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OPTIMAL ERROR ESTIMATE OF A DECOUPLED CONSERVATIVE LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE KLEIN-GORDON-SCHRÖDINGER EQUATIONS

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ABSTRACT. In this paper, we propose a decoupled local discontinuous Galerkin method for solving the Klein-Gordon-Schrödinger (KGS) equations. The KGS equations is a model of the Yukawa interaction of complex scalar nucleons and real scalar mesons. The advantage of our scheme is that the computation of the nucleon and meson field is fully decoupled, so that it is especially suitable for parallel computing. We present the conservation property of our fully discrete scheme, including the energy and Hamiltonian conservation, and establish the optimal error estimate.

1. INTRODUCTION

The Klein-Gordon-Schrödinger (KGS) Eqs. (1.1)

$$\begin{cases} i\psi_t + \psi_{xx} + \phi\psi = 0, \quad x \in I, \quad t > 0, \\ \phi_{tt} - \phi_{xx} + \phi - |\psi|^2 = 0, \quad x \in I, \quad t > 0, \end{cases} \tag{1.1}$$

is a classical model of the Yukawa interaction of the complex scalar nucleon field $\psi(x,t)$ and the real scalar meson field $\phi(x,t)$. In this paper, we consider the Eqs. (1.1) with initial conditions for $\psi(x,0)$, $\phi(x,0)$, $\phi_t(x,0)$, and periodic boundary conditions on ψ and ϕ. If we denote the real and imaginary part of the nucleon field $\psi(x,t)$ as $p(x,t)$ and $q(x,t)$, respectively, the KGS equations can be written as

$$\begin{cases} q_t = p_{xx} + \phi p, \quad p_t = -q_{xx} - \phi q, \\ \phi_{tt} = \phi_{xx} - \phi + p^2 + q^2, \end{cases} \tag{1.2}$$

There are two important conserved quantities for the KGS Eqs. (1.2), namely, the wave energy

$$\int_I \left(p^2 + q^2\right) dx, \tag{1.3}$$

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and the Hamiltonian

$$\int_I \left(\frac{1}{2}\phi^2 + \frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 + p_x^2 + q_x^2 - (p^2 + q^2)\phi \right) dx. \tag{1.4}$$

Accurate numerical methods of the KGS equations that preserve the invariant quantities (1.3) and (1.4) are of fundamental importance.

There has been numerous studies to investigate the numerical methods for the KGS equations. The finite difference methods for the KGS equations can be referred to [1, 2, 3, 4], and the spectral methods can be referred to [5, 6, 7, 8, 9, 10, 11]. Alternative numerical methods include the multi-symplectic schemes [12], the Sinc collocation methods [13], the generalized moving least squares methods [14], the spline collocation methods [15] and the local discontinuous Galerkin methods [16]. In particular, the author in [16] proposed energy- and Hamiltonian-preserving local discontinuous Galerkin methods for the KGS equations. The semi-discrete schemes are proved to conserve the energy and Hamiltonian exactly, and the conservation of the fully discrete Hamiltonian depends on the time step size Δt. Due to the fact that the full discrete schemes in [16] lead to undecoupled system about ψ and ϕ, one cannot update the numerical solutions of ψ and ϕ simultaneously. In this paper, we propose a new fully discrete local discontinuous Galerkin method such that the computation of the nucleon and meson field is fully decoupled, which is suitable for parallel computing at each time step.

Our scheme is based on the local discontinuous Galerkin (LDG) methods which were originally developed to solve a convection-diffusion equation in Cockburn and Shu's work [17]. The LDG methods were then used to solve KdV-type equations [18], PDEs with higher order derivatives [19], nonlinear wave equations [20, 21]. The LDG methods share the advantages of the standard discontinuous Galerkin methods, including high-order accuracy, suitability for parallel computing and complicated geometry. In addition, in recent years much attention has been paid to develop invariant-conserving LDG methods for various PDEs [22, 23, 24, 16]. In this paper, we design a fully discrete LDG method based on the semi-discrete LDG methods in [16], analyze the conservation property, and prove the optimal error estimate of the scheme. The key component of the error estimate is the assumption about the L^∞ norm of the numerical solutions. Such an assumption leads to the optimal error estimate of our fully discrete scheme, and it is further proved by mathematical induction in the time step. Numerical results show that our new fully discrete scheme leads to high-order, energy- and Hamiltonian-preserving numerical solutions.

The rest of the paper is organized as follows. In section 2, we present our fully discrete LDG method for the Klein-Gordon-Schrödinger equations, and prove the property of energy- and Hamiltonian conservation. In section 3, we establish the optimal error estimate of our proposed scheme. Some numerical results are given in section 5.

2. Fully Discrete Local Discontinuous Galerkin Methods

2.1. **Preliminaries.** Throughout this paper, we denote the L^2-Sobolev space of order m by H^m whose norm is represented by $\|\cdot\|_m$. When $m = 0$, the Sobolev space H^0 becomes L^2 whose equipped norm is simply $\|\cdot\|$. The L^∞-Sobolev space of order m is denoted by $W^{m,\infty}$

whose norm is $\|\cdot\|_{m,\infty}$. When $m=0$, we use $\|\cdot\|_\infty$ instead of $\|\cdot\|_{m,\infty}$. We consider the computational domain of the KGS equations to be $I := [a,b]$. Let $\{I_j\}_{j=1}^N$ to be a partition of I, where $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1,\dots,N$ with $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{\frac{2N+1}{2}} = b$. We denote the length of each interval I_j to be h_j, and the mesh size h is defined as the maximum of all h_j. We assume that our mesh is regular, such that $\max h_j / \min h_j$ is uniformly bounded with mesh refinement. Let $P^k(I_j)$ be a polynomial space of degree at most k on element I_j, and our approximation space is $V_h^k = \{v : v|_{I_j} \in P^k(I_j), \forall j\}$ which is a piecewise defined polynomial space. Note that for any $v \in V_h^k$, there are two values of v from both sides of $x_{j+\frac{1}{2}}$. Let $v^+_{j+\frac{1}{2}} = \lim_{\epsilon\to 0+} v(x_{j+\frac{1}{2}}+\epsilon)$, and $v^-_{j+\frac{1}{2}} = \lim_{\epsilon\to 0+} v(x_{j+\frac{1}{2}}-\epsilon)$. The jump of v at $x_{j+\frac{1}{2}}$ is denoted by $[v]_{j+\frac{1}{2}} = v^+_{j+\frac{1}{2}} - v^-_{j+\frac{1}{2}}$, and the average of v is denoted by $\{v\}_{j+\frac{1}{2}} = (v^+_{j+\frac{1}{2}} + v^-_{j+\frac{1}{2}})/2$. For any $f, g \in V_h^k$, the jumps of f and g satisfy the following equality

$$[fg]_{j+\frac{1}{2}} - f^-_{j+\frac{1}{2}}[g]_{j+\frac{1}{2}} - g^+_{j+\frac{1}{2}}[f]_{j+\frac{1}{2}} = 0. \tag{2.1}$$

We will use Gauss-Radau projections to define our scheme and establish error estimate. For any $v \in H^1(I)$, there are two types of Gauss-Radau projections of v onto V_h^k, i.e., $\Pi^+ v$ and $\Pi^- v$, defined as follows

$$\begin{aligned}
\Pi^+ : \quad & \textstyle\int_{I_j} (\Pi^+ v) w\, dx = \int_{I_j} vw\, dx, \quad \forall w \in P^{k-1}(I_j); \quad (\Pi^+ v)^+_{j-\frac{1}{2}} = v^+_{j-\frac{1}{2}}, \quad \forall j.\\
\Pi^- : \quad & \textstyle\int_{I_j} (\Pi^- v) w\, dx = \int_{I_j} vw\, dx, \quad \forall w \in P^{k-1}(I_j); \quad (\Pi^- v)^-_{j-\frac{1}{2}} = v^-_{j+\frac{1}{2}}, \quad \forall j.
\end{aligned}$$

The Gauss-Radau projections defined above, and the standard L^2 projection share some important approximation properties (2.2). For any $\mathcal{Q} = \Pi^+, \Pi^-$ or a standard L^2 projection, and for any sufficiently smooth function f, there are

$$\|f - \mathcal{Q}f\| \le Ch^{k+1}\|f\|_{k+1}, \quad \|f - \mathcal{Q}f\|_\infty \le Ch^{k+1}\|f\|_{k+1,\infty}. \tag{2.2}$$

Here C is a generic positive constant that is independent of h and f. Let g be any polynomial on any element I_j, then the inverse inequality is given by

$$\|g\|_\infty \le Ch^{-\frac{1}{2}}\|g\|. \tag{2.3}$$

Note that the $\|\cdot\|_\infty$ and $\|\cdot\|$ are two norms on I_j.

2.2. **Decoupled conservative local discontinuous Galerkin method.** We now define the decoupled conservative local discontinuous Galerkin method for Eqs. (1.2). To derive the LDG method, we first rewrite Eqs. (1.2) as a first-order PDE system:

$$\begin{cases}
q_t = u_x + \phi p, \quad p_t = -z_x - \phi q,\\
\phi_{tt} = s_x - \phi + p^2 + q^2,\\
u = p_x, \quad z = q_x, \quad s = \phi_x.
\end{cases} \tag{2.4}$$

We then obtain the semi-discrete LDG method of KGS equations based on (2.4). That is, we look for q_h, p_h, ϕ_h, u_h, z_h and $s_h \in V_h^k$, such that

$$
\begin{aligned}
\int_{I_j} q_{h,t} w_1 dx &= -\int_{I_j} u_h (w_1)_x dx + (\widehat{u_h})_{j+\frac{1}{2}} (w_1)^-_{j+\frac{1}{2}} - (\widehat{u_h})_{j-\frac{1}{2}} (w_1)^+_{j-\frac{1}{2}} \\
&\quad + \int_{I_j} \phi_h p_h w_1 dx, \\
\int_{I_j} p_{h,t} w_2 dx &= \int_{I_j} z_h (w_2)_x dx - (\widehat{z_h})_{j+\frac{1}{2}} (w_2)^-_{j+\frac{1}{2}} + (\widehat{z_h})_{j-\frac{1}{2}} (w_2)^+_{j-\frac{1}{2}} - \int_{I_j} \phi_h q_h w_2 dx, \\
\int_{I_j} \phi_{h,tt} w_3 dx &= -\int_{I_j} s_h (w_3)_x dx + (\widehat{s_h})_{j+\frac{1}{2}} (w_3)^-_{j+\frac{1}{2}} - (\widehat{s_h})_{j-\frac{1}{2}} (w_3)^+_{j-\frac{1}{2}} \\
&\quad + \int_{I_j} (p_h^2 + q_h^2) w_3 dx, \\
\int_{I_j} u_h w_4 dx &= -\int_{I_j} p_h (w_4)_x dx + (\widehat{p_h})_{j+\frac{1}{2}} (w_4)^-_{j+\frac{1}{2}} - (\widehat{p_h})_{j-\frac{1}{2}} (w_4)^+_{j-\frac{1}{2}}, \\
\int_{I_j} z_h w_5 dx &= -\int_{I_j} q_h (w_5)_x dx + (\widehat{q_h})_{j+\frac{1}{2}} (w_5)^-_{j+\frac{1}{2}} - (\widehat{q_h})_{j-\frac{1}{2}} (w_5)^+_{j-\frac{1}{2}}, \\
\int_{I_j} s_h w_6 dx &= -\int_{I_j} \phi_h (w_6)_x dx + (\widehat{\phi_h})_{j+\frac{1}{2}} (w_6)^-_{j+\frac{1}{2}} - (\widehat{\phi_h})_{j-\frac{1}{2}} (w_6)^+_{j-\frac{1}{2}},
\end{aligned}
$$

for all test functions $w_1, w_2, \ldots, w_6 \in V_h^k$ and all j. Here $q_{h,t}$ is the temporal derivative of q_h, and similar definitions hold for $p_{h,t}$ and $\phi_{h,tt}$. In addition, $\widehat{u_h}$, $\widehat{z_h}$, $\widehat{s_h}$, $\widehat{p_h}$, $\widehat{q_h}$ and $\widehat{\phi_h}$ are numerical fluxes, which are critical for the numerical stability and the order of convergence. One suitable choice of numerical fluxes is

$$
\widehat{u_h} = u_h^-, \quad \widehat{z_h} = z_h^-, \quad \widehat{p_h} = p_h^+, \quad \widehat{q_h} = q_h^+, \quad \widehat{s_h} = s_h^-, \quad \widehat{\phi_h} = \phi_h^+. \tag{2.5}
$$

Such a choice is not unique. As long as the flux $\widehat{u_h}$ and $\widehat{z_h}$ are taken from the same direction, $\widehat{p_h}$ and $\widehat{q_h}$ are taken from the same direction, $\widehat{u_h}$ and $\widehat{p_h}$ are taken from the opposite direction, and $\widehat{s_h}$ and $\widehat{\phi_h}$ are taken from the opposite direction, the semi-discrete energy and Hamiltonian can be conserved exactly [16].

The decoupled fully discrete local discontinuous Galerkin scheme is defined based on the semi-discrete scheme above with (2.5). Let ϕ_h^n, s_h^n, p_h^n, q_h^n, u_h^n and z_h^n be the numerical solutions at time $t^n = n\Delta t$, where Δt is the time step size. Let ϕ^n, s^n, p^n, q^n, u^n and z^n be the exact solutions at t^n. Our scheme consists of the following steps:

Step 1. Initialize numerical solutions at t^0 and t^1:

$$
u_h^0 = \Pi^- u^0, \quad z_h^0 = \Pi^- z^0, \quad p_h^0 = \Pi^+ p^0, \quad q_h^0 = \Pi^+ q^0, \quad s_h^0 = \Pi^- s^0, \quad \phi_h^0 = \Pi^+ \phi^0,
$$

$$
\phi_h^1 = \phi_h^0 + (\Delta t)\, \Pi^+ \phi_t^0 + \frac{(\Delta t)^2}{2} \Pi^+ \left((\phi^0)_{xx} - \phi^0 + (p^0)^2 + (q^0)^2 \right).
$$

$(q_h^1, p_h^1, u_h^1, z_h^1)$ is then updated by

$$\begin{cases} \displaystyle\int_{I_j} \frac{q_h^1 - q_h^0}{\Delta t} w_1 dx = -\frac{1}{2}\int_{I_j} (u_h^1 + u_h^0)(w_1)_x dx + \frac{1}{2}(u_h^1 + u_h^0)^-_{j+\frac{1}{2}}(w_1)^-_{j+\frac{1}{2}} \\ \qquad - \displaystyle\frac{1}{2}(u_h^1 + u_h^0)^-_{j-\frac{1}{2}}(w_1)^+_{j-\frac{1}{2}} + \frac{1}{4}\int_{I_j} (\phi_h^1 + \phi_h^0)(p_h^1 + p_h^0) w_1 dx, \\ \displaystyle\int_{I_j} \frac{p_h^1 - p_h^0}{\Delta t} w_2 dx = \frac{1}{2}\int_{I_j} (z_h^1 + z_h^0)(w_2)_x dx - \frac{1}{2}(z_h^1 + z_h^0)^-_{j+\frac{1}{2}}(w_2)^-_{j+\frac{1}{2}} \\ \qquad + \displaystyle\frac{1}{2}(z_h^1 + z_h^0)^-_{j-\frac{1}{2}}(w_2)^+_{j-\frac{1}{2}} - \frac{1}{4}\int_{I_j} (\phi_h^1 + \phi_h^0)(q_h^1 + q_h^0) w_2 dx, \\ \displaystyle\int_{I_j} u_h^1 w_3 dx = -\int_{I_j} p_h^1 (w_3)_x dx + (p_h^1)^+_{j+\frac{1}{2}}(w_3)^-_{j+\frac{1}{2}} - (p_h^1)^+_{j-\frac{1}{2}}(w_3)^+_{j-\frac{1}{2}}, \\ \displaystyle\int_{I_j} z_h^1 w_4 dx = -\int_{I_j} q_h^1 (w_4)_x dx + (q_h^1)^+_{j+\frac{1}{2}}(w_4)^-_{j+\frac{1}{2}} - (q_h^1)^+_{j-\frac{1}{2}}(w_4)^+_{j-\frac{1}{2}}, \end{cases} \tag{2.6}$$

For $n = 1, 2, \ldots$, the numerical solutions $(q_h^{n+1}, p_h^{n+1}, \phi_h^{n+1})$ are updated in step 2 and 3.

