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ON CO-WELL COVERED GRAPHS

Baha' Abughazaleh, Omar Abughneim, and Hasan Al-Ezeh

Abstract. A graph G is called a well covered graph if every maximal independent set in G is maximum, and co-well covered graph if its complement is a well covered graph. We study some properties of a co-well covered graph and we characterize when the join, the corona product, and cartesian product are co-well covered graphs. Also we characterize when powers of trees and cycles are co-well covered graphs. The line graph of a graph which is co-well covered is also studied.

1. Introduction

A set S of vertices in a graph G is called an independent set (stable set) if no pair of vertices are adjacent in S and the independence number is the cardinality of a maximum independent set in G. The independence number is denoted by $\alpha(G)$. A complete subgraph of a graph G is called a clique. A maximal clique in G is a clique which is not properly contained in any other clique in G. A maximum clique is a clique that has the largest number of vertices in G. The size of a maximum clique in G is called the clique number of G and is denoted by $\omega(G)$. A graph G is called a well covered graph if every maximal independent set in G is maximum. Well covered graphs were introduced by Plummer in 1970, see [6]. He also studied some properties of these graphs and he characterized when some important graphs are well covered, see [7]. Ravindra classified bipartite well-covered graphs, see [8]. Staples studied some subclasses of well-covered graphs, see [9]. Topp and Volkmann studied the relationship between graph operations and well covered graphs, see [10]. A graph G is called an equimatchable graph if every maximal matching is maximum; that means a graph G is an equimatchable graph if and only if its line graph $L(G)$ is a well covered graph. Equimatchable graphs were studied in [5]. A graph G is called a co-well covered graph if its complement is a well covered graph. We note that co-well covered graphs were first introduced in [2]. The authors in [2] were interested in characterizing which circulant graphs are CIS (A graph is called CIS graph if every maximal independent set intersects every maximal clique).

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In this paper we study co-well covered graphs and some of their properties. We prove that any bipartite graph is a co-well covered graph. Then we characterize when powers of trees and cycles are co-well covered graphs. Also, we study the relationship between some types of graph operations and co-well covered graphs. Finally, we characterize when a line graph $L(G)$ of a graph G is a co-well covered graph. For undefined notions and terminology, the reader is referred to [1] and [4]. Also for more details on well covered graphs, the reader is referred to [3], [6] and [10].

2. Co-well covered graphs

Lemma 2.1. *Let G be a graph. Then a set S is an independent set in its complement $\overline{G}$ if and only if $\langle S\rangle$ is a clique in G. In particular a set S is a maximal independent set in $\overline{G}$ if and only if $\langle S\rangle$ is a maximal clique in G (or we can say that a set S is an independent set in $\overline{G}$ if and only if $S = V(K)$, for some clique K in G).*

Proof. $S = \{v_1, v_2, \ldots, v_n\}$ is an independent set in $\overline{G}$ if and only if v_i and v_j are not adjacent in $\overline{G}$, for all $i, j = 1, 2, \ldots, n$ if and only if v_i and v_j are adjacent in G, for all $i, j = 1, 2, \ldots, n$ if and only if $\langle S\rangle$ is a clique in G. □

Corollary 2.2. *A graph G is a co-well covered graph if and only if every maximal clique in G is maximum.*

Theorem 2.3. *Let G be a graph. If there exists $v \in V(G)$ such that $v \notin V(K)$, for any maximum clique K in G, then G is not a co-well covered graph.*

Proof. Since $v \in V(G)$, then there exists a maximal clique K^* in G such that $v \in V(K^*)$. But $v \notin V(K)$, for any maximum clique K in G, so G has a maximal clique which is not maximum. Therefore G is not a co-well covered graph. □

Corollary 2.4. *Let G be a graph with $E(G) \neq \phi$. If G has an isolated vertex, then G is not a co-well covered graph.*

Proof. Let v be an isolated vertex in G. Then G has a maximal clique of order 1, but $E(G) \neq \phi$, that means $K_2 \subset G$ and thus $\omega(G) \geq 2$. So G has a maximal clique which is not maximum, and therefore G is not a co-well covered graph. □

Remark 2.5. The complete graphs and the null graphs are co-well covered graphs.

Theorem 2.6. *Let G be a triangle-free graph with $E(G) \neq \phi$. Then G is a co-well covered graph if and only if G has no isolated vertices.*

Proof. (⇒) Clear by Corollary 2.4.

(⇐) Let G be a graph that has no isolated vertices and let K be a maximal clique in G. Since G has no isolated vertices, then $|V(K)| \geq 2$, and since G is triangle-free graph, then $|V(K)| \leq 2$. So $|V(K)| = 2$ and hence any maximal clique in G is maximum. Therefore G is a co-well covered graph. □

We know that bipartite graphs and cycles (with more than three vertices) are triangle free graphs. Thus we get the following corollaries.

