dimensional standard representation of degree 6 as follows.

$$\begin{cases}
P_6 = (x_1-x_2)(15(x_1^5+x_2^5)+40x_3^5-5(x_1^4x_2+x_1x_2^4)+20(x_1^4x_3+x_2^4x_3) \\
\qquad -20(x_1^3x_3^2+x_2^3x_3^2)+20(x_1^2x_3^3+x_2^2x_3^3)-25(x_1x_3^4+x_2x_3^4) \\
\qquad +32(x_1^3x_2x_3+x_1x_2^3x_3)+24(x_1^2x_2^2x_3)+24(x_1x_2x_3^3) \\
\qquad -24(x_1^2x_2x_3^2+x_1x_2^2x_3^2)), \\
Q_6 = (x_1-x_2)(15(x_1^5+x_2^5)+40x_3^5-5(x_1^4x_2+x_1x_2^4)+20(x_1^4x_3+x_2^4x_3) \\
\qquad -20(x_1^3x_3^2+x_2^3x_3^2)+20(x_1^2x_3^3+x_2^2x_3^3)-25(x_1x_3^4+x_2x_3^4) \\
\qquad +32(x_1^3x_2x_3+x_1x_2^3x_3)+24(x_1^2x_2^2x_3)+24(x_1x_2x_3^3) \\
\qquad -24(x_1^2x_2x_3^2+x_1x_2^2x_3^2)).
\end{cases}$$

Applying F, we get 3 more standard representations.

We now work on the degree 7 cases. Let

$$\begin{aligned}
P_7 = (x_1-x_2)(&a(x_1^5x_2+x_1x_2^5)+b(x_1^5x_3+x_2^5x_3)+c(x_1^4x_2^2+x_1^2x_2^4) \\
&+d(x_1^4x_3^2+x_2^4x_3^2)+e(x_1^3x_2^3)+f(x_1^3x_3^3+x_2^3x_3^3)+g(x_1^2x_3^4+x_2^2x_3^4) \\
&+h(x_1x_3^5+x_2x_3^5)+p(x_1^4x_2x_3+x_1x_2^4x_3)+q(x_1x_2x_3^4) \\
&+r(x_1^3x_2^2x_3+x_1^2x_2^3x_3)+s(x_1^3x_2x_3^2+x_1x_2^3x_3^2) \\
&+t(x_1^2x_2x_3^3+x_1x_2^2x_3^3)+u(x_1^2x_2^2x_3^2))
\end{aligned}$$

be a candidate for a degree 7 highest weight vector, which is annihilated by F^2. Then we obtain a system of linear equations.

$$\begin{array}{lcl} a-e &=& 0,\\ 2a+b-2e+p-2r &=& 0,\\ c+d-e+2p-2r-u &=& 0,\\ a+2b-c+d-2r-s-u &=& 0,\\ -b+f+p+2s+t &=& 0,\\ 2d+f+p-r-2t-2u &=& 0,\\ b-f-p-2s-t &=& 0,\\ d+2f+g &=& 0,\\ -d+g+q+s+2t &=& 0,\\ 2f-q+s-2t-u &=& 0,\\ d-g-q-s-2t &=& 0,\\ -d-2f-g &=& 0,\\ f+2g+h &=& 0, \quad \text{and}\\ f-2h-2q-t &=& 0. \end{array}$$

If we take $a = e = h = r = 0$, then we get that

$$b = 12,\ c = 15,\ d = 15,\ f = -10,\ g = 5,$$
$$p = -12,\ q = -4,\ s = 18,\ t = -2,\ \text{and } u = 6.$$

Hence we have a 2-dimensional standard representation.