Step 2. Update q_h^{n+1}, p_h^{n+1}, u_h^{n+1} and $z_h^{n+1} \in V_h^k$:

$$\begin{cases} \displaystyle\int_{I_j} \frac{q_h^{n+1} - q_h^{n-1}}{2\Delta t} w_1 dx = -\frac{1}{2}\int_{I_j} (u_h^{n+1} + u_h^{n-1})(w_1)_x dx + \frac{1}{2}(u_h^{n+1} + u_h^{n-1})^-_{j+\frac{1}{2}}(w_1)^-_{j+\frac{1}{2}} \\ \qquad - \displaystyle\frac{1}{2}(u_h^{n+1} + u_h^{n-1})^-_{j-\frac{1}{2}}(w_1)^+_{j-\frac{1}{2}} + \frac{1}{2}\int_{I_j} \phi_h^n (p_h^{n+1} + p_h^{n-1}) w_1 dx, \\ \displaystyle\int_{I_j} \frac{p_h^{n+1} - p_h^{n-1}}{2\Delta t} w_2 dx = \frac{1}{2}\int_{I_j} (z_h^{n+1} + z_h^{n-1})(w_2)_x dx - \frac{1}{2}(z_h^{n+1} + z_h^{n-1})^-_{j+\frac{1}{2}}(w_2)^-_{j+\frac{1}{2}} \\ \qquad + \displaystyle\frac{1}{2}(z_h^{n+1} + z_h^{n-1})^-_{j-\frac{1}{2}}(w_2)^+_{j-\frac{1}{2}} - \frac{1}{2}\int_{I_j} \phi_h^n (q_h^{n+1} + q_h^{n-1}) w_2 dx, \\ \displaystyle\int_{I_j} u_h^{n+1} w_3 dx = -\int_{I_j} p_h^{n+1} (w_3)_x dx + (p_h^{n+1})^+_{j+\frac{1}{2}}(w_3)^-_{j+\frac{1}{2}} - (p_h^{n+1})^+_{j-\frac{1}{2}}(w_3)^+_{j-\frac{1}{2}}, \\ \displaystyle\int_{I_j} z_h^{n+1} w_4 dx = -\int_{I_j} q_h^{n+1} (w_4)_x dx + (q_h^{n+1})^+_{j+\frac{1}{2}}(w_4)^-_{j+\frac{1}{2}} - (q_h^{n+1})^+_{j-\frac{1}{2}}(w_4)^+_{j-\frac{1}{2}}, \end{cases} \tag{2.7}$$

for any w_1, w_2, w_3 and $w_4 \in V_h^k$.

Step 3. Update ϕ_h^{n+1} and $s_h^{n+1} \in V_h^k$:

$$\begin{cases} \displaystyle\int_{I_j} \frac{\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}}{(\Delta t)^2} v\, dx = -\frac{1}{2}\int_{I_j} (s_h^{n+1} + s_h^{n-1}) v_x dx + \frac{1}{2}(s_h^{n+1} + s_h^{n-1})^-_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} \\ \qquad \displaystyle - \frac{1}{2}(s_h^{n+1} + s_h^{n-1})^-_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} - \frac{1}{2}\int_{I_j} (\phi_h^{n+1} + \phi_h^{n-1}) v\, dx + \int_{I_j} \left((p_h^n)^2 + (q_h^n)^2\right) v\, dx, \\ \displaystyle\int_{I_j} s_h^{n+1} w\, dx = -\int_{I_j} \phi_h^{n+1} w_x dx + (\phi_h^{n+1})^+_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - (\phi_h^{n+1})^+_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}}, \end{cases} \tag{2.8}$$

for any v and $w \in V_h^k$.

Note that the step 2 and step 3 of our proposed fully discrete scheme are decoupled. In fact, step 1 leads to u_h^n, z_h^n, p_h^n, q_h^n, s_h^n and ϕ_h^n for $n = 0$ and 1. In step 2, we update u_h^2, z_h^2, p_h^2 and q_h^2 using u_h^0, z_h^0, p_h^0, q_h^0 and ϕ_h^1, which are obtained from step 1. In step 3, we update ϕ_h^2 and s_h^2 using p_h^1, q_h^1, ϕ_h^0 and s_h^0, which are also obtained from step 1. Since step 3 does not require the results from step 2, these two steps are decoupled for $n = 1$. For general n, suppose we have already computed all the numerical solutions for all the time level $\leq n$, then step 2 is to update u_h^{n+1}, z_h^{n+1}, p_h^{n+1} and q_h^{n+1} using ϕ_h^n, u_h^{n-1}, z_h^{n-1}, p_h^{n-1} and q_h^{n-1}, and step 3 is to update ϕ_h^{n+1} and s_h^{n+1} using p_h^n, q_h^n, ϕ_h^n, ϕ_h^{n-1} and s_h^{n-1}. Again, the step 3 does not require the results from step 2. Therefore, the step 2 and step 3 are decoupled at each time level. Because of that, these two steps can be updated simultaneously by parallel computing. However, the fully discrete scheme in [16] is not decoupled since the step 2 depends on the numerical solutions from step 1. Thus, the main advantage of the proposed method over the methods in [16] is the suitability for parallel computing. Another difference between our proposed scheme and undecoupled LDG method in [16] is that q_h^1 and p_h^1 need to be initialized. Our decoupled scheme leads to different conservation property and error estimates, which we will discuss in later sections.

2.3. **Conservation properties of the fully discrete decoupled LDG methods.** In this subsection, we present the energy- and Hamiltonian conservation properties of the fully discrete decoupled local discontinuous Galerkin methods (2.6)-(2.8).

Theorem 2.1. *(Energy Conservation) The numerical solutions to the fully discrete LDG methods* (2.6)-(2.8) *satisfy:*

$$\|q_h^n\|^2 + \|p_h^n\|^2 = \|q_h^0\|^2 + \|p_h^0\|^2, \quad \forall n. \tag{2.9}$$

Proof. For any $n \geq 1$, let $w_1 = q_h^{n+1} + q_h^{n-1}$ and $w_2 = p_h^{n+1} + p_h^{n-1}$ in (2.7), add the resulting equations and sum over j, one can get

$$\frac{1}{2\Delta t}(\|q_h^{n+1}\|^2 + \|p_h^{n+1}\|^2 - \|q_h^{n-1}\|^2 - \|p_h^{n-1}\|^2) = \Theta_1 + \Theta_2, \tag{2.10}$$

where

$$\Theta_1 = -\frac{1}{2}\int_I (u_h^{n+1} + u_h^{n-1})(q_h^{n+1} + q_h^{n-1})_x dx - \frac{1}{2}\sum_{j=1}^{N} (u_h^{n+1} + u_h^{n-1})^-_{j+\frac{1}{2}} [q_h^{n+1} + q_h^{n-1}]_{j+\frac{1}{2}},$$

and

$$\Theta_2 = \frac{1}{2}\int_I (z_h^{n+1}+z_h^{n-1})(p_h^{n+1}+p_h^{n-1})_x dx + \frac{1}{2}\sum_{j=1}^{N}(z_h^{n+1}+z_h^{n-1})^-_{j+\frac{1}{2}}[p_h^{n+1}+p_h^{n-1}]_{j+\frac{1}{2}}.$$

Here we have used the fact that $(u_h^{n+1}+u_h^{n-1})^-_{\frac{1}{2}}(q_h^{n+1}+q_h^{n-1})^+_{\frac{1}{2}} = (u_h^{n+1}+u_h^{n-1})^-_{N+\frac{1}{2}}(q_h^{n+1}+q_h^{n-1})^+_{N+\frac{1}{2}}$, and $(z_h^{n+1}+z_h^{n-1})^-_{\frac{1}{2}}(p_h^{n+1}+p_h^{n-1})^+_{\frac{1}{2}} = (z_h^{n+1}+z_h^{n-1})^-_{N+\frac{1}{2}}(p_h^{n+1}+p_h^{n-1})^+_{N+\frac{1}{2}}$ due to the periodic boundary conditions. Furthermore, we add the third and the fourth equation in (2.7) over all spatial intervals at time t^{n+1} and t^{n-1}, and get

$$\begin{aligned}
&\frac{1}{2}\int_I (u_h^{n+1}+u_h^{n-1})w_3 dx + \frac{1}{2}\int_I (z_h^{n+1}+z_h^{n-1})w_4 dx \\
= \quad & -\frac{1}{2}\int_I (p_h^{n+1}+p_h^{n-1})(w_3)_x dx - \frac{1}{2}\sum_{j=1}^{N}(p_h^{n+1}+p_h^{n-1})^+_{j+\frac{1}{2}}[w_3]_{j+\frac{1}{2}}, \\
& -\frac{1}{2}\int_I (q_h^{n+1}+q_h^{n-1})(w_4)_x dx - \frac{1}{2}\sum_{j=1}^{N}(q_h^{n+1}+q_h^{n-1})^+_{j+\frac{1}{2}}[w_4]_{j+\frac{1}{2}}.
\end{aligned} \tag{2.11}$$

Here we have used the periodic boundary conditions. We then let $w_3 = -(z_h^{n+1}+z_h^{n-1})$ and $w_4 = u_h^{n+1}+u_h^{n-1}$ in (2.11) and obtain

$$0 = \Theta_3 + \Theta_4, \tag{2.12}$$

where

$$\Theta_3 = \frac{1}{2}\int_I (p_h^{n+1}+p_h^{n-1})(z_h^{n+1}+z_h^{n-1})_x dx + \frac{1}{2}\sum_{j=1}^{N}(p_h^{n+1}+p_h^{n-1})^+_{j+\frac{1}{2}}[z_h^{n+1}+z_h^{n-1}]_{j+\frac{1}{2}},$$

and

$$\Theta_4 = -\frac{1}{2}\int_I (q_h^{n+1}+q_h^{n-1})(u_h^{n+1}+u_h^{n-1})_x dx - \frac{1}{2}\sum_{j=1}^{N}(q_h^{n+1}+q_h^{n-1})^+_{j+\frac{1}{2}}[u_h^{n+1}+u_h^{n-1}]_{j+\frac{1}{2}}.$$

We now add (2.10) and (2.12) to get

$$\frac{1}{2\Delta t}\left(\|q_h^{n+1}\|^2 + \|p_h^{n+1}\|^2 - \|q_h^{n-1}\|^2 - \|p_h^{n-1}\|^2\right) = (\Theta_1+\Theta_4) + (\Theta_2+\Theta_3). \tag{2.13}$$

Using the periodic boundary conditions and the property of jump (2.1), we can show that $\Theta_1 + \Theta_4 = 0$. Similarly, we can also show that $\Theta_2 + \Theta_3 = 0$. Therefore, we have

$$\|q_h^{n+1}\|^2 + \|p_h^{n+1}\|^2 = \|q_h^{n-1}\|^2 + \|p_h^{n-1}\|^2, \quad \forall n \ge 1. \tag{2.14}$$

Next, we can follow similar procedures described above to show that (2.6) leads to

$$\|q_h^1\|^2 + \|p_h^1\|^2 = \|q_h^0\|^2 + \|p_h^0\|^2. \tag{2.15}$$

Finally, we can conclude the proof by combining (2.14) and (2.15). □

Theorem 2.1 implies that our decoupled fully discrete local discontinuous Galerkin method leads to numerical solutions that preserve the total energy exactly. The fully discrete Hamiltonian property of the fully discrete LDG method is given in the next theorem.

Theorem 2.2. *(Hamiltonian Conservation) The fully discrete LDG method* (2.6)-(2.8) *satisfies:*

$$\begin{aligned}
&\frac{1}{4}\|\phi_h^{n+1}\|^2+\frac{1}{4}\|s_h^{n+1}\|^2+\frac{1}{2}\|u_h^{n+1}\|^2+\frac{1}{2}\|z_h^{n+1}\|^2+\frac{1}{2}\left\|\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right\|^2\\
&\quad-\frac{1}{2}\int_I((p_h^n)^2+(q_h^n)^2)\phi_h^{n+1}dx-\frac{1}{2}\int_I((p_h^{n+1})^2+(q_h^{n+1})^2)\phi_h^n dx\\
=\;&\frac{1}{4}\|\phi_h^{n-1}\|^2+\frac{1}{4}\|s_h^{n-1}\|^2+\frac{1}{2}\|u_h^{n-1}\|^2+\frac{1}{2}\|z_h^{n-1}\|^2+\frac{1}{2}\left\|\frac{\phi_h^{n}-\phi_h^{n-1}}{\Delta t}\right\|^2\\
&\quad-\frac{1}{2}\int_I((p_h^n)^2+(q_h^n)^2)\phi_h^{n-1}dx-\frac{1}{2}\int_I((p_h^{n-1})^2+(q_h^{n-1})^2)\phi_h^n dx,\quad\forall n\geq 1.
\end{aligned}\tag{2.16}$$

Proof. We only need to prove the following two equalities (2.17) and (2.18) to complete the proof:

$$\begin{aligned}
&\frac{1}{2}\|\phi_h^{n+1}\|^2+\frac{1}{2}\|s_h^{n+1}\|^2+\left\|\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right\|^2-\int_I((p_h^n)^2+(q_h^n)^2)\phi_h^{n+1}dx\\
=\;&\frac{1}{2}\|\phi_h^{n-1}\|^2+\frac{1}{2}\|s_h^{n-1}\|^2+\left\|\frac{\phi_h^{n}-\phi_h^{n-1}}{\Delta t}\right\|^2-\int_I((p_h^n)^2+(q_h^n)^2)\phi_h^{n-1}dx,
\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}
&\|u_h^{n+1}\|^2+\|z_h^{n+1}\|^2-\int_I((p_h^{n+1})^2+(q_h^{n+1})^2)\phi_h^n dx\\
=\;&\|u_h^{n-1}\|^2+\|z_h^{n-1}\|^2-\int_I((p_h^{n-1})^2+(q_h^{n-1})^2)\phi_h^n dx.
\end{aligned}\tag{2.18}$$

Since $(\phi_h^{n+1}-2\phi_h^n+\phi_h^{n-1})(\phi_h^{n+1}-\phi_h^{n-1})=(\phi_h^{n+1}-\phi_h^n)^2-(\phi_h^n-\phi_h^{n-1})^2$, we take $v=\phi_h^{n+1}-\phi_h^{n-1}$ in (2.8), sum over j and apply the periodic boundary conditions to get

$$\begin{aligned}
&\left\|\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right\|^2-\left\|\frac{\phi_h^{n}-\phi_h^{n-1}}{\Delta t}\right\|^2+\frac{1}{2}\|\phi_h^{n+1}\|^2-\frac{1}{2}\|\phi_h^{n-1}\|^2\\
&\quad-\int_I((p_h^n)^2+(q_h^n)^2)(\phi_h^{n+1}-\phi_h^{n-1})dx\\
=\;&-\frac{1}{2}\int_I(s_h^{n+1}+s_h^{n-1})(\phi_h^{n+1}-\phi_h^{n-1})_x dx-\frac{1}{2}\sum_{j=1}^{N}(s_h^{n+1}+s_h^{n-1})^-_{j+\frac{1}{2}}[\phi_h^{n+1}-\phi_h^{n-1}]_{j+\frac{1}{2}}.
\end{aligned}\tag{2.19}$$

We then take the sum of the second equation in (2.8) over all j, subtract the resulting equation at time t^{n-1} from the equation at time t^{n+1}, and let $w = (s_h^{n+1} + s_h^{n-1})/2$ to get

$$\begin{aligned} &\frac{1}{2}\|s_h^{n+1}\|^2 - \frac{1}{2}\|s_h^{n-1}\|^2 \qquad (2.20)\\ = &-\frac{1}{2}\int_I (\phi_h^{n+1} - \phi_h^{n-1})(s_h^{n+1} + s_h^{n-1})_x dx - \frac{1}{2}\sum_{j=1}^{N}(\phi_h^{n+1} - \phi_h^{n-1})^+_{j+\frac{1}{2}}[s_h^{n+1} + s_h^{n-1}]_{j+\frac{1}{2}}. \end{aligned}$$

Adding Eqs. (2.19)- (2.20) and applying (2.1), one can get (2.17).