Corollary 2.7. *Let G be a bipartite graph with $E(G) \neq \phi$. Then G is a co-well covered graph if and only if G has no isolated vertices.*

Corollary 2.8. *C_n is a co-well covered graph, for all $n \geq 3$.*

Proof. By Theorem 2.6 and Remark 2.5. □

3. Graph operations and co-well cover graphs

Now, we will study when graphs obtained by some graph operations on two graphs are co-well covered graphs.

We start by the graph join. For any two graphs G_1 and G_2. Let $G_1 + G_2$ be the join of G_1 and G_2.

Remark 3.1. Any clique in $G_1 + G_2$ has the form K_1, K_2 or $K_1 + K_2$, where K_1 and K_2 are cliques in G_1 and G_2 respectively. Moreover K_1 and K_2 are not maximal cliques in $G_1 + G_2$.

Theorem 3.2. *Let G_1 and G_2 be any two graphs. Then $K_1 + K_2$ is a maximal clique in $G_1 + G_2$ if and only if K_1 and K_2 are maximal cliques in G_1 and G_2 respectively.*

Proof. ($\Rightarrow$) Assume $K_1 + K_2$ is a maximal clique in $G_1 + G_2$. Since $K_1 + K_2$ is a clique in $G_1 + G_2$, then K_1 and K_2 are cliques in G_1 and G_2 respectively. We claim that K_1 and K_2 are maximal cliques in G_1 and G_2 respectively. Let $V(K_1) = \{v_1, v_2, \ldots, v_n\}$. Then $v_1, v_2, \ldots, v_n \in V(K_1 + K_2)$. Suppose K_1 is not a maximal clique in G_1. Then there exists $v \in V(G_1) - V(K_1)$ such that v is adjacent to v_i in G_1, for all $i = 1, 2, \ldots, n$. But v is adjacent to u in $G_1 + G_2$, for all $u \in V(G_2)$. Hence v is adjacent to u in $G_1 + G_2$, for all $u \in V(K_2)$. Therefore v is adjacent to u in $G_1 + G_2$, for all $u \in V(K_1 + K_2)$. But this is a contradiction, since $K_1 + K_2$ is a maximal clique in $G_1 + G_2$. Thus K_1 is a maximal clique in G_1. Similarly K_2 is a maximal clique in G_2.

($\Leftarrow$) Assume K_1 and K_2 are maximal cliques in G_1 and G_2 respectively such that $V(K_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_2) = \{u_1, u_2, \ldots, u_m\}$. Then $V(K_1 + K_2) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$ and it is clear that $K_1 + K_2$ is a clique in $G_1 + G_2$. Suppose that $K_1 + K_2$ is not a maximal clique in $G_1 + G_2$. Then there exists $v \in V(G_1 + G_2) - V(K_1 + K_2)$ such that v is adjacent to $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m$ in $G_1 + G_2$. Now since $v \in V(G_1 + G_2)$, then $v \in V(G_1)$ or $v \in V(G_2)$. If $v \in V(G_1)$, then since $v \notin V(K_1)$ and v is adjacent to $v_1, v_2, \ldots, v_n$ in G_1, we get K_1 is not a maximal clique in G_1, which is a contradiction. We get a similar contradiction if $v \in V(G_2)$. Therefore $K_1 + K_2$ is a maximal clique in $G_1 + G_2$. □

Corollary 3.3. *The graph $G_1 + G_2$ is a co-well covered graph if and only if G_1 and G_2 are co-well covered graphs. Moreover $\omega(G_1 + G_2) = \omega(G_1) + \omega(G_2)$.*

Now, we will discuss when the corona product of two graphs is a co-well covered graph. For any two graphs G and H. Let $G \circ H$ be the corona product of G and H.

Remark 3.4. Let G and H be any two graphs. Then for all $v \in V(G)$, let H_v be the copy of H in $G \circ H$ such that v is adjacent to u in $G \circ H$, for all $u \in V(H_v)$.

Theorem 3.5. *Let G and H be two graphs. Then $G \circ H$ is a co-well covered graph if and only if H is a co-well covered graph and for every component G_i of G either G_i is a co-well covered graph with $\omega(G_i) = \omega(H) + 1$ or G_i is an isolated vertex. Moreover $\omega(G \circ H) = \omega(H) + 1$.*