$$\begin{cases} P_7 = 15x_1^5x_2^2 - 15x_1^4x_2^3 + 15x_1^3x_2^4 - 15x_1^2x_2^5 - 24x_1^5x_2x_3 + 12x_1^4x_2^2x_3 \\ \qquad - 12x_1^2x_2^4x_3 + 24x_1x_2^5x_3 + 15x_1^5x_3^2 + 3x_1^4x_2x_3^2 - 12x_1^3x_2^2x_3^2 \\ \qquad + 12x_1^2x_2^3x_3^2 - 3x_1x_2^4x_3^2 - 15x_2^5x_3^2 - 10x_1^4x_3^3 + 8x_1^3x_2x_3^3 \\ \qquad - 8x_1x_2^3x_3^3 + 10x_2^4x_3^3 + 5x_1^3x_3^4 - 9x_1^2x_2x_3^4 + 9x_1x_2^2x_3^4 - 5x_2^3x_3^4, \\ Q_7 = 15x_1^5x_3^2 - 15x_1^4x_3^3 + 15x_1^3x_3^4 - 15x_1^2x_3^5 - 24x_1^5x_2x_3 + 12x_1^4x_2x_3^2 \\ \qquad - 12x_1^2x_2x_3^4 + 24x_1x_2x_3^5 + 15x_1^5x_2^2 + 3x_1^4x_2^2x_3 - 12x_1^3x_2^2x_3^2 \\ \qquad + 12x_1^2x_2^2x_3^3 - 3x_1x_2^2x_3^4 - 15x_2^2x_3^5 - 10x_1^4x_2^3 + 8x_1^3x_2^3x_3 - 8x_1x_2^3x_3^3 \\ \qquad + 10x_2^3x_3^4 + 5x_1^3x_2^4 - 9x_1^2x_2^4x_3 + 9x_1x_2^4x_3^2 - 5x_2^4x_3^3. \end{cases}$$

Thus we have constructed the basis of all 70 standard representations inside A in Figure 1 according to highest weight vectors in degrees $1, 2, 3, 4, 5$, and 6.

3.3. 20 sign representations

We consider the sign representations. We already know that the cubic polynomial

$$D = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

generates the sign representation in degree 3 and multiplying by F repeatedly, we get 9 more sign representations.

We now consider a sign representation in degree 5. As a candidate, we may take a product of D and a symmetric quadratic polynomial

$$Q = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(a(x_1^2 + x_2^2 + x_3^2) + b(x_1x_2 + x_1x_3 + x_2x_3)).$$

Imposing the condition $F^6(Q) = 0$, we get that $a = 1$ and $b = 0$, and thus we have a sign representation as follows.

$$Q = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1^2 + x_2^2 + x_3^2).$$

Now we have 6 sign representations in degree 5.

Now consider a sign representation in degree 6. As a candidate, we may take a product of D and a symmetric cubic polynomial

$$\begin{aligned} S = {} & (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \\ & (a(x_1^3 + x_2^3 + x_3^3) + b(x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2) + cx_1x_2x_3). \end{aligned}$$

Imposing the condition $F^4(Q) = 0$, we obtain the following equation

$$3a - 4b - 2c = 0 \quad \text{and} \quad 3a - 2b - 3c = 0.$$

Taking $a = 8,\ b = 3,\ c = 6$, we have a sign representation as

$$\begin{aligned} S = {} & 8(x_1^5x_2 - x_1^5x_3 + x_2^5x_3 - x_1x_2^5 + x_1x_3^5 - x_2x_3^5) \\ & + 8(x_1^3x_2^2x_3 - x_1^2x_2^3x_3 - x_1^3x_2x_3^2 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 - x_1x_2^2x_3^3) \\ & + 5(-x_1^4x_2^2 + x_1^2x_2^4 + x_1^4x_3^2 - x_2^4x_3^2 - x_1^2x_3^4 + x_2^2x_3^4). \end{aligned}$$

Therefore, we have 3 more sign representations. So we have constructed the basis of all 20 sign representations inside A in Figure 1 according to highest weight vectors in degrees $3, 5$, and 6.

Using the above trivial, standard, and sign representations all we have found, we obtain the following theorem.