Next we will show equality (2.18) to conclude the proof. Let $w_1 = -(p_h^{n+1} - p_h^{n-1})$, $w_2 = q_h^{n+1} - q_h^{n-1}$ in (2.7), and add these two equations over all j to get

$$\begin{aligned} 0 = &\frac{1}{2}\int_{I_j}(u_h^{n+1} + u_h^{n-1})(p_h^{n+1} - p_h^{n-1})_x dx + \frac{1}{2}\sum_{j=1}^{N}(u_h^{n+1} + u_h^{n-1})^-_{j+\frac{1}{2}}[p_h^{n+1} - p_h^{n-1}]_{j+\frac{1}{2}}\\ + &\frac{1}{2}\int_{I_j}(z_h^{n+1} + z_h^{n-1})(q_h^{n+1} - q_h^{n-1})_x dx + \frac{1}{2}\sum_{j=1}^{N}(z_h^{n+1} + z_h^{n-1})^-_{j+\frac{1}{2}}[q_h^{n+1} - q_h^{n-1}]_{j+\frac{1}{2}}\\ - &\frac{1}{2}\int_I \phi_h^n((p_h^{n+1})^2 - (p_h^{n-1})^2)dx - \frac{1}{2}\int_I \phi_h^n((q_h^{n+1})^2 - (q_h^{n-1})^2)dx. \qquad (2.21) \end{aligned}$$

Due to the third and fourth equation in (2.7), it is easy to show that

$$\begin{aligned} \int_I (u_h^{n+1} - u_h^{n-1})w_3 dx &= -\int_I (p_h^{n+1} - p_h^{n-1})(w_3)_x dx - \sum_{j=1}^{N}(p_h^{n+1} - p_h^{n-1})^+_{j+\frac{1}{2}}[w_3]_{j+\frac{1}{2}},\\ \int_I (z_h^{n+1} - z_h^{n-1})w_4 dx &= -\int_I (q_h^{n+1} - q_h^{n-1})(w_4)_x dx - \sum_{j=1}^{N}(q_h^{n+1} - q_h^{n-1})^+_{j+\frac{1}{2}}[w_4]_{j+\frac{1}{2}}. \end{aligned}$$

Let $w_3 = \frac{1}{2}(u_h^{n+1} + u_h^{n-1})$ and $w_4 = \frac{1}{2}(z_h^{n+1} + z_h^{n-1})$ in the two equalities above, respectively, and add the resulting equations, we can get

$$\begin{aligned} &\frac{1}{2}\|u_h^{n+1}\|^2 + \frac{1}{2}\|z_h^{n+1}\|^2 - \frac{1}{2}\|u_h^{n-1}\|^2 - \frac{1}{2}\|z_h^{n-1}\|^2 \qquad (2.22)\\ = &-\frac{1}{2}\int_I (p_h^{n+1} - p_h^{n-1})(u_h^{n+1} + u_h^{n-1})_x dx - \frac{1}{2}\sum_{j=1}^{N}(p_h^{n+1} - p_h^{n-1})^+_{j+\frac{1}{2}}[u_h^{n+1} + u_h^{n-1}]_{j+\frac{1}{2}}\\ &-\frac{1}{2}\int_I (q_h^{n+1} - q_h^{n-1})(z_h^{n+1} + z_h^{n-1})_x dx - \frac{1}{2}\sum_{j=1}^{N}(q_h^{n+1} - q_h^{n-1})^+_{j+\frac{1}{2}}[z_h^{n+1} + z_h^{n-1}]_{j+\frac{1}{2}}. \end{aligned}$$

We then subtract Eq. (2.22) and (2.21), apply the property of jump (2.1) to get

$$\begin{aligned}&\frac{1}{2}\|u_h^{n+1}\|^2+\frac{1}{2}\|z_h^{n+1}\|^2-\frac{1}{2}\|u_h^{n-1}\|^2-\frac{1}{2}\|z_h^{n-1}\|^2 \\ =\ &\frac{1}{2}\int_I \phi_h^n((p_h^{n+1})^2-(p_h^{n-1})^2)dx+\frac{1}{2}\int_I \phi_h^n((q_h^{n+1})^2-(q_h^{n-1})^2)dx.\end{aligned} \tag{2.23}$$

Since (2.23) is equivalent to (2.18), we can conclude this proof. □

Remark 2.1. *Since*

$$\begin{aligned}\frac{1}{4}\|\phi_h^{n+1}\|^2-\frac{1}{4}\|\phi_h^{n-1}\|^2 &= \frac{1}{2}\int_I \frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\phi_h^{n+1}dx-\frac{1}{2}\int_I \frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\phi_h^{n-1}dx \\ &\approx \frac{1}{2}\|\phi_h^{n+\frac{1}{2}}\|^2-\frac{1}{2}\|\phi_h^{n-\frac{1}{2}}\|^2, \\ \frac{1}{4}\|s_h^{n+1}\|^2-\frac{1}{4}\|s_h^{n-1}\|^2 &\approx \frac{1}{2}\|s_h^{n+\frac{1}{2}}\|^2-\frac{1}{2}\|s_h^{n-\frac{1}{2}}\|^2, \\ \frac{1}{2}\|u_h^{n+1}\|^2-\frac{1}{2}\|u_h^{n-1}\|^2 &\approx \|u_h^{n+\frac{1}{2}}\|^2-\|u_h^{n-\frac{1}{2}}\|^2, \\ \frac{1}{2}\|z_h^{n+1}\|^2-\frac{1}{2}\|z_h^{n-1}\|^2 &\approx \|z_h^{n+\frac{1}{2}}\|^2-\|z_h^{n-\frac{1}{2}}\|^2,\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\int_I((p_h^n)^2+(q_h^n)^2)\phi_h^{n+1}dx+\frac{1}{2}\int_I((p_h^{n+1})^2+(q_h^{n+1})^2)\phi_h^n dx \\ \approx\ &\int_I((p_h^{n+\frac{1}{2}})^2+(q_h^{n+\frac{1}{2}})^2)\phi_h^{n+\frac{1}{2}}dx,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\int_I((p_h^n)^2+(q_h^n)^2)\phi_h^{n-1}dx+\frac{1}{2}\int_I((p_h^{n-1})^2+(q_h^{n-1})^2)\phi_h^n dx \\ \approx\ &\int_I((p_h^{n-\frac{1}{2}})^2+(q_h^{n-\frac{1}{2}})^2)\phi_h^{n-\frac{1}{2}}dx,\end{aligned}$$

we can regard (2.16) *as an approximation of the Hamiltonian conservation* (1.4) *from time* $t^{n-\frac{1}{2}}$ *to time* $t^{n+\frac{1}{2}}$. *Moreover, if we compare the Hamiltonian conservation property of our decoupled fully discrete LDG method with that of the undecoupled fully discrete LDG method in* [16], *we can see that only the terms* $-\frac{1}{2}\int_I((p_h^{n+1})^2+(q_h^{n+1})^2)\phi_h^n dx$ *and* $-\frac{1}{2}\int_I((p_h^{n-1})^2+(q_h^{n-1})^2)\phi_h^n dx$ *are different. Therefore, our decoupled LDG method leads to quite similar Hamiltonian property to the undecoupled method.*

3. Error Estimates of the Decoupled Fully Discrete LDG Method

In this section, we give the error estimates for the L^2 errors of the decoupled fully discrete LDG method (2.6)-(2.8). Theorem 3.1 is the main theorem, which is proved based on error equations, energy equations and a key component, i.e., the L^∞-assumption. Throughout the section, we use C to denote a generic constant, and we assume that the time step size Δt and the mesh size h are both less than or equal to 1. The main theorem is given as follows.

Theorem 3.1. *For any $T > 0$, let q^n, p^n and ϕ^n be the sufficiently smooth exact solutions to the Klein-Gordon-Schrödinger equations* (1.2) *at $t^n = n\Delta t$ for all $n \leq \lfloor T/\Delta t \rfloor$. Let q_h^n, p_h^n and ϕ_h^n be the numerical solutions to the scheme* (2.6)-(2.8) *with numerical flux* (2.5)*. Suppose there exists a constant $\gamma > 0$ such that $(\Delta t)^2 \leq \gamma h^{\frac{1}{2}}$, then the following error estimates hold:*

$$\|p^n - p_h^n\|^2 + \|q^n - q_h^n\|^2 + \|\phi^n - \phi_h^n\|^2 \leq Ch^{2k+2} + C(\Delta t)^4, \tag{3.1}$$

where C is a positive constant that depends on T and the exact solutions, and it is independent of Δt, h and n.

3.1. **Truncation Errors and Error equations.** In order to prove Theorem 3.1, we need some results about truncations errors and error equations.

Lemma 3.1. *For $n \geq 1$, let $T_q^n(x)$, $T_p^n(x)$ and $T_\phi^n(x)$ be the truncation errors that satisfy*

$$\begin{aligned}
\frac{q^{n+1} - q^{n-1}}{2\Delta t} &= \frac{1}{2}(u_x^{n+1} + u_x^{n-1}) + \frac{1}{2}\phi^n(p^{n+1} + p^{n-1}) + T_q^n(x),\\
\frac{p^{n+1} - p^{n-1}}{2\Delta t} &= -\frac{1}{2}(z_x^{n+1} + z_x^{n-1}) - \frac{1}{2}\phi^n(q^{n+1} + q^{n-1}) + T_p^n(x),\\
\frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{(\Delta t)^2} &= \frac{1}{2}(s_x^{n+1} + s_x^{n-1}) - \frac{1}{2}(\phi^{n+1} + \phi^{n-1}) + (p^n)^2 + (q^n)^2 + T_\phi^n(x).
\end{aligned}$$

Let $T_q^{\frac{1}{2}}(x)$, and $T_p^{\frac{1}{2}}(x)$ be the truncation errors at the first step that satisfy

$$\begin{aligned}
\frac{q^1 - q^0}{\Delta t} &= \frac{1}{2}(u_x^1 + u_x^0) + \frac{1}{4}(\phi^1 + \phi^0)(p^1 + p^0) + T_q^{\frac{1}{2}}(x),\\
\frac{p^1 - p^0}{\Delta t} &= -\frac{1}{2}(z_x^1 + z_x^0) - \frac{1}{4}(\phi^1 + \phi^0)(q^1 + q^0) + T_p^{\frac{1}{2}}(x).
\end{aligned} \tag{3.2}$$

Then the following estimates hold

$$\|T_q^{\frac{1}{2}}\|, \|T_p^{\frac{1}{2}}\|, \|T_q^n\|, \|T_p^n\|, \|T_\phi^n\| \leq C(\Delta t)^2, \quad \forall n \geq 1.$$

In addition, there are

$$\|T_q^{n+1} - T_q^{n-1}\|, \|T_q^{n+1} - T_q^{n-1}\| \leq C(\Delta t)^3, \quad \forall n \geq 1.$$

The lemma above can be proved simply by Taylor expansions of the exact solutions and we skip its proof. Next we derive the error equations for our scheme. We will use e_p^n to denote the error of p at t^n, that is, $e_p^n = p^n - p_h^n = \eta_p^n - \xi_p^n$, where $\eta_p^n = p^n - \Pi^+ p^n$ and $\xi_p^n = p_h^n - \Pi^+ p^n$. Note that Π^+ is the Gauss-Radau projection, such that $(\eta_p^n)_{j+\frac{1}{2}} = (p^n - \Pi^+ p^n)^+_{j+\frac{1}{2}} = 0, \forall j$. Similarly, we define $\eta_q^n = q^n - \Pi^+ q^n$, $\xi_q^n = q_h^n - \Pi^+ q^n$, $\eta_\phi^n = \phi^n - \Pi^+ \phi^n$, $\xi_\phi^n = \phi_h^n - \Pi^+ \phi^n$, $\eta_u^n = u^n - \Pi^- u^n$, $\xi_u^n = u_h^n - \Pi^- u^n$, $\eta_z^n = z^n - \Pi^- z^n$, $\xi_z^n = z_h^n - \Pi^- z^n$, $\eta_s^n = s^n - \Pi^- s^n$ and $\xi_s^n = s_h^n - \Pi^- s^n$. Thus we have $e_q^n = \eta_q^n - \xi_q^n$, $e_\phi^n = \eta_\phi^n - \xi_\phi^n$, $e_u^n = \eta_u^n - \xi_u^n$, $e_z^n = \eta_z^n - \xi_z^n$

and $e_s^n = \eta_s^n - \xi_s^n$. Also note that the terms $\|\eta_p^n\|$, $\|\eta_q^n\|$, $\|\eta_\phi^n\|$, $\|\eta_u^n\|$, $\|\eta_z^n\|$ and $\|\eta_s^n\|$ are independent of the numerical scheme, and can be estimated directly by the approximation properties in Eq. (2.2). Thus we only need to focus on the estimates of $\|\xi_p^n\|$, $\|\xi_q^n\|$, $\|\xi_\phi^n\|$, $\|\xi_u^n\|$, $\|\xi_z^n\|$, $\|\xi_s^n\|$ using error equations and energy equations. To derive the error equations, we first multiply the test function w_1 on both sides of the first equation in (3.2), take integral over interval I and apply integration by parts to get

$$\begin{aligned}\int_I \frac{q^1 - q^0}{\Delta t} w_1 dx &= -\frac{1}{2}\int_I (u^1 + u^0)(w_1)_x dx - \frac{1}{2}\sum_{j=1}^{N}(u^1+u^0)_{j+\frac{1}{2}}[w_1]_{j+\frac{1}{2}} \\ &\quad + \frac{1}{4}\int_I (\phi^1+\phi^0)(p^1+p^0) w_1 dx + \int_I T_q^{\frac{1}{2}} w_1 dx. \qquad (3.3)\end{aligned}$$

We then use the first equation in (2.6), sum over j, and subtract the resulting equation and (3.3) to get

$$\begin{aligned}\int_I \frac{\xi_q^1 - \xi_q^0}{\Delta t} w_1 dx &= -\frac{1}{2}\int_I (\xi_u^1 + \xi_u^0)(w_1)_x dx - \frac{1}{2}\sum_{j=1}^{N}(\xi_u^1+\xi_u^0)^-_{j+\frac{1}{2}}[w_1]_{j+\frac{1}{2}} \\ &\quad + \frac{1}{4}\int_I (\phi_h^1+\phi_h^0)(p_h^1+p_h^0) w_1 dx + \int_I \frac{\eta_q^1 - \eta_q^0}{\Delta t} w_1 dx \\ &\quad - \frac{1}{4}\int_I (\phi^1+\phi^0)(p^1+p^0) w_1 dx - \int_I T_q^{\frac{1}{2}} w_1 dx. \qquad (3.4)\end{aligned}$$