Proof. ($\Rightarrow$) Assume $G \circ H$ is a co-well covered graph. To prove that H is a co-well covered graph, it is enough to show that H_v is a co-well covered graph, and this is because $H_v \approx H$, for all $v \in V(G)$. Let K be a maximal clique in H_v, for some $v \in V(G)$. Then $K + \langle\{v\}\rangle$ is a maximal clique in $H_v + \langle\{v\}\rangle$. Hence $K + \langle\{v\}\rangle$ is a maximal clique in $G \circ H$. But $G \circ H$ is a co-well covered graph, therefore $K + \langle\{v\}\rangle$ is a maximum clique in $G \circ H$. Thus $K + \langle\{v\}\rangle$ is a maximum clique in $H_v + \langle\{v\}\rangle$. Hence K is a maximum clique in H_v. So H_v is a co-well covered graph. We get H is a co-well covered graph with $\omega(H) = \omega(G \circ H) - 1$. Now to show that either G_i is a co-well covered with $\omega(G_i) = \omega(H) + 1$ or G_i is an isolated vertex. Let G_i be any component of G with $|V(G_i)| \geq 2$. It is enough to show that G_i is a co-well covered graph with $\omega(G_i) = \omega(H) + 1$. Let K be a maximal clique in G_i. Then K is a maximal clique in $G \circ H$. Therefore K is a maximum clique in $G \circ H$. So K is a maximum clique in G_i. Thus G_i is a co-well covered graph with $\omega(G_i) = \omega(G \circ H) = \omega(H) + 1$.

($\Leftarrow$) Let K be a maximal clique in $G \circ H$. Then either K is contained in G_i, for some i and $|V(G_i)| \geq 2$ or K is contained in $H_v + \langle\{v\}\rangle$, for some $v \in V(G)$. This is because there is no $u \in V(H_v)$ and $w \in V(G - v)$ that are adjacent in $G \circ H$. Firstly, assume K is contained in G_i, for some i and $|V(G_i)| \geq 2$. Then K is a maximal clique in G_i. But G_i is a co-well covered graph. So K is a maximum clique in G_i. Therefore

$$|V(K)| = \omega(G_i) = \omega(H) + 1. \tag{1}$$

Secondly, assume K is contained in $H_v + \langle\{v\}\rangle$, for some $v \in V(G)$. Then K is a maximal clique in $H_v + \langle\{v\}\rangle$. So $K - v$ is a maximal clique in H_v. But H is a co-well covered graph and $H_v \approx H$. Then H_v is co-well covered graph. Therefore $K - v$ is a maximum clique in H_v, and we get K is a maximum clique in $H_v + \langle\{v\}\rangle$. Thus

$$|V(K)| = \omega(H_v) + 1 = \omega(H) + 1. \tag{2}$$

From (1) and (2), we get every maximal clique in $G \circ H$ has order $\omega(H) + 1$. Thus $G \circ H$ is a co-well covered graph and $\omega(G \circ H) = \omega(H) + 1$. □

Now, we will study when the cartesian product of two graphs is a co-well covered graph. For any two graphs G_1 and G_2. Let $G_1 \Box G_2$ be the cartesian product of G_1 and G_2.

Lemma 3.6. *Let G_1 and G_2 be any two graphs such that $G_1 \Box G_2$ has a maximal clique of order n. Then either G_1 or G_2 has a maximal clique of order n.*

Proof. Let K be a maximal clique in $G_1 \Box G_2$ of order n. Then either $V(K) = \{(v_1, u), (v_2, u), \ldots, (v_n, u)\}$, where $v_1, v_2, \ldots, v_n$ are mutually adjacent in G_1 and $u \in V(G_2)$ or $V(K) = \{(v, u_1), (v, u_2), \ldots, (v, u_n)\}$, where $v \in V(G_1)$ and $u_1, u_2, \ldots, u_n$ are mutually adjacent in G_2. If $V(K) = \{(v_1, u), (v_2, u), \ldots, (v_n, u)\}$, where $v_1, v_2, \ldots, v_n$ are mutually adjacent in G_1 and $u \in V(G_2)$, then let $K_1 = \langle\{v_1, v_2, \ldots, v_n\}\rangle$ in G_1. Observe that K_1 is a clique in G_1. Suppose K_1 is not a maximal clique in G_1, then there exists $v \in V(G_1 - K_1)$ such that v is adjacent to v_i, for all $i = 1, 2, \ldots, n$ in G_1. So (v, u) is adjacent to (v_i, u), for all $i = 1, 2, \ldots, n$ in $G_1 \Box G_2$. But this is a contradiction, since K is a maximal clique in $G_1 \Box G_2$. Thus K_1 is a maximal clique in G_1 of order n. If $V(K) = \{(v, u_1), (v, u_2), \ldots, (v, u_n)\}$, then similarly we show that G_2 has a maximal clique of order n. □

Lemma 3.7. *Let G_1 and G_2 be any two graphs such that G_1 or G_2 has a maximal clique of order n where $n \geq 2$. Then $G_1 \Box G_2$ has a maximal clique of order n.*