Theorem 3.1. *Let $A = \mathbb{k}[x_1, x_2, x_3]/(x_1^6, x_2^6, x_3^6)$. Then the S_3-module structure of A is completely determined by the following representations.*

(a) *Trivial representations*
- (i) *degree* $0 : \mathbb{k}(1)$.
- (ii) *degree* $2 : \mathbb{k}(5(x_1^2 + x_2^2 + x_3^2) - 4(x_1x_2 + x_1x_3 + x_2x_3))$.
- (iii) *degree* $3 : \mathbb{k}(50(x_1^3 + x_2^3 + x_3^3) - 45(x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2) + 144(x_1x_2x_3))$.
- (iv) *degree* $4 : \mathbb{k}(10(x_1^4 + x_2^4 + x_3^4) - 8(x_1^3x_2 + x_1^3x_3 + x_2^3x_3 + x_1x_2^3 + x_1x_3^3 + x_2x_3^3) + 9(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2))$.
- (v) *degree* $5 : \mathbb{k}(150(x_1^5 + x_2^5 + x_3^5) - 75(x_1^4x_2 + x_1^4x_3 + x_2^4x_3 + x_1x_2^4 + x_1x_3^4 + x_2x_3^4) + 15(x_1^3x_2^2 + x_1^3x_3^2 + x_2^3x_3^2 + x_1^2x_2^3 + x_1^2x_3^3 + x_2^2x_3^3) + 96(x_1^3x_2x_3 + x_1x_2^3x_3 + x_1x_2x_3^3) - 54(x_1^2x_2^2x_3 + x_1x_2^2x_3^2 + x_1^2x_2x_3^2))$.
- (vi) *degree* $6 : \mathbb{k}(15(x_1^4x_2^2 + x_1^4x_3^2 + x_2^4x_3^2 + x_1^2x_2^4 + x_1^2x_3^4 + x_2^2x_3^4) - 24(x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) - 20(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) + 12(x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^2x_2^3x_3 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3) - 12(x_1^2x_2^2x_3^2))$.

(b) *Sign representations*

(i) *degree* $3 : \mathbb{k}(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.
(ii) *degree* $5 : \mathbb{k}((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1^2 + x_2^2 + x_3^2))$.
(ii) *degree* $6 : \mathbb{k}(8(x_1^5x_2 - x_1^5x_3 + x_2^5x_3 - x_1x_2^5 + x_1x_3^5 - x_2x_3^5) + 8(x_1^3x_2^2x_3 - x_1^2x_2^3x_3 - x_1^3x_2x_3^2 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 - x_1x_2^2x_3^3) + 5(-x_1^4x_2^2 + x_1^2x_2^4 + x_1^4x_3^2 - x_2^4x_3^2 - x_1^2x_3^4 + x_2^2x_3^4))$.