Here we have used the property of Gauss-Radau projection, i.e., $\int_I (\eta_u^1 + \eta_u^0)(w_1)_x = 0$ and $(\xi_u)^-_{j+\frac{1}{2}} = 0$ for all j. Similarly, we can derive the error equations for ξ_p^1, ξ_u^1, ξ_z^1 using other equations in (2.6), and the error equations for ξ_q^{n+1}, ξ_p^{n+1}, ξ_u^{n+1}, ξ_z^{n+1}, ξ_ϕ^{n+1} and ξ_s^{n+1} using (2.7) and (2.8). That is,

$$\begin{aligned}\int_I \frac{\xi_p^1 - \xi_p^0}{\Delta t} w_2 dx &= \frac{1}{2}\int_I (\xi_z^1 + \xi_z^0)(w_2)_x dx + \frac{1}{2}\sum_{j=1}^{N}(\xi_z^1+\xi_z^0)^-_{j+\frac{1}{2}}[w_2]_{j+\frac{1}{2}} \\ &\quad - \frac{1}{4}\int_I (\phi_h^1+\phi_h^0)(q_h^1+q_h^0) w_2 dx + \int_I \frac{\eta_p^1 - \eta_p^0}{\Delta t} w_2 dx \\ &\quad + \frac{1}{4}\int_I (\phi^1+\phi^0)(q^1+q^0) w_2 dx - \int_I T_p^{\frac{1}{2}} w_2 dx, \qquad (3.5)\end{aligned}$$

$$\begin{aligned}\int_I (\xi_u^1 + \xi_u^0) w_3\, dx &= -\int_I (\xi_p^1 + \xi_p^0)(w_3)_x\, dx - \sum_{j=1}^{N}(\xi_p^1+\xi_p^0)^+_{j+\frac{1}{2}}[w_3]_{j+\frac{1}{2}} \\ &\quad + \int_I (\eta_u^1 + \eta_u^0) w_3\, dx, \qquad (3.6)\end{aligned}$$

$$\int_I (\xi_z^1 + \xi_z^0) w_4 \, dx = -\int_I (\xi_q^1 + \xi_q^0)(w_4)_x \, dx - \sum_{j=1}^{N} (\xi_q^1 + \xi_q^0)^+_{j+\frac{1}{2}} [w_4]_{j+\frac{1}{2}} + \int_I (\eta_z^1 + \eta_z^0) w_4 \, dx, \tag{3.7}$$

for all w_1, w_2, w_3 and $w_4 \in V_h^k$. In addition, for $n \geq 1$, there are

$$\int_I \frac{\xi_q^{n+1} - \xi_q^{n-1}}{\Delta t} w_1 \, dx = -\frac{1}{2} \int_I (\xi_u^{n+1} + \xi_u^{n-1})(w_1)_x \, dx - \frac{1}{2} \sum_{j=1}^{N} (\xi_u^{n+1} + \xi_u^{n-1})^-_{j+\frac{1}{2}} [w_1]_{j+\frac{1}{2}} + \frac{1}{2} \int_I \phi_h^n (p_h^{n+1} + p_h^{n-1}) w_1 \, dx - \frac{1}{2} \int_I \phi^n (p^{n+1} + p^{n-1}) w_1 \, dx + \int_I \frac{\eta_q^{n+1} - \eta_q^{n-1}}{2\Delta t} w_1 \, dx - \int_I T_q^n w_1 \, dx, \tag{3.8}$$

$$\int_I \frac{\xi_p^{n+1} - \xi_p^{n-1}}{2\Delta t} w_2 \, dx = \frac{1}{2} \int_I (\xi_z^{n+1} + \xi_z^{n-1})(w_2)_x \, dx + \frac{1}{2} \sum_{j=1}^{N} (\xi_z^{n+1} + \xi_z^{n-1})^-_{j+\frac{1}{2}} [w_2]_{j+\frac{1}{2}} - \frac{1}{2} \int_I \phi_h^n (q_h^{n+1} + q_h^{n-1}) w_2 \, dx + \frac{1}{2} \int_I \phi^n (q^{n+1} + q^{n-1}) w_2 \, dx + \int_I \frac{\eta_p^{n+1} - \eta_p^{n-1}}{2\Delta t} w_2 \, dx - \int_I T_p^n w_2 \, dx, \tag{3.9}$$

$$\int_I (\xi_u^{n+1} + \xi_u^{n-1}) w_3 \, dx = -\int_I (\xi_p^{n+1} + \xi_p^{n-1})(w_3)_x \, dx - \sum_{j=1}^{N} (\xi_p^{n+1} + \xi_p^{n-1})^+_{j+\frac{1}{2}} [w_3]_{j+\frac{1}{2}} + \int_I (\eta_u^{n+1} + \eta_u^{n-1}) w_3 \, dx, \tag{3.10}$$

$$\int_I (\xi_z^{n+1} + \xi_z^{n-1}) w_4 \, dx = -\int_I (\xi_q^{n+1} + \xi_q^{n-1})(w_4)_x \, dx - \sum_{j=1}^{N} (\xi_q^{n+1} + \xi_q^{n-1})^+_{j+\frac{1}{2}} [w_4]_{j+\frac{1}{2}} + \int_I (\eta_z^{n+1} + \eta_z^{n-1}) w_4 \, dx, \tag{3.11}$$

$$\begin{aligned}
&\int_I \frac{\xi_\phi^{n+1} - 2\xi_\phi^n + \xi_\phi^{n-1}}{(\Delta t)^2} w_5\,dx \\
=\ & -\frac{1}{2}\int_I (\xi_s^{n+1} + \xi_s^{n-1})(w_5)_x\,dx - \frac{1}{2}\int_I (\xi_\phi^{n+1} + \xi_\phi^{n-1}) w_5\,dx - \int_I T_\phi^n w_5\,dx \\
& -\frac{1}{2}\sum_{j=1}^{N} (\xi_s^{n+1} + \xi_s^{n-1})^-_{j+\frac{1}{2}} [w_5]_{j+\frac{1}{2}} + \int_I \left((p_h^n)^2 + (q_h^n)^2 - (p^n)^2 - (q^n)^2\right) w_5\,dx \\
& + \int_I \frac{\eta_\phi^{n+1} - 2\eta_\phi^n + \eta_\phi^{n-1}}{(\Delta t)^2} w_5\,dx + \frac{1}{2}\int_I (\eta_\phi^{n+1} + \eta_\phi^{n-1}) w_5\,dx, \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\int_I (\xi_s^{n+1} - \xi_s^{n-1}) w_6\,dx \ =\ & -\int_I (\xi_\phi^{n+1} - \xi_\phi^{n-1})(w_6)_x\,dx - \sum_{j=1}^{N} (\xi_\phi^{n+1} - \xi_\phi^{n-1})^+_{j+\frac{1}{2}} [w_6]_{j+\frac{1}{2}} \\
& + \int_I (\eta_s^{n+1} - \eta_s^{n-1}) w_6\,dx, \qquad (3.13)
\end{aligned}$$

for all $w_1, w_2, \ldots, w_6 \in V_h^k$. Note that the error equations for ξ_q^1 and ξ_p^1 are different from that for ξ_q^{n+1} and ξ_p^{n+1} with $n \geq 1$. This is due to the fact that we are using decoupled LDG method such that q_h^{n+1}, p_h^{n+1} and ϕ_h^{n+1} must be computed using q_h^{n-1}, p_h^{n-1} and ϕ_h^{n-1} for $n \geq 1$. Thus q_h^1 and p_h^1 need to be updated using a different method (2.6). To deal with the linear combination of truncation errors from the error equations, we introduce the following lemma which will be used frequently in the proof of the main theorem. The detail of the proof is similar to that of lemma 3.3 in [25], thus we omit its proof.

Lemma 3.2. *For $\star = q$, p or ϕ, we have the following estimates:*

$$\|\eta_\star^{n+1} - 2\eta_\star^n + \eta_\star^{n-1}\| \leq C(\Delta t)^2 h^{k+1}, \quad \|\eta_\star^{n+1} - \eta_\star^n\| \leq C(\Delta t) h^{k+1},$$

and

$$\|\eta_\star^{n+1} - \eta_\star^{n-1}\| \leq C(\Delta t) h^{k+1}.$$

We now use the error Eqs. (3.4)-(3.7) to estimate ξ_q^1, ξ_p^1, ξ_u^1 and ξ_z^1. The results are given in the next lemma.

Lemma 3.3.

$$\begin{aligned}
&\|\xi_q^0\| = \|\xi_p^0\| = \|\xi_\phi^0\| = \|\xi_u^0\| = \|\xi_z^0\| = \|\xi_s^0\| = 0, \\
&\|\xi_\phi^1\| \leq C(\Delta t)^3, \quad \|\xi_q^1\|^2 + \|\xi_p^1\|^2 \leq C(\Delta t) h^{2k+2} + C(\Delta t)^5.
\end{aligned}$$

The proof of Theorem 3.1 consists of three important components, i.e., the L^∞ assumption, the estimates from the Klein-Gordon equation part, and the estimates from the Schrödinger equation part. We first give the L^∞ assumption as follows.

L^∞**-Assumption:** For any integer $n \le T/\Delta t - 1$, the following inequalities hold;

$$\|p_h^n\|_\infty, \|q_h^n\|_\infty, \|\phi_h^n\|_\infty, \|(\phi_h^n - \phi_h^{n-1})/\Delta t\|_\infty \le C.$$

The assumption above will be used for the estimates of both the Klein-Gordon equation part and the Schrödinger equation part. It will eventually be proved by mathematical induction. Another important technique for the estimates of the Schrödinger equation part is the summation-by-parts formula:

$$\sum_{i=1}^{n} f_i(g_{i+1} - g_{i-1}) = -f_1 g_0 - f_2 g_1 - \sum_{i=2}^{n}(f_{i+1} - f_{i-1})g_i + f_{n-1}g_n + f_n g_{n+1}, \tag{3.14}$$

which can be easily verified.

3.2. **The estimates for the Schrödinger and the Klein-Gordon equation parts.** To make the proof of Theorem 3.1 easy to follow, we only present the error estimates for Schrödinger and the Klein-Gordon equation parts in the next two lemmas without proof. The detailed proof can be found in section 4. We first give Lemma 3.4 and 3.5, which are the estimates from the Schrödinger equation part.

Lemma 3.4. *Under the L^∞-Assumption, there is the following inequality for $n \ge 1$:*

$$\begin{aligned} & \|\xi_q^{n+1}\|^2 + \|\xi_p^{n+1}\|^2 - \|\xi_q^{n-1}\|^2 - \|\xi_p^{n-1}\|^2 \\ \le\ & C(\Delta t)h^{2k+2} + C(\Delta t)^5 + C(\Delta t)h^{k+1}\left(\|\xi_u^{n+1} + \xi_u^{n-1}\| + \|\xi_z^{n+1} + \xi_z^{n-1}\|\right) \\ & + C(\Delta t)h^{k+1}(1 + \|\phi_h^n\|_\infty)(\|\xi_q^{n+1}\| + \|\xi_q^{n-1}\| + \|\xi_p^{n+1}\| + \|\xi_p^{n-1}\|) \\ & + C(\Delta t)\left(\|\xi_q^{n+1}\|^2 + \|\xi_q^{n-1}\|^2 + \|\xi_p^{n+1}\|^2 + \|\xi_p^{n-1}\|^2 + \|\xi_\phi^n\|^2\right). \end{aligned} \tag{3.15}$$

Lemma 3.5. *Under the L^∞-Assumption, there is the following inequality:*

$$\begin{aligned} & \max_{1\le i\le n+1}(\|\xi_u^i\| + \|\xi_z^i\|) \\ \le\ & Ch^{k+1} + C(\Delta t)^2 + C\max_{1\le i\le n}\|\xi_\phi^i\| + C\max_{2\le i\le n-1}\left\|\frac{\xi_\phi^{i+1} - \xi_\phi^{i-1}}{\Delta t}\right\| \\ & + C\left(1 + \max_{1\le i\le n}\|\phi_h^i\|_\infty + \max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1} - \phi_h^{i-1}}{\Delta t}\right\|_\infty\right)^{\frac{1}{2}} \times \\ & \quad (h^{k+1} + \max_{1\le i\le n+1}\left(\|\xi_p^i\| + \|\xi_q^i\|\right)). \end{aligned}$$

From the error equations for the Klein-Gordon equation part (3.12)-(3.13), we can derive the following estimate:

Lemma 3.6. *Under the L^∞-Assumption, the following estimate holds:*

$$
\begin{aligned}
&\frac{1}{2}\left(\|\xi_s^{n+1}\|^2+\|\xi_\phi^{n+1}\|^2-\|\xi_s^{n-1}\|^2-\|\xi_\phi^{n-1}\|^2\right)+\left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|^2-\left\|\frac{\xi_\phi^{n}-\xi_\phi^{n-1}}{\Delta t}\right\|^2\\
\leq\;& C(\Delta t)\left(1+\|p_h^n\|_\infty+\|q_h^n\|_\infty\right)\left(h^{2k+2}+\|\xi_p^n\|^2+\|\xi_q^n\|^2+\left\|\frac{\xi_\phi^{n+1}-\xi_\phi^{n-1}}{\Delta t}\right\|^2\right)\\
&+C(\Delta t)(\|\xi_s^{n+1}\|^2+\|\xi_s^{n-1}\|^2)+C(\Delta t)h^{2k+2}+C(\Delta t)^5. \qquad (3.16)
\end{aligned}
$$

Remark 3.1. *Lemma 3.4 and 3.5 indicate that the estimates for the Schrödinger equation part depend on the a priori assumption about $\|\phi_h^n\|_\infty$ and $\max_{2\le i\le n-1}\|(\phi_h^{i+1}-\phi_h^{i-1})/\Delta t\|_\infty$. Lemma 3.6 indicates that the estimate for the Klein-Gordon equation part depends on the a priori assumption about $\|p_h^n\|_\infty$ and $\|q_h^n\|_\infty$. Note that $\max_{2\le i\le n-1}\|(\phi_h^{i+1}-\phi_h^{i-1})/\Delta t\|_\infty$ is bounded by $\max_{0\le i\le n-1}\|(\phi_h^{i+1}-\phi_h^{i})/\Delta t\|_\infty$. Therefore, under the L^∞-assumption, we can combine Lemma 3.4 and 3.5 to obtain the error estimates for $\|\xi_p^{n+1}\|$, $\|\xi_q^{n+1}\|$ and $\|\xi_\phi^{n+1}\|$. The L^∞-assumption can later be proved by mathematical induction. Follow this idea, we can prove Theorem 3.1 in the next section.*

3.3. **Proof of Theorem 3.1.**

Proof. Applying Lemma 3.5, the L^∞-Assumption for $n\le T/\Delta t-1$ and $h\le 1$ to Eq. (3.15), we obtain

$$
\begin{aligned}
&\|\xi_q^{n+1}\|^2+\|\xi_p^{n+1}\|^2-\|\xi_q^{n-1}\|^2-\|\xi_p^{n-1}\|^2\\
\leq\;& C(\Delta t)h^{2k+2}+C(\Delta t)^5+C(\Delta t)h^{k+1}\left(\|\xi_u^{n+1}+\xi_u^{n-1}\|+\|\xi_z^{n+1}+\xi_z^{n-1}\|\right)\\
&+C(\Delta t)\left(h^{2k+2}+\|\xi_q^{n+1}\|^2+\|\xi_q^{n-1}\|^2+\|\xi_p^{n+1}\|^2+\|\xi_p^{n-1}\|^2+\|\xi_\phi^n\|^2\right)\\
\leq\;& C(\Delta t)h^{2k+2}+C(\Delta t)^5+C(\Delta t)h^{k+1}\times\\
&\left(h^{k+1}+(\Delta t)^2+\max_{1\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|+\|\xi_\phi^i\|+\|\frac{\xi_\phi^i-\xi_\phi^{i-1}}{\Delta t}\|)\right)\\
&+C(\Delta t)\left(\|\xi_q^{n+1}\|^2+\|\xi_q^{n-1}\|^2+\|\xi_p^{n+1}\|^2+\|\xi_p^{n-1}\|^2+\|\xi_\phi^n\|^2\right). \qquad (3.17)
\end{aligned}
$$

Here we have used the fact that $\|(\xi_\phi^{i+1}-\xi_\phi^{i-1})/\Delta t\|\le\|(\xi_\phi^{i+1}-\xi_\phi^{i})/\Delta t\|+\|(\xi_\phi^{i}-\xi_\phi^{i-1})/\Delta t\|$ in the second inequality above. Note that $(\Delta t)^3h^{k+1}\le\frac{1}{2}(\Delta t)h^{2k+2}+\frac{1}{2}(\Delta t)^5$, inequality (3.17) leads to

$$
\begin{aligned}
&\|\xi_q^{n+1}\|^2+\|\xi_p^{n+1}\|^2-\|\xi_q^{n-1}\|^2-\|\xi_p^{n-1}\|^2\\
\leq\;& C(\Delta t)h^{2k+2}+C(\Delta t)^5+C(\Delta t)h^{k+1}\max_{1\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|+\|\xi_\phi^i\|+\|\frac{\xi_\phi^i-\xi_\phi^{i-1}}{\Delta t}\|)\\
&+C(\Delta t)\left(\|\xi_q^{n+1}\|^2+\|\xi_q^{n-1}\|^2+\|\xi_p^{n+1}\|^2+\|\xi_p^{n-1}\|^2+\|\xi_\phi^n\|^2\right). \qquad (3.18)
\end{aligned}
$$