Proof. Without loss of generality, assume G_1 has a maximal clique K_1 of order n where $n \geq 2$, say $V(K_1) = \{v_1, v_2, \ldots, v_n\}$. Let $u \in V(G_2)$. Then $(v_1, u), (v_2, u), \ldots, (v_n, u)$ are mutually adjacent in $G_1 \Box G_2$. So,

$$\langle\{(v_1, u), (v_2, u), \ldots, (v_n, u)\}\rangle$$

is a clique in $G_1 \Box G_2$, say K. We claim that K is a maximal clique in $G_1 \Box G_2$. Assume K is not a maximal clique in $G_1 \Box G_2$, then there exists $(v, w) \in V(G_1 \Box G_2 - K)$ such that (v, w) is adjacent to (v_i, u), for all $i = 1, 2, \ldots, n$ in $G_1 \Box G_2$. So if $w = u$, then v is adjacent to v_i, for all $i = 1, 2, \ldots, n$ in G_1, which is a contradiction, since K_1 is a maximal clique in G_1. If $w \neq u$, then $v = v_i$, for all $i = 1, 2, \ldots, n$ in G_1, which is a contradiction, since $n \geq 2$. Thus K is a maximal clique in $G_1 \Box G_2$ and $|V(K)| = n$. □

Corollary 3.8. *Let G_1 and G_2 be any two graphs with $E(G_1) = \phi$. Then $G_1 \Box G_2$ is a co-well covered graph if and only if G_2 is a co-well covered graph.*

Theorem 3.9. *Let G_1 and G_2 be any two graphs with $E(G_1) \neq \phi$. Then $G_1 \Box G_2$ is a co-well covered graph with $\omega(G_1 \Box G_2) = n$ if and only if G_1 or G_2 has no isolated vertices and any maximal clique in G_1 and in G_2 has order 1 or n.*

Proof. ($\Leftarrow$) Since G_1 or G_2 has no isolated vertices, then $G_1 \Box G_2$ has no isolated vertices. Thus, if K is a maximal clique in $G_1 \Box G_2$ of order m, then $m \geq 2$. So by Lemma 3.6 G_1 or G_2 has a maximal clique of order m. Since any maximal clique in G_1 and in G_2 has order 1 or n and $m \geq 2$, then $m = n$. Hence $G_1 \Box G_2$ is a co-well covered graph with $\omega(G_1 \Box G_2) = n$.

($\Rightarrow$) Assume $G_1 \Box G_2$ is a co-well covered graph with $\omega(G_1 \Box G_2) = n$. Since $E(G_1) \neq \phi$. Then $E(G_1 \Box G_2) \neq \phi$. But $G_1 \Box G_2$ is a co-well covered graph, thus $G_1 \Box G_2$ has no isolated vertices. Therefore G_1 or G_2 has no isolated vertices. Let K_1 be a maximal clique in G_1 of order m. We want to show that $m = 1$ or $m = n$. Assume $m \neq 1$ and so by Lemma 3.7 $G_1 \Box G_2$ has a maximal clique of order m. But $G_1 \Box G_2$ is a co-well covered graph with $\omega(G_1 \Box G_2) = n$. Therefore $m = n$. Similarly, we show that any maximal clique in G_2 has order 1 or n. □

4. Powers of trees and cycles

Let G be any graph. The distance between any two vertices u and v in G is denoted by $d_G(u, v)$. For any positive integer k, the graph power G^k of G has vertex set $V(G)$ and any two distinct vertices u and v are adjacent in G^k if $d_G(u, v) \leq k$. Let $N_G^k[u] = \{v \in V(G) : d_G(u, v) \leq k\}$ which equals $N_{G^k}[u]$ and $N_G^k(u) = N_G^k[u] - \{u\}$.

Now we want to discuss when a power of a tree T is a co-well covered graph. Firstly, we will characterize when even powers of a tree T are co-well covered graphs. We start with the following lemma that characterizes maximal cliques in T^{2k}.

Lemma 4.1. *Let T be a tree with $diam(T) \geq 2k$. Then K is a maximal clique in T^{2k} if and only if $V(K) = N_T^k[u]$, for some $u \in V(T)$ and there exist $v_1, v_2 \in N_T^k[u]$ with $d_T(v_1, v_2) = 2k$.*

Proof. ($\Leftarrow$) Assume $u \in V(T)$ such that there exist $v_1, v_2 \in N_T^k[u]$ with $d_T(v_1, v_2) = 2k$. So $d_T(u, v_1) = k$ and $d_T(u, v_2) = k$. Now, let $S = N_T^k[u]$. Then $\langle S \rangle$ is clique in T^{2k}, say K. We want to show that $K = \langle S \rangle$ is a maximal clique in T^{2k}. Let $w \in V(T) - S$. Then $d_T(u, w) \geq k + 1$. Since T is a tree, then either $d_T(v_1, w) \geq 2k + 1$ or $d_T(v_2, w) \geq 2k + 1$. Thus v_1 and w or v_2 and w are not adjacent in T^{2k}. Therefore $K = \langle S \rangle$ is a maximal clique in T^{2k}.