(c) *standard representations*
(i) *degree* $1 : \mathbb{k}(x_1 - x_2) \oplus \mathbb{k}(x_1 - x_3)$.
(ii) *degree* $2 : \mathbb{k}(5x_1^2 - 5x_2^2 - 8x_1x_3 + 8x_2x_3) \oplus \mathbb{k}(5x_1^2 - 5x_3^2 - 8x_1x_2 + 8x_2x_3)$.
(iii) *degree* $3 : \mathbb{k}(5x_1^3 - 5x_2^3 - 9x_1^2x_2 + 9x_1x_2^2) \oplus \mathbb{k}(5x_1^3 - 5x_3^2 - 9x_1^2x_3 + 9x_1x_3^2)$.
(iv) *degree* $4 : \mathbb{k}(25x_1^4 - 25x_2^4 - 20x_1^3x_2 + 20x_1x_2^3 - 20x_1^3x_3 + 20x_2^3x_3 + 36x_1^2x_2x_3 - 36x_1x_2^2x_3) \oplus \mathbb{k}(25x_1^4 - 25x_3^4 - 20x_1^3x_3 + 20x_1x_3^3 - 20x_1^3x_2 + 20x_2x_3^3 + 36x_1^2x_2x_3 - 36x_1x_2x_3^2)$, *and*
$\mathbb{k}(5x_1^4 - 5x_2^4 - 8x_1^3x_3 + 8x_2^3x_3 + 9x_1^2x_3^2 - 9x_2^2x_3^2 - 8x_1x_3^3 + 8x_2x_3^3) \oplus \mathbb{k}(5x_1^4 - 5x_3^4 - 8x_1^3x_2 + 8x_2x_3^3 + 9x_1^2x_2^2 - 9x_2^2x_3^2 - 8x_1x_2^3 + 8x_2^3x_3)$.
(iv) *degree* $5 : \mathbb{k}(x_1^5 - x_2^5 + x_1^4x_2 - x_1x_2^4 - 2x_1^4x_3 + 2x_2^4x_3 - x_1^3x_2^2 + x_1^2x_2^3 + 2x_1^3x_3^2 - 2x_2^3x_3^2 - 2x_1^2x_3^3 + 2x_2^2x_3^3 + 2x_1x_3^4 - 2x_2x_3^4) \oplus \mathbb{k}(x_1^5 - x_3^5 + x_1^4x_3 - x_1x_3^4 - 2x_1^4x_2 + 2x_2x_3^4 - x_1^3x_3^2 + x_1^2x_3^3 + 2x_1^3x_2^2 - 2x_2^2x_3^3 - 2x_1^2x_2^3 + 2x_2^3x_3^2 + 2x_1x_2^4 - 2x_2^4x_3)$, *and*
$\mathbb{k}(15x_1^5 - 15x_2^5 - 5x_1^4x_2 + 5x_1x_2^4 - 10x_1^4x_3 + 10x_2^4x_3 + 5x_1^3x_3^2 - 5x_2^3x_3^2 + 8x_1^3x_2x_3 - 8x_1x_2^3x_3 - 9x_1^2x_2x_3^2 + 9x_1x_2^2x_3^2) \oplus \mathbb{k}(15x_1^5 - 15x_3^5 - 5x_1^4x_3 + 5x_1x_3^4 - 10x_1^4x_2 + 10x_2x_3^4 + 5x_1^3x_2^2 - 5x_2^2x_3^3 + 8x_1^3x_2x_3 - 8x_1x_2x_3^3 - 9x_1^2x_2^2x_3 + 9x_1x_2^2x_3^2)$.
(v) *degree* $6 : \mathbb{k}((x_1 - x_2)(15(x_1^5 + x_2^5) + 40x_3^5 - 5(x_1^4x_2 + x_1x_2^4) + 20(x_1^4x_3 + x_2^4x_3) - 20(x_1^3x_3^2 + x_2^3x_3^2) + 20(x_1^2x_3^3 + x_2^2x_3^3) - 25(x_1x_3^4 + x_2x_3^4) + 32(x_1^3x_2x_3 + x_1x_2^3x_3) + 24(x_1^2x_2^2x_3) + 24(x_1x_2x_3^3) - 24(x_1^2x_2x_3^2 + x_1x_2^2x_3^2))) \oplus \mathbb{k}((x_1 - x_2)(15(x_1^5 + x_2^5) + 40x_3^5 - 5(x_1^4x_2 + x_1x_2^4) + 20(x_1^4x_3 + x_2^4x_3) - 20(x_1^3x_3^2 + x_2^3x_3^2) + 20(x_1^2x_3^3 + x_2^2x_3^3) - 25(x_1x_3^4 + x_2x_3^4) + 32(x_1^3x_2x_3 + x_1x_2^3x_3) + 24(x_1^2x_2^2x_3) + 24(x_1x_2x_3^3) - 24(x_1^2x_2x_3^2 + x_1x_2^2x_3^2)))$.

Remark 3.2. In [8], the authors found an explicit basis for

$$A := \mathbb{k}[x_1, x_2, x_3]/(x_1^d, x_2^d, x_3^d),$$

which is compatible with the S_3-module structure for $d = 3, 4, 5$. In this paper, we extend the result to $d = 6$.

The following question is worth further study for a complete generalization.

Question 3.3. What is an explicit basis for $A := \mathbb{k}[x_1, x_2, x_3]/(x_1^d, x_2^d, x_3^d)$ which is compatible with the S_3-module structure for $d \geq 7$?

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