We then apply L^∞-assumption for $n \le T/\Delta t - 1$ to (3.16), and get

$$\begin{aligned}
&\frac{1}{2}\left(\|\xi_s^{n+1}\|^2 + \|\xi_\phi^{n+1}\|^2 - \|\xi_s^{n-1}\|^2 - \|\xi_\phi^{n-1}\|^2\right) + \left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|^2 - \left\|\frac{\xi_\phi^{n}-\xi_\phi^{n-1}}{\Delta t}\right\|^2 \\
\le\ & C(\Delta t)(h^{2k+2} + \|\xi_p^n\|^2 + \|\xi_q^n\|^2 + \left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|^2 + \left\|\frac{\xi_\phi^{n}-\xi_\phi^{n-1}}{\Delta t}\right\|^2) \\
&+ C(\Delta t)(\|\xi_s^{n+1}\|^2 + \|\xi_s^{n-1}\|^2) + C(\Delta t)h^{2k+2} + C(\Delta t)^5. \qquad (3.19)
\end{aligned}$$

We further add (3.18) and (3.19) to get

$$\begin{aligned}
&\|\xi_q^{n+1}\|^2 + \|\xi_p^{n+1}\|^2 + \frac{1}{2}\|\xi_s^{n+1}\|^2 + \frac{1}{2}\|\xi_\phi^{n+1}\|^2 + \left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|^2 \\
&\qquad -\|\xi_q^{n-1}\|^2 - \|\xi_p^{n-1}\|^2 - \frac{1}{2}\|\xi_s^{n-1}\|^2 - \frac{1}{2}\|\xi_\phi^{n-1}\|^2 - \left\|\frac{\xi_\phi^{n}-\xi_\phi^{n-1}}{\Delta t}\right\|^2 \\
\le\ & C(\Delta t)h^{2k+2} + C(\Delta t)^5 + C(\Delta t)h^{k+1} \max_{1\le i\le T/\Delta t}(\|\xi_p^i\| + \|\xi_q^i\| + \|\xi_\phi^i\| + \|\frac{\xi_\phi^{i}-\xi_\phi^{i-1}}{\Delta t}\|) \\
&+ C(\Delta t)\left(\frac{1}{2}\|\xi_s^{n+1}\|^2 + \frac{1}{2}\|\xi_s^{n-1}\|^2 + \|\xi_p^{n+1}\|^2 + \|\xi_q^{n+1}\|^2 + \|\xi_p^{n-1}\|^2 + \|\xi_q^{n-1}\|^2\right) \\
&+ C(\Delta t)(\left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|^2 + \left\|\frac{\xi_\phi^{n}-\xi_\phi^{n-1}}{\Delta t}\right\|^2) + C(\Delta t)(\|\xi_p^n\|^2 + \|\xi_q^n\|^2 + \|\xi_\phi^n\|^2). \qquad (3.20)
\end{aligned}$$

Let

$$\begin{aligned}
E_{n+1} &= \|\xi_q^{n+1}\|^2 + \|\xi_p^{n+1}\|^2 + \|\xi_q^n\|^2 + \|\xi_p^n\|^2 + \left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|^2 \\
&\quad + \frac{1}{2}(\|\xi_s^{n+1}\|^2 + \|\xi_\phi^{n+1}\|^2 + \|\xi_s^n\|^2 + \|\xi_\phi^n\|^2),
\end{aligned}$$

then (3.20) leads to

$$\begin{aligned}
E_{n+1} - E_n &\le C(\Delta t)h^{2k+2} + C(\Delta t)^5 + C(\Delta t)(E_{n+1} + E_n) \\
&\quad + C(\Delta t)h^{k+1} \max_{1\le i\le T/\Delta t}\left(\|\xi_p^i\| + \|\xi_q^i\| + \|\xi_\phi^i\| + \left\|\frac{\xi_\phi^{i}-\xi_\phi^{i-1}}{\Delta t}\right\|\right).
\end{aligned}$$

Suppose we choose sufficiently small Δt such that $C\Delta t \le 1/2$. Thus we have

$$\begin{aligned}
&(1 - C\Delta t)E_{n+1} - (1 + C\Delta t)E_n \qquad (3.21) \\
\le\ & C(\Delta t)h^{2k+2} + C(\Delta t)^5 + C(\Delta t)h^{k+1} \max_{1\le i\le T/\Delta t}(\|\xi_p^i\| + \|\xi_q^i\| + \|\xi_\phi^i\| + \left\|\frac{\xi_\phi^{i}-\xi_\phi^{i-1}}{\Delta t}\right\|).
\end{aligned}$$

We then denote $\theta = \frac{1+C(\Delta t)}{1-C(\Delta t)}$. Since $0 < C\Delta t \leq 1/2$, it is easy to show that $\frac{1}{1-C(\Delta t)} \leq 2$ and $\theta > 1$. We then divide both sides of the inequality (3.21) by $(1 - C\Delta t)\theta^{n+1}$, and obtain

$$\begin{aligned} \frac{E_{n+1}}{\theta^{n+1}} - \frac{E_n}{\theta^n} \quad \leq \quad & \frac{C\Delta t}{1 - C\Delta t} \cdot \frac{1}{\theta^{n+1}} \Big(h^{2k+2} + (\Delta t)^4 + h^{k+1} \times \\ & \max_{1 \leq i \leq T/\Delta t} \left(\|\xi_p^i\| + \|\xi_q^i\| + \|\xi_\phi^i\| + \left\| \frac{\xi_\phi^i - \xi_\phi^{i-1}}{\Delta t} \right\| \right) \Big). \end{aligned} \tag{3.22}$$

Note that (3.22) is satisfied for all $n \geq 1$. We thus add these equations over n, and get

$$\begin{aligned} \frac{E_{n+1}}{\theta^{n+1}} - \frac{E_1}{\theta} \quad \leq \quad & \frac{C\Delta t}{1 - C\Delta t} \cdot \frac{\frac{1}{\theta^2}(1 - \frac{1}{\theta^n})}{1 - \frac{1}{\theta}} \Big(h^{2k+2} + (\Delta t)^4 + h^{k+1} \times \\ & \max_{1 \leq i \leq T/\Delta t} \left(\|\xi_p^i\| + \|\xi_q^i\| + \|\xi_\phi^i\| + \left\| \frac{\xi_\phi^i - \xi_\phi^{i-1}}{\Delta t} \right\| \right) \Big). \end{aligned} \tag{3.23}$$

Using the definition of θ and the fact $\theta > 1$, we have

$$\frac{C\Delta t}{1 - C\Delta t} \cdot \frac{\frac{1}{\theta^2}(1 - \frac{1}{\theta^n})}{1 - \frac{1}{\theta}} \leq \frac{C\Delta t}{1 - C\Delta t} \cdot \frac{1}{\theta(\theta - 1)} = \frac{1 - C\Delta t}{2(1 + C\Delta t)} \leq \frac{1}{2}.$$

Thus (3.23) leads to

$$\begin{aligned} \frac{E_{n+1}}{\theta^{n+1}} - \frac{E_1}{\theta} \quad \leq \quad & \frac{1}{2} \left(h^{k+1} \max_{1 \leq i \leq T/\Delta t} \left(\|\xi_p^i\| + \|\xi_q^i\| + \|\xi_\phi^i\| + \left\| \frac{\xi_\phi^i - \xi_\phi^{i-1}}{\Delta t} \right\| \right) \right) \\ & + \frac{1}{2} \left(h^{2k+2} + (\Delta t)^4 \right). \end{aligned} \tag{3.24}$$

Also we have $\theta^{n+1} = (1 + \frac{2C}{1-C\Delta t}\Delta t)^{n+1} \leq (1 + 4C\Delta t)^{n+1} \leq \exp(4C\Delta t(n+1)) \leq \exp(4CT)$. We then apply this inequality to (3.24), and obtain

$$E_{n+1} \quad \leq \quad C \left(h^{2k+2} + (\Delta t)^4 \right) + Ch^{k+1} \max_{1 \leq i \leq T/\Delta t} \left(\|\xi_p^i\| + \|\xi_q^i\| + \|\xi_\phi^i\| + \left\| \frac{\xi_\phi^i - \xi_\phi^{i-1}}{\Delta t} \right\| \right), \tag{3.25}$$

where we have used Lemma 3.3 and Lemma 3.5 with $n = 0$ in the inequality above, i.e.

$$\begin{aligned} E_1 \quad = \quad & \|\xi_q^1\|^2 + \|\xi_p^1\|^2 + \frac{1}{2}(\|\xi_s^1\|^2 + \|\xi_\phi^1\|^2) + \left\| \frac{\xi_\phi^1 - \xi_\phi^0}{\Delta t} \right\|^2 \\ \leq \quad & C(\Delta t)h^{2k+2} + C(\Delta t)^5 + \frac{1}{2}\|\xi_s^1\|^2 + C(\Delta t)^6 + C(\Delta t)^4 \\ \leq \quad & C(\Delta t)h^{2k+2} + C(\Delta t)^4 + Ch^{2k+2} + C(\Delta t)^4 + C(\|\xi_\phi^1\|^2 + \|\xi_p^1\|^2 + \|\xi_q^1\|^2) \\ \leq \quad & Ch^{2k+2} + C(\Delta t)^4. \end{aligned}$$

Note that

$$\frac{1}{8}\left(\|\xi_q^{n+1}\|+\|\xi_p^{n+1}\|+\|\xi_\phi^{n+1}\|+\left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|\right)^2\le E_{n+1},$$

then from (3.25) we obtain

$$\begin{aligned}&\frac{1}{8}\left(\|\xi_q^{n+1}\|+\|\xi_p^{n+1}\|+\|\xi_\phi^{n+1}\|+\left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|\right)^2\\ \le\ & Ch^{2k+2}+C(\Delta t)^4+\frac{1}{16}\max_{1\le i\le T/\Delta t}\left(\|\xi_p^i\|+\|\xi_q^i\|+\|\xi_\phi^i\|+\left\|\frac{\xi_\phi^i-\xi_\phi^{i-1}}{\Delta t}\right\|\right)^2\end{aligned}\tag{3.26}$$

Here we have used Young's inequality. Since (3.26) holds for all $n\le T/\Delta t-1$, there is

$$\begin{aligned}&\frac{1}{8}\max_{1\le i\le T/\Delta t}\left(\|\xi_q^i\|+\|\xi_p^i\|+\|\xi_\phi^i\|+\left\|\frac{\xi_\phi^i-\xi_\phi^{i-1}}{\Delta t}\right\|\right)^2\\ \le\ & Ch^{2k+2}+C(\Delta t)^4+\frac{1}{16}\max_{1\le i\le T/\Delta t}\left(\|\xi_p^i\|+\|\xi_q^i\|+\|\xi_\phi^i\|+\left\|\frac{\xi_\phi^i-\xi_\phi^{i-1}}{\Delta t}\right\|\right)^2\end{aligned}\tag{3.27}$$

Inequality (3.27) implies that

$$\|\xi_p^n\|+\|\xi_q^n\|+\|\xi_\phi^n\|+\left\|\frac{\xi_\phi^n-\xi_\phi^{n-1}}{\Delta t}\right\|\le Ch^{k+1}+C(\Delta t)^2$$

for any $(n+1)\Delta t\le T$. Thus we can obtain inequality (3.1) if we use the approximation property and the triangle inequality.

Finally we will prove the L^∞-assumption using mathematical induction. For $n=0$, there is $\|p_h^0\|_\infty=\|\Pi^+p^0-p^0+p^0\|_\infty\le\|\Pi^+p^0-p^0\|_\infty+\|p^0\|_\infty\le C$. Similarly, we can show that $\|q_h^0\|_\infty,\|\phi_h^0\|_\infty\le C$. For $n=1$, we have $\|p_h^1\|_\infty=\|\xi_p^1-\eta_p^1+p^1\|_\infty\le\|\xi_p^1\|_\infty+\|\eta_p^1\|_\infty+\|p^1\|_\infty\le Ch^{-1/2}\|\xi_p^1\|+Ch^{k+1}+C$, where we have used the inverse inequality (2.3) in the last inequality. We further apply Lemma 3.3 to get $\|p_h^1\|_\infty\le Ch^{-1/2}(h^{k+1}+(\Delta t)^2)+Ch^{k+1}+C\le C$, since $(\Delta t)^2\le\gamma h^{1/2}$ and $\Delta t,h\le 1$.Similarly, we can also show that $\|q_h^1\|_\infty,\|\phi_h^1\|_\infty\le C$. In addition, $\|(\phi_h^1-\phi_h^0)/(\Delta t)\|_\infty\le\|\phi_t(\cdot,0)\|_\infty+\|(\phi^0)_{xx}-\phi^0+(p^0)^2+(q^0)^2\|_\infty(\Delta t)/2\le C$. Thus the L^∞-assumption is satisfied for $n=0$ and 1.

Now we assume the L^∞-assumption is satisfied up to n. That is, $\|p_h^i\|_\infty$, $\|q_h^i\|_\infty$, $\|\phi_h^i\|_\infty$, $\|(\phi_h^i-\phi_h^{i-1})/(\Delta t)\|_\infty\le C$ for $i\le n$. Therefore, we have $\|\xi_p^{n+1}\|$, $\|\xi_q^{n+1}\|$, $\|\xi_\phi^{n+1}\|$, $\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\|\le Ch^{k+1}+C(\Delta t)^2$, based on the previous proof. Thus

$$\begin{aligned}\|p_h^{n+1}\|_\infty &\le \|\xi_p^{n+1}\|_\infty+\|\eta_p^{n+1}\|_\infty+\|p^{n+1}\|_\infty\le Ch^{-\frac12}((\Delta t)^2+h^{k+1})+Ch^{k+1}+C\\ &\le C(1+h^{k+1}+h^{k+\frac12}+(\Delta t)^2h^{-\frac12})\le C.\end{aligned}$$

Here we have applied the inverse inequality to the second inequality above. The last inequality above is due to $h < 1$ and $(\Delta t)^2 \leq \gamma h^{\frac{1}{2}}$. Similarly $\|q_h^{n+1}\|_\infty$, $\|\phi_h^{n+1}\|_\infty \leq C$. For $\|(\phi_h^{n+1} - \phi_h^n)/(\Delta t)\|_\infty$, we have

$$\begin{aligned}\left\|\frac{\phi_h^{n+1} - \phi_h^n}{\Delta t}\right\|_\infty &\leq \left\|\frac{\Pi^+(\phi^{n+1} - \phi^n)}{\Delta t}\right\|_\infty + \left\|\frac{\xi_\phi^{n+1} - \xi_\phi^n}{\Delta t}\right\|_\infty \leq C + Ch^{-\frac{1}{2}}(h^{k+1} + (\Delta t)^2)\\ &\leq C.\end{aligned}$$

Thus we have proved the L^∞-assumption at $n+1$, which concludes the proof of this theorem. □

4. PROOFS OF SOME LEMMAS

In this section, we present the proofs of some lemmas in Section 3.