($\Rightarrow$) Let K be a maximal clique in T^{2k}. Since $diam(T) \geq 2k$, then there exist $v_1, v_2 \in V(K)$ such that $d_T(v_1, v_2) = 2k$. So there exists $u \in V(K)$ such that $d_T(u, v_1) = k$ and $d_T(u, v_2) = k$. Now let $S = N_T^k[u]$. If $v \notin S$, then $d_T(u, v) \geq k + 1$. Thus $d_T(v_1, v) \geq 2k + 1$ or $d_T(v_2, v) \geq 2k + 1$. So $v \notin V(K)$. Thus $V(K) \subset S$, and hence $K \leq \langle S \rangle$ in T^{2k}. But $\langle S \rangle$ is a clique in T^{2k} That contains a maximal clique K in T^{2k}. Therefore $K = \langle S \rangle$ in T^{2k}. □

Now, the following theorem characterizes when even powers of a tree are co-well covered graphs.

Theorem 4.2. *Let T be a tree with $diam(T) \geq 2k$. Then T^{2k} is a co-well covered graph with $\omega(T^{2k}) = m$ if and only if whenever $u \in V(T)$ and there exist $v_1, v_2 \in N_T^k[u]$ with $d_T(v_1, v_2) = 2k$, then $\left|N_T^k[u]\right| = m$.*

Proof. ($\Leftarrow$) Let K be a maximal clique in T^{2k}. Then using Lemma 4.1 $V(K) = N_T^k[u]$, for some $u \in V(T)$ and there exist $v_1, v_2 \in N_T^k[u]$ with $d_T(v_1, v_2) = 2k$. Thus $\left|N_T^k[u]\right| = m$ and hence $|V(K)| = m$. Therefore T^{2k} is a co-well covered graph with $\omega(T^{2k}) = m$.

($\Rightarrow$) Assume $u \in V(T)$ and there exist $v_1, v_2 \in N_T^k[u]$ with $d_T(v_1, v_2) = 2k$. We claim that $\left|N_T^k[u]\right| = m$. Let $S = N_T^k[u]$. Then by Lemma 4.1 $\langle S\rangle$ is a maximal clique in T^{2k}. Since T^{2k} is a co-well covered graph with $\omega(T^{2k}) = m$. Then $|S| = \left|N_T^k[u]\right| = m$. $\square$

Secondly, we will characterize when odd powers of a tree are co-well covered graphs.

Lemma 4.3. *Let T be a tree with $diam(T) \geq 2k+1$. Then K is a maximal clique in T^{2k+1} if and only if $V(K) = N_T^k[u] \cup N_T^k[v]$, for some adjacent vertices u and v in T and there exist $u_1 \in N_T^k[u], v_1 \in N_T^k[v]$ with $d_T(u_1, v_1) = 2k+1$.*

Proof. ($\Leftarrow$) Assume u and v are adjacent vertices in T such that there exist $u_1 \in N_T^k[u], v_1 \in N_T^k[v]$ with $d_T(u_1, v_1) = 2k+1$, i.e., $d_T(u_1, u) = k$ and $d_T(v_1, v) = k$. Let $S = N_T^k[u] \cup N_T^k[v]$. Then $K = \langle S\rangle$ is a clique in T^{2k+1}. Now, let $w \in V(T) - S$. Then $d_T(u, w) \geq k+1$ and $d_T(v, w) \geq k+1$. Since u and v are adjacent in the tree T, then $|d_T(u, w) - d_T(v, w)| = 1$. Without loss of generality assume $d_T(u, w) < d_T(v, w)$. Then the unique path between w and v in T must contain u, and hence the unique path between w and v_1 in T must contain u. So $d_T(v_1, w) = d_T(u, w) + d_T(u, v_1) \geq k+1+k+1 = 2k+2$. Thus w and v_1 are not adjacent in T^{2k+1}. Therefore $K = \langle S\rangle$ is a maximal clique in T^{2k+1}.

($\Rightarrow$) Assume K is a maximal clique in T^{2k+1}. Since $diam\,(T) \geq 2k+1$, then there exist $u_1, v_1 \in V(K)$ such that $d_T(u_1, v_1) = 2k+1$. Let P be the path between u_1 and v_1 in T. Since the length of P is odd, then $|C(P)| = 2$, where $C(P)$ is the center of the path P. Now, let $C(P) = \{u, v\}$ such that $d_T(u_1, u) = k$ and $d_T(v_1, v) = k$ and let $S = N_T^k[u] \cup N_T^k[v]$. We claim that $K = \langle S\rangle$ in T^{2k+1}. Suppose $w \notin S$. Then $d_T(u_1, w) \geq 2k+2$ or $d_T(v_1, w) \geq 2k+2$ as above. Thus either w and u_1 or w and v_1 are not adjacent in T^{2k+1}. Therefore $w \notin V(K)$. Thus $V(K) \subset S$ and hence $K \leq \langle S\rangle$ in T^{2k+1}. But $\langle S\rangle$ is a clique in T^{2k+1}. Therefore $K = \langle S\rangle$ in T^{2k+1}. $\square$

Now, the following theorem characterizes when odd powers of a tree are co-well covered graphs.

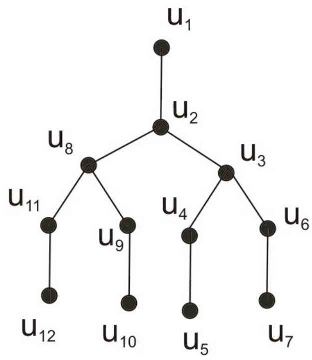
Theorem 4.4. *Let T be a tree with $diam\,(T) \geq 2k+1$. Then T^{2k+1} is co-well covered graph with $\omega(T^{2k+1}) = m$ if and only if whenever u and v are adjacent vertices in T and there exist $u_1 \in N_T^k[u], v_1 \in N_T^k[v]$ with $d_T(u_1, v_1) = 2k+1$, then $\left|N_T^k[u] \cup N_T^k[v]\right| = m$.*

Proof. ($\Leftarrow$) Let K be a maximal clique in T^{2k+1}. Then by Lemma 4.3 $V(K) = N_T^k[u] \cup N_T^k[v]$, for some adjacent vertices u and v in T and there exist $u_1 \in$

$N_T^k[u], v_1 \in N_T^k[v]$ with $d_T(u_1, v_1) = 2k+1$. So $|V(K)| = \left|N_T^k[u] \cup N_T^k[v]\right| = m$. Therefore T^{2k+1} is a co-well covered graph with $\omega\left(T^{2k+1}\right) = m$.

($\Rightarrow$) Assume u and v are adjacent vertices in T and there exist $u_1 \in N_T^k[u], v_1 \in N_T^k[v]$ with $d_T(u_1, v_1) = 2k+1$. Let $S = N_T^k[u] \cup N_T^k[v]$. Then using Lemma 4.3 $\langle S \rangle$ is a maximal clique in T^{2k+1}. Since T^{2k+1} is a co-well covered graph with $\omega(T^{2k+1}) = m$. Then $|S| = \left|N_T^k[u] \cup N_T^k[v]\right| = m$. □

Example 4.5. The following figure is a tree T. We will show that T^4 and T^5 are co-well covered graphs, whereas T^2 and T^3 are not co-well covered graphs.



A tree T

Note that $u_3, u_7 \in N_T[u_6]$ with $d_T(u_3, u_7) = 2$ and $u_2, u_4 \in N_T[u_3]$ with $d_T(u_2, u_4) = 2$. But $|N_T[u_6]| = 3$ and $|N_T[u_3]| = 4$ which are not equal. So by Theorem 4.2 T^2 is not a co-well covered graph. Note that u_3, u_6 are adjacent vertices in T and $u_2, u_7 \in N_T[u_3] \cup N_T[u_6]$ with $d_T(u_2, u_7) = 3$. Also, u_2, u_8 are adjacent vertices in T and $u_1, u_9 \in N_T[u_2] \cup N_T[u_8]$ with $d_T(u_1, u_9) = 3$. But $|N_T[u_3] \cup N_T[u_6]| = 5$ and $|N_T[u_2] \cup N_T[u_8]| = 6$ which are not equal. So by Theorem 4.4 T^3 is not a co-well covered graph. The graph T^4 is a co-well covered graph. To show that, let $u \in V(T)$. Then we need to compute $\left|N_T^2[u]\right|$ whenever there exist $v_1, v_2 \in N_T^2[u]$ with $d_T(v_1, v_2) = 4$. This holds only for $N_T^2[u_2]$, $N_T^2[u_3]$ and $N_T^2[u_8]$ and we have $\left|N_T^2[u_2]\right| = \left|N_T^2[u_3]\right| = \left|N_T^2[u_8]\right| = 8$. Thus by Theorem 4.2 T^4 is a co-well covered graph with $\omega(T^4) = 8$. Also T^5 is a co-well covered graph. To show that, let u and v be adjacent vertices in T. Then we need to compute $\left|N_T^2[u] \cup N_T^2[v]\right|$ whenever there exist $v_1, v_2 \in N_T^2[u] \cup N_T^2[v]$ and we have $d_T(v_1, v_2) = 5$. This holds only for $N_T^2[u_2] \cup N_T^2[u_3]$ and $N_T^2[u_2] \cup N_T^2[u_8]$ with $\left|N_T^2[u_2] \cup N_T^2[u_3]\right| = \left|N_T^2[u_2] \cup N_T^2[u_8]\right| = 10$. Therefore by Theorem 4.4 T^5 is a co-well covered graph with $\omega(T^5) = 10$.