4.1. **Proof of Lemma 3.3.** Based on the definition of q_h^0, p_h^0, ϕ_h^0, u_h^0, z_h^0 and s_h^0, it is easy to show that $\|\xi_q^0\| = \|\xi_p^0\| = \|\xi_\phi^0\| = \|\xi_u^0\| = \|\xi_z^0\| = \|\xi_s^0\| = 0$. Using the definition of ϕ_h^1 and Taylor expansion, we can also verify that $\|\xi_\phi^1\| \leq C(\Delta t)^3$. Next we show the upper bound of $\|\xi_q^1\|^2 + \|\xi_p^1\|^2$. Let $w_1 = (\xi_q^1 + \xi_q^0)\Delta t$ in (3.4), $w_2 = (\xi_p^1 + \xi_p^0)\Delta t$ in (3.5), $w_3 = -\frac{1}{2}(\xi_z^1 + \xi_z^0)\Delta t$ in (3.6), $w_4 = \frac{1}{2}(\xi_u^1 + \xi_u^0)\Delta t$ in (3.6), and add the resulting equations to get

$$\|\xi_q^1\|^2 + \|\xi_p^1\|^2 - \|\xi_q^0\|^2 - \|\xi_p^0\|^2 = \Lambda_1 + \Lambda_2 + \Lambda_3,$$

where

$$\begin{aligned}\Lambda_1 &= \frac{\Delta t}{2}\sum_{j=1}^N [(\xi_u^1 + \xi_u^0)(\xi_q^1 + \xi_q^0)]_{j+\frac{1}{2}} - \frac{\Delta t}{2}\sum_{j=1}^N (\xi_u^1 + \xi_u^0)^-_{j+\frac{1}{2}}[\xi_q^1 + \xi_q^0]_{j+\frac{1}{2}}\\ &\quad -\frac{\Delta t}{2}\sum_{j=1}^N (\xi_q^1 + \xi_q^0)^+_{j+\frac{1}{2}}[\xi_u^1 + \xi_u^0]_{j+\frac{1}{2}} - \frac{\Delta t}{2}\sum_{j=1}^N [(\xi_z^1 + \xi_z^0)(\xi_p^1 + \xi_p^0)]_{j+\frac{1}{2}}\\ &\quad +\frac{\Delta t}{2}\sum_{j=1}^N (\xi_z^1 + \xi_z^0)^-_{j+\frac{1}{2}}[\xi_p^1 + \xi_p^0]_{j+\frac{1}{2}} + \frac{\Delta t}{2}\sum_{j=1}^N (\xi_p^1 + \xi_p^0)^+_{j+\frac{1}{2}}[\xi_z^1 + \xi_z^0]_{j+\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}\Lambda_2 &= \int_I (\eta_q^1 - \eta_q^0)(\xi_q^1 + \xi_q^0)dx + \int_I (\eta_p^1 - \eta_p^0)(\xi_p^1 + \xi_p^0)dx - (\Delta t)\int_I T_q^{\frac{1}{2}}(\xi_q^1 + \xi_q^0)dx\\ &\quad -(\Delta t)\int_I T_p^{\frac{1}{2}}(\xi_p^1 + \xi_p^0)dx - \frac{\Delta t}{2}\int_I \left((\eta_u^1 + \eta_u^0)(\xi_z^1 + \xi_z^0) - (\eta_z^1 + \eta_z^0)(\xi_u^1 + \xi_u^0)\right)dx,\end{aligned}$$

and

$$\begin{aligned}\Lambda_3 &= \frac{\Delta t}{4}\int_I (\phi_h^1 + \phi_h^0)(p_h^1 + p_h^0)(\xi_q^1 + \xi_q^0)dx - \frac{\Delta t}{4}\int_I (\phi^1 + \phi^0)(p^1 + p^0)(\xi_q^1 + \xi_q^0)dx\\ &\quad -\frac{\Delta t}{4}\int_I (\phi_h^1 + \phi_h^0)(q_h^1 + q_h^0)(\xi_p^1 + \xi_p^0)dx + \frac{\Delta t}{4}\int_I (\phi^1 + \phi^0)(q^1 + q^0)(\xi_p^1 + \xi_p^0)dx.\end{aligned}$$

We can show that $\Lambda_1 = 0$ due to the property of jump (2.1). For Λ_2, we have

$$\begin{aligned}\Lambda_2 \quad &\leq \quad \|\eta_q^1 - \eta_q^0\|\|\xi_q^1 + \xi_q^0\| + \|\eta_p^1 - \eta_p^0\|\|\xi_p^1 + \xi_p^0\| + (\Delta t)\|T_q^{\frac{1}{2}}\|\|\xi_q^1 + \xi_q^0\| \\ &\quad +(\Delta t)\|T_p^{\frac{1}{2}}\|\|\xi_p^1 + \xi_p^0\| + \frac{\Delta t}{2}\|\eta_u^1 + \eta_u^0\|\|\xi_z^1 + \xi_z^0\| + \frac{\Delta t}{2}\|\eta_z^1 + \eta_z^0\|\|\xi_u^1 + \xi_u^0\|, \\ &\leq \quad C(\Delta t)h^{k+1}(\|\xi_q^1\| + \|\xi_p^1\|) + C(\Delta t)^3(\|\xi_q^1\| + \|\xi_p^1\|) + C(\Delta t)h^{k+1}(\|\xi_z^1\| + \|\xi_u^1\|).\end{aligned}$$

Here we have used Cauchy-Schwartz inequality in the first inequality above, and applied Lemma 3.1 and Lemma 3.2, as well as the fact that $\|\xi_q^0\| = \|\xi_p^0\| = \|\xi_z^0\| = \|\xi_u^0\| = 0$ in the second inequality. For Λ_3, we first rewrite $(\phi_h^1 + \phi_h^0)(p_h^1 + p_h^0) - (\phi^1 + \phi^0)(p^1 + p^0)$ as $(\phi_h^1 + \phi_h^0)(e_p^1 + e_p^0) + (p^1 + p^0)(e_\phi^1 + e_\phi^0) = (\phi_h^1 + \phi_h^0)(\eta_p^1 + \eta_p^0) - (\phi_h^1 + \phi_h^0)(\xi_p^1 + \xi_p^0) + (p^1 + p^0)(e_\phi^1 + e_\phi^0)$, and rewrite $(\phi_h^1 + \phi_h^0)(q_h^1 + q_h^0) - (\phi^1 + \phi^0)(q^1 + q^0)$ as $(\phi_h^1 + \phi_h^0)(\eta_q^1 + \eta_q^0) - (\phi_h^1 + \phi_h^0)(\xi_q^1 + \xi_q^0) + (q^1 + q^0)(e_\phi^1 + e_\phi^0)$. We then substitute these equations to the formulation of Λ_3, and it can be estimated as

$$\begin{aligned}\Lambda_3 \quad &= \quad \frac{\Delta t}{4}\int_I(\phi_h^1 + \phi_h^0)(\eta_p^1 + \eta_p^0)(\xi_q^1 + \xi_q^0)dx + \frac{\Delta t}{4}\int_I(p^1 + p^0)(e_\phi^1 + e_\phi^0)(\xi_q^1 + \xi_q^0)dx \\ &\quad -\frac{\Delta t}{4}\int_I(\phi_h^1 + \phi_h^0)(\eta_q^1 + \eta_q^0)(\xi_p^1 + \xi_p^0)dx - \frac{\Delta t}{4}\int_I(q^1 + q^0)(e_\phi^1 + e_\phi^0)(\xi_p^1 + \xi_p^0)dx \\ &\leq \quad C(\Delta t)h^{k+1}\|\phi_h^1 + \phi_h^0\|_\infty\left(\|\xi_q^1 + \xi_q^0\| + \|\xi_p^1 + \xi_p^0\|\right) \\ &\quad +\frac{\Delta t}{4}\|p^1 + p^0\|_\infty\|e_\phi^1 + e_\phi^0\|\|\xi_q^1 + \xi_q^0\| + \frac{\Delta t}{4}\|q^1 + q^0\|_\infty\|e_\phi^1 + e_\phi^0\|\|\xi_p^1 + \xi_p^0\| \\ &\leq \quad C(\Delta t)h^{k+1}(\|\phi_h^1\|_\infty + \|\phi_h^0\|_\infty)(\|\xi_q^1\| + \|\xi_p^1\|) \\ &\quad +C(\Delta t)(\|\xi_\phi^1\| + Ch^{k+1})(\|\xi_q^1\| + \|\xi_p^1\|).\end{aligned}$$

Here we have used Cauchy-Schwartz inequality and approximation property (2.2) in the first inequality above. Note that $\|\phi_h^0\|_\infty = \|\Pi^+\phi^0 - \phi^0 + \phi^0\|_\infty \leq \|\Pi^+\phi^0 - \phi^0\|_\infty + \|\phi^0\|_\infty \leq C$, $\|\phi_h^1 - \phi_h^0\|_\infty \leq (\Delta t)\|\Pi^+\phi_t(\cdot, 0)\|_\infty + (\Delta t)^2\|\Pi^+((\phi^0)_{xx} - \phi^0 + (p^0)^2 + (q^0)^2)\|/2 \leq C$. Thus $\|\phi_h^1\|_\infty \leq C$. Therefore, we can estimate Λ_3 as

$$\Lambda_3 \leq C(\Delta t)(\|\xi_\phi^1\| + Ch^{k+1})(\|\xi_q^1\| + \|\xi_p^1\|).$$

We now combine all the estimates of Λ_1, Λ_2 and Λ_3 above to get

$$\begin{aligned}\|\xi_q^1\|^2 + \|\xi_p^1\|^2 \quad &\leq \quad C(\Delta t)\left(h^{k+1} + (\Delta t)^2 + \|\xi_\phi^1\|\right)\left(\|\xi_q^1\| + \|\xi_p^1\|\right) \\ &\quad +C(\Delta t)h^{k+1}\left(\|\xi_z^1\| + \|\xi_u^1\|\right) \\ &\leq \quad C(\Delta t)\left(h^{k+1} + (\Delta t)^2\right)\left(\|\xi_q^1\| + \|\xi_p^1\|\right) \\ &\quad +C(\Delta t)h^{k+1}\left(\|\xi_z^1\| + \|\xi_u^1\|\right). \qquad (4.1)\end{aligned}$$

Here we have used the fact that $\|\xi_\phi^1\| \le C(\Delta t)^3$. In order to estimate $\|\xi_q^1\|^2 + \|\xi_p^1\|^2$, we need to further estimate $\|\xi_z^1\| + \|\xi_u^1\|$. Just as we derive (3.6) and (3.7), we can show that

$$\begin{aligned}\int_I (\xi_u^1 - \xi_u^0) w_3\, dx &= -\int_I (\xi_p^1 - \xi_p^0)(w_3)_x\, dx - \sum_{j=1}^N (\xi_p^1 - \xi_p^0)^+_{j+\frac{1}{2}} [w_3]_{j+\frac{1}{2}} \\ &\quad + \int_I (\eta_u^1 - \eta_u^0) w_3\, dx, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\int_I (\xi_z^1 - \xi_z^0) w_4\, dx &= -\int_I (\xi_q^1 - \xi_q^0)(w_4)_x\, dx - \sum_{j=1}^N (\xi_q^1 - \xi_q^0)^+_{j+\frac{1}{2}} [w_4]_{j+\frac{1}{2}} \\ &\quad + \int_I (\eta_z^1 - \eta_z^0) w_4\, dx. \end{aligned} \tag{4.3}$$

Let $w_3 = \xi_u^1 + \xi_u^0$ in (4.2), $w_4 = \xi_z^1 + \xi_z^0$ in (4.3) and add these equations, we have

$$\begin{aligned}\|\xi_u^1\|^2 + \|\xi_z^1\|^2 &= -\int_I (\xi_p^1 - \xi_p^0)(\xi_u^1 + \xi_u^0)_x\, dx - \sum_{j=1}^N (\xi_p^1 - \xi_p^0)^+_{j+\frac{1}{2}} [\xi_u^1 + \xi_u^0]_{j+\frac{1}{2}} \\ &\quad + \int_I (\eta_u^1 - \eta_u^0)(\xi_u^1 + \xi_u^0)\, dx - \int_I (\xi_q^1 - \xi_q^0)(\xi_z^1 + \xi_z^0)_x\, dx \\ &\quad - \sum_{j=1}^N (\xi_q^1 - \xi_q^0)^+_{j+\frac{1}{2}} [\xi_z^1 + \xi_z^0]_{j+\frac{1}{2}} + \int_I (\eta_z^1 - \eta_z^0)(\xi_z^1 + \xi_z^0)\, dx. \end{aligned} \tag{4.4}$$

To simplify the right side of the equality above, we take $w_1 = 2(\xi_p^1 - \xi_p^0)$ in (3.4), $w_2 = -2(\xi_q^1 - \xi_q^0)$ in (3.5), and add the resulting equations to obtain

$$\begin{aligned}0 &= -\int_I (\xi_u^1 + \xi_u^0)(\xi_p^1 - \xi_p^0)_x dx - \sum_{j=1}^N (\xi_u^1 + \xi_u^0)^-_{j+\frac{1}{2}} [\xi_p^1 - \xi_p^0]_{j+\frac{1}{2}} - 2\int_I T_q^{\frac{1}{2}} (\xi_p^1 - \xi_p^0) dx \\ &\quad - \int_I (\xi_z^1 + \xi_z^0)(\xi_q^1 - \xi_q^0)_x dx - \sum_{j=1}^N (\xi_z^1 + \xi_z^0)^-_{j+\frac{1}{2}} [\xi_q^1 - \xi_q^0]_{j+\frac{1}{2}} + 2\int_I T_p^{\frac{1}{2}} (\xi_q^1 - \xi_q^0) dx \\ &\quad + \frac{1}{2}\int_I ((\phi_h^1 + \phi_h^0)(p_h^1 + p_h^0) - (\phi^1 + \phi^0)(p^1 + p^0))(\xi_p^1 - \xi_p^0) dx \\ &\quad + \frac{1}{2}\int_I ((\phi_h^1 + \phi_h^0)(q_h^1 + q_h^0) - (\phi^1 + \phi^0)(q^1 + q^0))(\xi_q^1 - \xi_q^0) dx \\ &\quad + 2\int_I \frac{\eta_q^1 - \eta_q^0}{\Delta t} (\xi_p^1 - \xi_p^0) dx - 2\int_I \frac{\eta_p^1 - \eta_p^0}{\Delta t} (\xi_q^1 - \xi_q^0) dx. \end{aligned} \tag{4.5}$$

We then add (4.4) and (4.5) and apply the property of jump (2.1) to get

$$\begin{aligned}\|\xi_u^1\|^2+\|\xi_z^1\|^2 &= \int_I(\eta_u^1-\eta_u^0)(\xi_u^1+\xi_u^0)dx+\int_I(\eta_z^1-\eta_z^0)(\xi_z^1+\xi_z^0)dx\\ &\quad -2\int_I T_q^{\frac{1}{2}}(\xi_p^1-\xi_p^0)dx+2\int_I T_p^{\frac{1}{2}}(\xi_q^1-\xi_q^0)dx\\ &\quad +\frac{1}{2}\int_I((\phi_h^1+\phi_h^0)(p_h^1+p_h^0)-(\phi^1+\phi^0)(p^1+p^0))(\xi_p^1-\xi_p^0)dx\\ &\quad +\frac{1}{2}\int_I((\phi_h^1+\phi_h^0)(q_h^1+q_h^0)-(\phi^1+\phi^0)(q^1+q^0))(\xi_q^1-\xi_q^0)dx\\ &\quad +2\int_I\frac{\eta_q^1-\eta_q^0}{\Delta t}(\xi_p^1-\xi_p^0)dx-2\int_I\frac{\eta_p^1-\eta_p^0}{\Delta t}(\xi_q^1-\xi_q^0)dx. \qquad (4.6)\end{aligned}$$

To estimate the upper bound of $\|\xi_u^1\|^2+\|\xi_z^1\|^2$, we denote each line of the right side of (4.6) as Ψ_1, Ψ_2 and Ψ_3, respectively. Ψ_1 can be estimated using Lemma 3.1 and 3.2. That is,

$$\Psi_1\le C(\Delta t)h^{k+1}(\|\xi_u^1\|+\|\xi_z^1\|)+C(\Delta t)^2(\|\xi_p^1\|+\|\xi_q^1\|). \qquad (4.7)$$