Now, we will prove that any power C_n^k of a cycle C_n is a co-well covered graph. First observe that if $k \geq \frac{n-1}{2}$, then C_n^k is a complete graph which is a co-well covered graph. In the following theorem, we will prove that C_n^k is a co-well covered graph whenever $1 < k < \frac{n-1}{2}$.

Theorem 4.6. *Let C_n be the cycle on n vertices. Then C_n^k is a co-well covered graph with $\omega(C_n^k) = k+1$, where $1 < k < \frac{n-1}{2}$.*

Proof. Let $C_n : u_0u_1 \cdots u_{n-1}u_0$, and let K be a maximal clique in C_n^k. Then there exist $u, v \in V(K)$ such that $d_{C_n}(u,v) = k$. Suppose that $u = u_i$, $v = u_{i+k}$ where $i+k$ is taken mod(n) and $0 \leq i \leq n-1$. Now, we can rename the vertices of C_n as $v_j = u_{i+j-1}$ where $i+j-1$ is taken mod (n) and $1 \leq j \leq n$. Thus $C_n : v_1v_2 \cdots v_nv_1$, where $v_1 = u_i$, $v_{k+1} = u_{i+k}$, $i+k$ is taken mod(n) and $v_1, v_{k+1} \in V(K)$. Let $S = \{v_1, v_2, \ldots, v_{k+1}\}$. Then $\langle S\rangle$ is a clique in C_n^k which is contained in K. We want to show that any vertex of the set $\{v_j : \ k+2 \leq j \leq n\}$ is not in $V(K)$. Firstly, there are two paths in C_n between v_i and v_{i+k+1}, for all $i = 1, 2, \ldots, k+1$. The first path is $v_i, v_{i+1}, \ldots, v_{i+k+1}$ of length $k+1$ and the second path $C_n - \{v_{i+1}, v_{i+2}, \ldots, v_{i+k}\}$ of length $n-(k+1)$. Since $n > 2k+1$, then $n-(k+1) > k$, and thus $d_{C_n}(v_i, v_{i+k+1}) = k+1$, for all $i = 1, 2, \ldots, k+1$. Therefore

$$v_j \notin V(K), \text{ for all } j = k+2, k+3, \ldots, 2k+2. \tag{3}$$

Secondly, $d_{C_n}(v_{2k+1+i}, v_{k+1}) > k$, for all $i = 1, 2, \ldots, n-2k-1$. Thus

$$v_j \notin V(K), \text{ for all } j = 2k+2, 2k+3, \ldots, n. \tag{4}$$

Therefore from (3) and (4), we get $V(K) \subset S$. But K is a maximal clique in C_n^k, then $K = \langle S\rangle$ in C_n^k and $|V(K)| = |S| = k+1$. Thus C_n^k is a co-well covered graph and $\omega(C_n^k) = k+1$. □

5. Line graph of a graph

Now, we will study when the line graph $L(G)$ of a graph G is a co-well covered graph.

First observe that if G is a connected graph with at least two edges, then there are two kinds of maximal cliques of $L(G)$, cliques resulting from stars and cliques resulting from triangles in G. Firstly, if K is a maximal clique in $L(G)$ that results from a star in G, then there exists $u \in V(G)$ with either $\deg(u) \geq 3$ or $\deg(u) = 2$ and u is not contained in a triangle in G. Thus $V(K) = \{e \in E(G) : e \text{ is incident with } u\}$, say S_u. Secondly, if K is a maximal clique in $L(G)$ that results from a triangle in G, then it is clear that K is a triangle in $L(G)$.

Now, the following two theorems address when the line graph of a graph is a co-well covered graph.

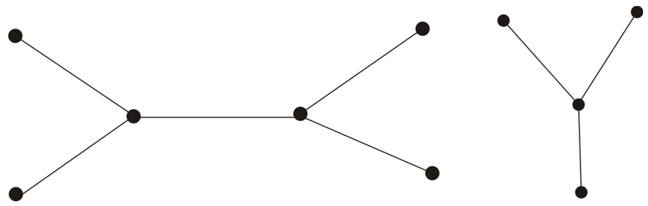
Theorem 5.1. *Let G be a triangle-free graph with $E(G) \neq \emptyset$. Then the line graph $L(G)$ is a co-well covered graph with $\omega(L(G)) = m$ if and only if for each component G_i of G that has more than one vertex (i.e., which is not an isolated vertex), we have $\deg(u) \in \{1, m\}$ for all $u \in V(G_i)$ and at least one of these vertices has degree m.*

Proof. If $m = 1$, then each vertex in G has degree either 0 or 1. Hence each component of G is a complete graph of order 1 or 2. So $L(G)$ is the null graph which is a co-well covered graph.

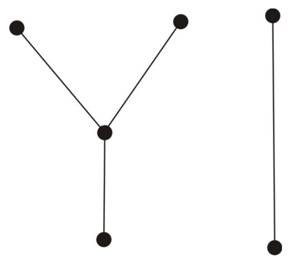
Suppose $m \geq 2$. ($\Leftarrow$) Let K be a maximal clique in $L(G)$. Since G has no triangle, then K is a maximal clique in $L(G)$ that results from a star in G. So $V(K) = \{e \in E(G) : e \text{ is incident with } u\}$, for some $u \in V(G)$ such that $\deg(u) \geq 2$ in G. Therefore $\deg(u) = m$ in G. Thus $|V(K)| = m$, and so $L(G)$ is a co-well covered graph.

($\Rightarrow$) Assume $L(G)$ is a co-well covered graph with $\omega(L(G)) = m$. Then for each component $L(G_i)$ of $L(G)$, we have $\omega(L(G_i)) = m \geq 2$. So for each component $L(G_i)$ of $L(G)$, there exists $u_i \in V(G_i)$ such that $\deg(u_i) = m \geq 2$. Now we want to show that if $\deg(v) \geq 2$ in G, then $\deg(v) = m$ in G. Let $v \in V(G)$ with $\deg(v) \geq 2$, and let $S_v = \{e \in E(G) : e \text{ is incident with } v\}$. Then $\langle S_v \rangle$ is a maximal clique in $L(G)$, since $L(G)$ is a co-well covered graph, then $|S_v| = m$. Therefore $\deg(v) = m$. □

Example 5.2. The following figures are the graphs G and H.



G



H

Since G is triangle free, $\deg(u) \in \{1, 3\}$ for all $u \in V(G)$ and each component of G has at least one vertex of degree 3, then by Theorem 5.1 $L(G)$ is a co-well covered graph. The graph H is triangle free. But one of the two components of H has a vertex of degree 3 and the other component has no vertices of degree 3. So by Theorem 5.1 $L(H)$ is not a co-well covered graph.

Theorem 5.3. *Let G be a graph with at least one triangle. Then the line graph $L(G)$ is a co-well covered graph if and only if* $\deg(u) \le 3$, *for all* $u \in V(G)$ *and whenever* $\deg(u) = 2$, *then* u *must be contained in a triangle of* G, *and* G *has no components of order* 2.

Proof. ($\Leftarrow$) Let K be a maximal clique in $L(G)$. Then K either results from a star in G or results from a triangle in G. Firstly, if K results from a triangle in G, then $|V(K)| = 3$. Secondly, if K results from a star in G, then $V(K) = \{e \in E(G) : e \text{ is incident with } u\}$, for some $u \in V(G)$ such that either $\deg(u) \ge 3$ in G or $\deg(u) = 2$ in G such that u is not contained in a triangle in G. But according to the assumption $\deg(u) \le 3$ and whenever $\deg(u) = 2$, then u is contained in a triangle of G, we get that $\deg(u) = 3$ in G. Thus $|V(K)| = 3$. Therefore $L(G)$ is a co-well covered graph with $\omega(L(G) = 3$.

($\Rightarrow$) Assume $L(G)$ is a co-well covered graph. Since G has a triangle, then $L(G)$ has a maximal clique of order 3. But $L(G)$ is a co-well covered graph, then $\omega(L(G)) = 3$. We claim that $\deg(u) \le 3$, for all $u \in V(G)$. Suppose there exists $v \in V(G)$ such that $\deg(v) \ge 4$ in G, and let $S_v = \{e \in E(G) : e \text{ is incident with } v\}$. Then $\langle S_v \rangle$ is a maximal clique in $L(G)$ with $|S_v| \ge 4$. But this is a contradiction, because $L(G)$ is a co-well covered graph with $\omega(L(G)) = 3$. Now, let $v \in V(G)$ with $\deg(v) = 2$ in G. We claim that v is contained in a triangle in G. Suppose v is not contained in a triangle in G, and let $S_v = \{e \in E(G) : e \text{ is incident with } v\}$. Then $\langle S_v \rangle$ is a maximal clique in $L(G)$ with $|S_v| = 2$. But this is a contradiction, because $L(G)$ is a co-well covered graph with $\omega(L(G)) = 3$. Finally, we will prove that G has no components of order 2. Suppose G has a component of order 2, say G_1. Then $L(G_1)$ is a component of $L(G)$ of order 1 and hence $L(G)$ has a maximal clique of order 1. But again this is a contradiction, because $L(G)$ is a co-well covered graph with $\omega(L(G)) = 3$. □

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Baha' Abughazaleh
Department of Mathematics
The University of Jordan
Amman 11942, Jordan
Email address: babughazaleh@uop.edu.jo

Omar Abughneim
Department of Mathematics
The University of Jordan
Amman 11942, Jordan
Email address: o.abughneim@ju.edu.jo

Hasan Al-Ezeh
Department of Mathematics
The University of Jordan
Amman 11942, Jordan
Email address: alezehh@ju.edu.jo