Since $(\phi_h^1+\phi_h^0)(p_h^1+p_h^0)-(\phi^1+\phi^0)(p^1+p^0)=(\phi_h^1+\phi_h^0)(e_p^1+e_p^0)+(p^1+p^0)(e_\phi^1+e_\phi^0)$, we have

$$\begin{aligned}\Psi_2 &\le \frac{1}{2}(\|\phi_h^1\|_\infty+\|\phi_h^0\|_\infty)(\|e_p^1\|+\|e_p^0\|)\|\xi_p^1\|+\frac{1}{2}(\|p^1\|_\infty+\|p^0\|_\infty)(\|e_\phi^1\|+\|e_\phi^0\|)\|\xi_p^1\|\\ &\quad +\frac{2}{\Delta t}\|\eta_q^1-\eta_q^0\|\|\xi_p^1\|\\ &\le C(\|\xi_p^1\|+Ch^{k+1})\|\xi_p^1\|+C(h^{k+1}+(\Delta t)^3)\|\xi_p^1\|+Ch^{k+1}\|\xi_p^1\|\\ &\le C\|\xi_p^1\|^2+C(h^{k+1}+(\Delta t)^3)\|\xi_p^1\|, \qquad (4.8)\end{aligned}$$

where the second inequality is due to the fact that $\|\phi_h^1\|_\infty$, $\|\phi_h^0\|_\infty\le C$ and $\|\eta_q^1-\eta_q^0\|\le C(\Delta t)h^{k+1}$. Similarly, we can also derive that

$$\Psi_3 \le C\|\xi_q^1\|^2+C(h^{k+1}+(\Delta t)^3)\|\xi_q^1\|. \qquad (4.9)$$

Now we apply all the estimates (4.7)-(4.9) to (4.6), and have

$$\begin{aligned}&\frac{1}{2}(\|\xi_u^1\|+\|\xi_z^1\|)^2\le\|\xi_u^1\|^2+\|\xi_z^1\|^2\\ &\le C(\|\xi_p^1\|^2+\|\xi_q^1\|^2)+C(h^{k+1}+(\Delta t)^2)(\|\xi_p^1\|+\|\xi_q^1\|)+C(\Delta t)h^{k+1}(\|\xi_u^1\|+\|\xi_z^1\|)\\ &\le C(\|\xi_p^1\|^2+\|\xi_q^1\|^2+h^{2k+2}+(\Delta t)^4)+\frac{1}{4}(\|\xi_u^1\|+\|\xi_z^1\|)^2+C^2(\Delta t)^2h^{2k+2}. \qquad (4.10)\end{aligned}$$

The last inequality above is derived using Young's inequality. Therefore, (4.10) leads to $(\|\xi_u^1\|+\|\xi_z^1\|)^2\le C(\|\xi_p^1\|^2+\|\xi_q^1\|^2+h^{2k+2}+(\Delta t)^4)\le C(\|\xi_p^1\|+\|\xi_q^1\|+h^{k+1}+(\Delta t)^2)^2$. We thus have

$$\|\xi_u^1\|+\|\xi_z^1\|\le C(\|\xi_p^1\|+\|\xi_q^1\|+h^{k+1}+(\Delta t)^2). \qquad (4.11)$$

Now we apply (4.11) to the right side of (4.1) to get

$$\begin{aligned}\|\xi_q^1\|^2+\|\xi_p^1\|^2 &\le C(\Delta t)(h^{k+1}+(\Delta t)^2)(\|\xi_q^1\|+\|\xi_p^1\|+h^{k+1})\\ &\le C(\Delta t)(h^{2k+2}+(\Delta t)^4)\ +\frac{1}{4}(\|\xi_q^1\|+\|\xi_p^1\|)^2.\end{aligned}$$

It is easy to show that the inequality above leads to that $\|\xi_q^1\|^2+\|\xi_p^1\|^2 \le C(\Delta t)h^{2k+2}+C(\Delta t)^5$, which concludes the proof.

4.2. **Proof of Lemma 3.4.** This proof is similar to the proof for the estimate of $\|\xi_q^1\|^2+\|\xi_p^1\|^2$ in Lemma 3.3. That is, we first let $w_1=(\xi_q^{n+1}+\xi_q^{n-1})(\Delta t)$ in (3.8), $w_2=(\xi_p^{n+1}+\xi_p^{n-1})(\Delta t)$ in (3.9), $w_3=-\frac{1}{2}(\xi_z^{n+1}+\xi_z^{n-1})(\Delta t)$ in (3.10), $w_4=\frac{1}{2}(\xi_u^{n+1}+\xi_u^{n-1})(\Delta t)$ in (3.11), add these equations and get the energy equations for p and q:

$$\|\xi_q^{n+1}\|^2+\|\xi_p^{n+1}\|^2-\|\xi_q^{n-1}\|^2-\|\xi_p^{n-1}\|^2=\Theta_1+\Theta_2+\Theta_3,$$

where

$$\begin{aligned}&\Theta_1\\ =\ &\frac{\Delta t}{2}\sum_{j=1}^{N}\left([(\xi_u^{n+1}+\xi_u^{n-1})(\xi_q^{n+1}+\xi_q^{n-1})]_{j+\frac12}-(\xi_u^{n+1}+\xi_u^{n-1})^-_{j+\frac12}[\xi_q^{n+1}+\xi_q^{n-1}]_{j+\frac12}\right)\\ -\ &\frac{\Delta t}{2}\sum_{j=1}^{N}\left((\xi_q^{n+1}+\xi_q^{n-1})^+_{j+\frac12}[\xi_u^{n+1}+\xi_u^{n-1}]_{j+\frac12}-[(\xi_z^{n+1}+\xi_z^{n-1})(\xi_p^{n+1}+\xi_p^{n-1})]_{j+\frac12}\right)\\ +\ &\frac{\Delta t}{2}\sum_{j=1}^{N}\left((\xi_z^{n+1}+\xi_z^{n-1})^-_{j+\frac12}[\xi_p^{n+1}+\xi_p^{n-1}]_{j+\frac12}+(\xi_p^{n+1}+\xi_p^{n-1})^+_{j+\frac12}[\xi_z^{n+1}+\xi_z^{n-1}]_{j+\frac12}\right),\end{aligned}$$

$$\begin{aligned}\Theta_2 &= \int_I(\eta_q^{n+1}-\eta_q^{n-1})(\xi_q^{n+1}+\xi_q^{n-1})dx+\int_I(\eta_p^{n+1}-\eta_p^{n-1})(\xi_p^{n+1}+\xi_p^{n-1})dx\\ &- (\Delta t)\int_I T_q^n(\xi_q^{n+1}+\xi_q^{n-1})dx-(\Delta t)\int_I T_p^n(\xi_p^{n+1}+\xi_p^{n-1})dx\\ &- \frac{\Delta t}{2}\int_I\left((\eta_u^{n+1}+\eta_u^{n-1})(\xi_z^{n+1}+\xi_z^{n-1})-(\eta_z^{n+1}+\eta_z^{n-1})(\xi_u^{n+1}+\xi_u^{n-1})\right)dx,\end{aligned}$$

and

$$\begin{aligned}&\Theta_3\\ =\ &\frac{\Delta t}{2}\int_I\phi_h^n(p_h^{n+1}+p_h^{n-1})(\xi_q^{n+1}+\xi_q^{n-1})dx-\frac{\Delta t}{2}\int_I\phi^n(p^{n+1}+p^{n-1})(\xi_q^{n+1}+\xi_q^{n-1})dx\\ -\ &\frac{\Delta t}{2}\int_I\phi_h^n(q_h^{n+1}+q_h^{n-1})(\xi_p^{n+1}+\xi_p^{n-1})dx+\frac{\Delta t}{2}\int_I\phi^n(q^{n+1}+q^{n-1})(\xi_p^{n+1}+\xi_p^{n-1})dx.\end{aligned}$$

It is easy to show that $\Theta_1 = 0$ due to the property of jump (2.1). Θ_2 can then be estimated using Cauchy-Schwartz inequality, Lemma 3.1 and Lemma 3.2. That is,

$$\begin{aligned}
\Theta_2 \leq\ & \|\eta_q^{n+1} - \eta_q^{n-1}\|\|\xi_q^{n+1} + \xi_q^{n-1}\| + \|\eta_p^{n+1} - \eta_p^{n-1}\|\|\xi_p^{n+1} + \xi_p^{n-1}\| \\
& -(\Delta t)\|T_q^n\|\|\xi_q^{n+1} + \xi_q^{n-1}\| + (\Delta t)\|T_p^n\|\|\xi_p^{n+1} + \xi_p^{n-1}\| \\
& + \frac{\Delta t}{2}(\|\eta_u^{n+1} + \eta_u^{n-1}\|\|\xi_z^{n+1} + \xi_z^{n-1}\| + \|\eta_z^{n+1} + \eta_z^{n-1}\|\|\xi_u^{n+1} + \xi_u^{n-1}\|) \\
\leq\ & (C(\Delta t)h^{k+1} + C(\Delta t)^3)(\|\xi_q^{n+1} + \xi_q^{n-1}\| + \|\xi_p^{n+1} + \xi_p^{n-1}\|) \\
& + C(\Delta t)h^{k+1}(\|\xi_u^{n+1} + \xi_u^{n-1}\| + \|\xi_z^{n+1} + \xi_z^{n-1}\|) \\
\leq\ & C(\Delta t)h^{2k+2} + C(\Delta t)^5 + C(\Delta t)(\|\xi_q^{n+1}\|^2 + \|\xi_q^{n-1}\|^2 + \|\xi_p^{n+1}\|^2 + \|\xi_p^{n-1}\|^2) \\
& + C(\Delta t)h^{k+1}(\|\xi_u^{n+1} + \xi_u^{n-1}\| + \|\xi_z^{n+1} + \xi_z^{n-1}\|).
\end{aligned}$$

The last inequality above is due to Cauchy-Schwartz inequality. We then estimate Θ_3 as

$$\begin{aligned}
& \Theta_3 \\
=\ & -\frac{\Delta t}{2}\left(\int_I \phi_h^n(e_p^{n+1} + e_p^{n-1})(\xi_q^{n+1} + \xi_q^{n-1})dx + \int_I e_\phi^n(p^{n+1} + p^{n-1})(\xi_q^{n+1} + \xi_q^{n-1})dx\right) \\
& + \frac{\Delta t}{2}\left(\int_I \phi_h^n(e_q^{n+1} + e_q^{n-1})(\xi_p^{n+1} + \xi_p^{n-1})dx + \int_I e_\phi^n(q^{n+1} + q^{n-1})(\xi_p^{n+1} + \xi_p^{n-1})dx\right) \\
=\ & -\frac{\Delta t}{2}\left(\int_I \phi_h^n(\eta_p^{n+1} + \eta_p^{n-1})(\xi_q^{n+1} + \xi_q^{n-1})dx - \int_I \phi_h^n(\eta_q^{n+1} + \eta_q^{n-1})(\xi_p^{n+1} + \xi_p^{n-1})dx\right) \\
& -\frac{\Delta t}{2}\left(\int_I e_\phi^n(p^{n+1} + p^{n-1})(\xi_q^{n+1} + \xi_q^{n-1})dx - \int_I e_\phi^n(q^{n+1} + q^{n-1})(\xi_p^{n+1} + \xi_p^{n-1})dx\right) \\
\leq\ & C(\Delta t)h^{k+1}\|\phi_h^n\|_\infty(\|\xi_q^{n+1}\| + \|\xi_q^{n-1}\| + \|\xi_p^{n+1}\| + \|\xi_p^{n-1}\|) \\
& + C(\Delta t)(\|\xi_\phi^n\| + Ch^{k+1})(\|\xi_q^{n+1}\| + \|\xi_q^{n-1}\| + \|\xi_p^{n+1}\| + \|\xi_p^{n-1}\|) \\
\leq\ & C(\Delta t)h^{k+1}(1 + \|\phi_h^n\|_\infty)(\|\xi_q^{n+1}\| + \|\xi_q^{n-1}\| + \|\xi_p^{n+1}\| + \|\xi_p^{n-1}\|) \\
& + C(\Delta t)(\|\xi_\phi^n\|^2 + \|\xi_q^{n+1}\|^2 + \|\xi_q^{n-1}\|^2 + \|\xi_p^{n+1}\|^2 + \|\xi_p^{n-1}\|^2).
\end{aligned}$$

Finally, we combine the estimates of Θ_1, Θ_2 and Θ_3 to conclude the proof.

4.3. **Proof of Lemma 3.5.** Similar to the derivation of (3.10) and (3.11), it is easy to show the following two inequalities

$$\begin{aligned}
\int_I (\xi_u^{n+1} - \xi_u^{n-1})w_3\, dx \ =\ & -\int_I (\xi_p^{n+1} - \xi_p^{n-1})(w_3)_x\, dx - \sum_{j=1}^{N}(\xi_p^{n+1} - \xi_p^{n-1})^+_{j+\frac{1}{2}}[w_3]_{j+\frac{1}{2}} \\
& + \int_I (\eta_u^{n+1} - \eta_u^{n-1})w_3\, dx, \qquad (4.12)
\end{aligned}$$

$$\int_I (\xi_z^{n+1} - \xi_z^{n-1}) w_4 \, dx = -\int_I (\xi_q^{n+1} - \xi_q^{n-1})(w_4)_x \, dx - \sum_{j=1}^{N} (\xi_q^{n+1} - \xi_q^{n-1})^+_{j+\frac{1}{2}} [w_4]_{j+\frac{1}{2}} + \int_I (\eta_z^{n+1} - \eta_z^{n-1}) w_4 \, dx. \quad (4.13)$$

We then let $w_3 = \xi_u^{n+1} + \xi_u^{n-1}$ in (4.12), let $w_4 = \xi_z^{n+1} + \xi_z^{n-1}$ in (4.13), and add these two equations to obtain

$$\begin{aligned} & \|\xi_u^{n+1}\|^2 + \|\xi_z^{n+1}\|^2 - \|\xi_u^{n-1}\|^2 - \|\xi_z^{n-1}\|^2 \\ = & \int_I (\eta_u^{n+1} - \eta_u^{n-1})(\xi_u^{n+1} + \xi_u^{n-1}) dx + \int_I (\eta_z^{n+1} - \eta_z^{n-1})(\xi_z^{n+1} + \xi_z^{n-1}) dx \\ & - \int_I (\xi_p^{n+1} - \xi_p^{n-1})(\xi_u^{n+1} + \xi_u^{n-1})_x \, dx - \sum_{j=1}^{N} \left(\xi_p^{n+1} - \xi_p^{n-1}\right)^+_{j+\frac{1}{2}} [\xi_u^{n+1} + \xi_u^{n-1}]_{j+\frac{1}{2}} \\ & - \int_I (\xi_q^{n+1} - \xi_q^{n-1})(\xi_z^{n+1} + \xi_z^{n-1})_x \, dx - \sum_{j=1}^{N} \left(\xi_q^{n+1} - \xi_q^{n-1}\right)^+_{j+\frac{1}{2}} [\xi_z^{n+1} + \xi_z^{n-1}]_{j+\frac{1}{2}}. \end{aligned} \quad (4.14)$$

To simplify the second and the third line on the right side of Eq. (4.14), we take $w_1 = 2(\xi_p^{n+1} - \xi_p^{n-1})$ in (3.8), and take $w_2 = -2(\xi_q^{n+1} - \xi_q^{n-1})$ in and (3.9), and add the equations to get

$$\begin{aligned} 0 = & -\int_I (\xi_u^{n+1} + \xi_u^{n-1})(\xi_p^{n+1} - \xi_p^{n-1})_x \, dx - \sum_{j=1}^{N} \left(\xi_u^{n+1} + \xi_u^{n-1}\right)^-_{j+\frac{1}{2}} [\xi_p^{n+1} - \xi_p^{n-1}]_{j+\frac{1}{2}} \\ & - \int_I (\xi_z^{n+1} + \xi_z^{n-1})(\xi_q^{n+1} - \xi_q^{n-1})_x \, dx - \sum_{j=1}^{N} \left(\xi_z^{n+1} + \xi_z^{n-1}\right)^-_{j+\frac{1}{2}} [\xi_q^{n+1} - \xi_q^{n-1}]_{j+\frac{1}{2}} \\ & + \int_I \left(\phi_h^n (p_h^{n+1} + p_h^{n-1}) - \phi^n (p^{n+1} + p^{n-1})\right) (\xi_p^{n+1} - \xi_p^{n-1}) dx \\ & + \int_I \left(\phi_h^n (q_h^{n+1} + q_h^{n-1}) - \phi^n (q^{n+1} + q^{n-1})\right) (\xi_q^{n+1} - \xi_q^{n-1}) dx \\ & + \int_I \frac{\eta_q^{n+1} - \eta_q^{n-1}}{\Delta t} (\xi_p^{n+1} - \xi_p^{n-1}) \, dx - \int_I \frac{\eta_p^{n+1} - \eta_p^{n-1}}{\Delta t} (\xi_q^{n+1} - \xi_q^{n-1}) \, dx \\ & -2 \int_I T_q^n (\xi_p^{n+1} - \xi_p^{n-1}) \, dx + 2 \int_I T_p^n (\xi_q^{n+1} - \xi_q^{n-1}) \, dx. \end{aligned} \quad (4.15)$$

We now add (4.14) and (4.15), and apply the property of jump (2.1) to get

$$\begin{aligned}
&\|\xi_u^{n+1}\|^2 + \|\xi_z^{n+1}\|^2 - \|\xi_u^{n-1}\|^2 - \|\xi_z^{n-1}\|^2 \\
= & \int_I (\eta_u^{n+1} - \eta_u^{n-1})(\xi_u^{n+1} + \xi_u^{n-1})dx + \int_I (\eta_z^{n+1} - \eta_z^{n-1})(\xi_z^{n+1} + \xi_z^{n-1})dx \\
& + \int_I \frac{\eta_q^{n+1} - \eta_q^{n-1}}{\Delta t}(\xi_p^{n+1} - \xi_p^{n-1})\, dx - \int_I \frac{\eta_p^{n+1} - \eta_p^{n-1}}{\Delta t}(\xi_q^{n+1} - \xi_q^{n-1})\, dx \\
& -2\int_I T_q^n(\xi_p^{n+1} - \xi_p^{n-1})\, dx + 2\int_I T_p^n(\xi_q^{n+1} - \xi_q^{n-1})\, dx \\
& + \int_I \left(\phi_h^n(p_h^{n+1} + p_h^{n-1}) - \phi^n(p^{n+1} + p^{n-1})\right)(\xi_p^{n+1} - \xi_p^{n-1})dx \\
& + \int_I \left(\phi_h^n(q_h^{n+1} + q_h^{n-1}) - \phi^n(q^{n+1} + q^{n-1})\right)(\xi_q^{n+1} - \xi_q^{n-1})dx.
\end{aligned}$$

We then take the sum of the equation above over $n = 1, 2, \ldots$. Thus we have

$$\|\xi_u^{n+1}\|^2 + \|\xi_z^{n+1}\|^2 = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5,$$

where

$$\begin{aligned}
\Lambda_1 &= \sum_{i=1}^n \int_I (\eta_u^{i+1} - \eta_u^{i-1})(\xi_u^{i+1} + \xi_u^{i-1})dx + \sum_{i=1}^n \int_I (\eta_z^{i+1} - \eta_z^{i-1})(\xi_z^{i+1} + \xi_z^{i-1})dx, \\
\Lambda_2 &= \sum_{i=1}^n \int_I \frac{\eta_q^{i+1} - \eta_q^{i-1}}{\Delta t}(\xi_p^{i+1} - \xi_p^{i-1})dx - \sum_{i=1}^n \int_I \frac{\eta_p^{i+1} - \eta_p^{i-1}}{\Delta t}(\xi_q^{i+1} - \xi_q^{i-1})dx, \\
\Lambda_3 &= -2\sum_{i=1}^n \int_I T_q^i(\xi_p^{i+1} - \xi_p^{i-1})\, dx + 2\sum_{i=1}^n \int_I T_p^i(\xi_q^{i+1} - \xi_q^{i-1})\, dx, \\
\Lambda_4 &= \sum_{i=1}^n \int_I \left(\phi_h^i(p_h^{i+1} + p_h^{i-1}) - \phi^i(p^{i+1} + p^{i-1})\right)(\xi_p^{i+1} - \xi_p^{i-1})dx,
\end{aligned}$$

and

$$\Lambda_5 = \sum_{i=1}^n \int_I \left(\phi_h^i(q_h^{i+1} + q_h^{i-1}) - \phi^i(q^{i+1} + q^{i-1})\right)(\xi_q^{i+1} - \xi_q^{i-1})dx.$$

Next we estimate the upper bound of each Λ_i for $i = 1, \ldots, 5$. For Λ_1, we apply Lemma 3.2 and Cauchy-Schwartz inequality to get

$$\begin{aligned}
\Lambda_1 &\leq C(\Delta t)h^{k+1}\sum_{i=1}^n \left(\|\xi_u^{i+1}\| + \|\xi_u^{i-1}\| + \|\xi_z^{i+1}\| + \|\xi_z^{i-1}\|\right) \\
&\leq Ch^{k+1}\max_{1\leq i\leq n+1}\left(\|\xi_u^i\| + \|\xi_z^i\|\right), \qquad (4.16)
\end{aligned}$$

where we have used the fact that $(n+1)\Delta t \leq T$.

For Λ_2, we apply the summation-by-parts formula (3.14) to get

$$\begin{aligned}
\Lambda_2 &= \frac{1}{\Delta t}\left(-\int_I(\eta_q^2-\eta_q^0)\xi_p^0dx-\int_I(\eta_q^3-\eta_q^1)\xi_p^1dx+\int_I(\eta_q^n-\eta_q^{n-2})\xi_p^ndx\right)\\
&\quad+\frac{1}{\Delta t}\left(\int_I(\eta_q^{n+1}-\eta_q^{n-1})\xi_p^{n+1}dx+\int_I(\eta_p^2-\eta_p^0)\xi_q^0dx+\int_I(\eta_p^3-\eta_p^1)\xi_q^1dx\right)\\
&\quad-\frac{1}{\Delta t}\left(\int_I(\eta_p^n-\eta_p^{n-2})\xi_q^ndx+\int_I(\eta_p^{n+1}-\eta_p^{n-1})\xi_q^{n+1}dx\right)\\
&\quad-\frac{1}{\Delta t}\sum_{i=2}^{n-1}\int_I(\eta_q^{i+2}-2\eta_q^i+\eta_q^{i-2})\xi_p^idx+\frac{1}{\Delta t}\sum_{i=2}^{n-1}\int_I(\eta_p^{i+2}-2\eta_p^i+\eta_p^{i-2})\xi_q^idx\\
&\le Ch^{k+1}(\|\xi_p^1\|+\|\xi_p^n\|+\|\xi_p^{n+1}\|+\|\xi_q^1\|+\|\xi_q^n\|+\|\xi_q^{n+1}\|)\\
&\quad+CTh^{k+1}\max_{0\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|).
\end{aligned}$$

Note that we have used Lemma 3.2 and $(n+1)\Delta t \le T$ in the last inequality above. Thus we have

$$\Lambda_2 \le Ch^{k+1}\max_{0\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|). \tag{4.17}$$

Next we consider Λ_3. Using summation-by-parts, there is

$$\begin{aligned}
\Lambda_3 &= -2\left(-\int_I T_q^1\xi_p^0dx-\int_I T_q^2\xi_p^1dx-\sum_{i=2}^{n-1}\int_I(T_q^{i+1}-T_q^{i-1})\xi_p^idx\right)\\
&\quad-2\left(\int_I T_q^{n-1}\xi_p^ndx+\int_I T_q^n\xi_p^{n+1}dx+\int_I T_p^1\xi_q^0dx-\int_I T_p^2\xi_q^1dx\right)\\
&\quad+2\left(-\sum_{i=2}^{n-1}\int_I(T_p^{i+1}-T_p^{i-1})\xi_q^idx+\int_I T_p^{n-1}\xi_q^ndx+\int_I T_p^n\xi_q^{n+1}dx\right)\\
&\le C(\Delta t)^2(\|\xi_p^1\|+\|\xi_p^n\|+\|\xi_p^{n+1}\|+\|\xi_q^1\|+\|\xi_q^n\|+\|\xi_q^{n+1}\|)\\
&\quad+CT(\Delta t)^2\max_{2\le i\le n-1}(\|\xi_p^i\|+\|\xi_q^i\|)\\
&\le C(\Delta t)^2\max_{1\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|).
\end{aligned} \tag{4.18}$$

For Λ_4, since

$$\begin{aligned}
&\phi_h^i(p_h^{i+1}+p_h^{i-1})-\phi^i(p^{i+1}+p^{i-1})\\
=\;&\phi_h^i(\xi_p^{i+1}+\xi_p^{i-1})-\phi_h^i(\eta_p^{i+1}+\eta_p^{i-1})+(p^{i+1}+p^{i-1})\xi_\phi^i-(p^{i+1}+p^{i-1})\eta_\phi^i,
\end{aligned}$$

we have

$$
\begin{aligned}
\Lambda_4 &= \sum_{i=1}^{n}\int_I \phi_h^i \left((\xi_p^{i+1})^2-(\xi_p^{i-1})^2\right)dx-\sum_{i=1}^{n}\int_I \phi_h^i(\eta_p^{i+1}+\eta_p^{i-1})(\xi_p^{i+1}-\xi_p^{i-1})dx\\
&\quad+\sum_{i=1}^{n}\int_I (p^{i+1}+p^{i-1})\xi_\phi^i(\xi_p^{i+1}-\xi_p^{i-1})dx-\sum_{i=1}^{n}\int_I (p^{i+1}+p^{i-1})\eta_\phi^i(\xi_p^{i+1}-\xi_p^{i-1})dx.
\end{aligned}
\tag{4.19}
$$

We denote each term on the right side of (4.19) as Λ_{41}, Λ_{42}, Λ_{43} and Λ_{44}, respectively. We can apply summation-by-parts to estimate each of these terms. Firstly, we have

$$
\begin{aligned}
\Lambda_{41} &= -\int_I \phi_h^1(\xi_p^0)^2dx-\int_I \phi_h^2(\xi_p^1)^2dx-\sum_{i=2}^{n-1}\int_I(\phi_h^{i+1}-\phi_h^{i-1})(\xi_p^i)^2dx+\int_I\phi_h^{n-1}(\xi_p^n)^2dx\\
&\quad+\int_I\phi_h^n(\xi_p^{n+1})^2dx\\
&\le \|\phi_h^1\|_\infty\|\xi_p^0\|^2+\|\phi_h^2\|_\infty\|\xi_p^1\|^2+\sum_{i=2}^{n-1}\|\phi_h^{i+1}-\phi_h^{i-1}\|_\infty\|\xi_p^i\|^2+\|\phi_h^{n-1}\|_\infty\|\xi_p^n\|^2\\
&\quad+\|\phi_h^n\|_\infty\|\xi_p^{n+1}\|^2\\
&\le C\max_{1\le i\le n}\|\phi_h^i\|_\infty\max_{1\le i\le n+1}\|\xi_h^i\|^2+(\Delta t)\max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty n\max_{2\le i\le n-1}\|\xi_p^i\|^2\\
&\le \left(C\max_{1\le i\le n}\|\phi_h^i\|_\infty+T\max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)\cdot\max_{1\le i\le n+1}\|\xi_p^i\|^2.
\end{aligned}
$$

Secondly, there is

$$
\begin{aligned}
\Lambda_{42} &= -\sum_{i=1}^{n}\int_I\phi_h^i(\eta_p^{i+1}+\eta_p^{i-1})(\xi_p^{i+1}-\xi_p^{i-1})dx\\
&= \int_I\phi_h^1(\eta_p^2+\eta_p^0)\xi_p^0dx+\int_I\phi_h^2(\eta_p^3+\eta_p^1)\xi_p^1dx-\int_I\phi_h^{n-1}(\eta_p^n+\eta_p^{n-2})\xi_p^ndx\\
&\quad+\sum_{i=2}^{n-1}\int_I\left(\phi_h^{i+1}(\eta_p^{i+2}+\eta_p^i)-\phi_h^{i-1}(\eta_p^i+\eta_p^{i-2})\right)\xi_p^idx-\int_I\phi_h^n(\eta_p^{n+1}+\eta_p^{n-1})\xi_p^{n+1}dx.
\end{aligned}
$$

Since

$$
\phi_h^{i+1}(\eta_p^{i+2}+\eta_p^2)-\phi_h^{i-1}(\eta_p^i+\eta_p^{i-2})=(\phi_h^{i+1}-\phi_h^{i-1})(\eta_p^{i+2}+\eta_p^i)+\phi_h^{i-1}(\eta_p^{i+2}-\eta_p^{i-2}),
$$

we can estimate Λ_{42} as

$$\begin{aligned}
\Lambda_{42} &\leq Ch^{k+1}\left(\|\phi_h^1\|_\infty\|\xi_p^0\| + \|\phi_h^2\|_\infty\|\xi_p^1\| + \|\phi_h^{n-1}\|_\infty\|\xi_p^n\| + \|\phi_h^n\|_\infty\|\xi_p^{n+1}\|\right) \\
&\quad + \sum_{i=2}^{n-1}\|\phi_h^{i+1}-\phi_h^{i-1}\|_\infty\|\eta_p^{i+2}+\eta_p^i\|\|\xi_p^i\| + \sum_{i=2}^{n-1}\|\phi_h^{i-1}\|_\infty\|\eta_p^{i+2}-\eta_p^{i-2}\|\|\xi_p^i\| \\
&\leq Ch^{k+1}\max_{1\leq i\leq n}\|\phi_h^i\|_\infty \max_{1\leq i\leq n+1}\|\xi_p^i\| + Ch^{k+1}(\Delta t)\max_{2\leq i\leq n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty \sum_{i=2}^{n-1}\|\xi_p^i\| \\
&\quad + \max_{1\leq i\leq n-2}\|\phi_h^i\|_\infty C(\Delta t)h^{k+1}\sum_{i=2}^{n-1}\|\xi_p^i\| \\
&\leq Ch^{k+1}\max_{1\leq i\leq n}\|\phi_h^i\|_\infty \max_{1\leq i\leq n+1}\|\xi_p^i\| + Ch^{k+1}\max_{2\leq i\leq n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty \max_{2\leq i\leq n-1}\|\xi_p^i\|.
\end{aligned}$$

Thirdly, we can follow the same idea to estimate Λ_{43}, i.e.,

$$\begin{aligned}
\Lambda_{43} &= \sum_{i=1}^{n}\int_I (p^{i+1}+p^{i-1})\xi_\phi^i(\xi_p^{i+1}-\xi_p^{i-1})dx \\
&= -\int_I (p^2+p^0)\xi_\phi^1\xi_p^0 dx - \int_I (p^3+p^1)\xi_\phi^2\xi_p^1 dx + \int_I (p^n+p^{n-2})\xi_\phi^{n-1}\xi_p^n dx \\
&\quad -\sum_{i=2}^{n-1}\int_I\left((p^{i+2}+p^i)\xi_\phi^{i+1}-(p^i+p^{i-2})\xi_\phi^{i-1}\right)\xi_p^i dx + \int_I (p^{n+1}+p^{n-1})\xi_\phi^n\xi_p^{n+1}dx.
\end{aligned}$$

We can rewrite $(p^{i+2}+p^i)\xi_\phi^{i+1}-(p^i+p^{i-2})\xi_\phi^{i-1}$ as $(p^{i+2}+p^i)(\xi_\phi^{i+1}-\xi_\phi^{i-1})+\xi_\phi^{i-1}(p^{i+2}-p^{i-2})$. Thus, the inequality above leads to

$$\begin{aligned}
\Lambda_{43} &\leq C\|\xi_\phi^1\|\|\xi_p^0\| + C\|\xi_\phi^2\|\|\xi_p^1\| + C\sum_{i=2}^{n-1}\|\xi_\phi^{i+1}-\xi_\phi^{i-1}\|\|\xi_p^i\| \\
&\quad + C(\Delta t)h^{k+1}\sum_{i=2}^{n-1}\|\xi_\phi^{i-1}\|\|\xi_p^i\| + C\|\xi_\phi^{n-1}\|\|\xi_p^n\| + C\|\xi_\phi^n\|\|\xi_p^{n+1}\| \\
&\leq C\max_{1\leq i\leq n}\|\xi_\phi^i\|\cdot\max_{1\leq i\leq n+1}\|\xi_p^i\| + C\max_{2\leq i\leq n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|\cdot\max_{2\leq i\leq n-1}\|\xi_p^i\| \\
&\quad + Ch^{k+1}\max_{2\leq i\leq n-1}\|\xi_\phi^{i-1}\|\cdot\max_{2\leq i\leq n-1}\|\xi_p^i\| \\
&\leq C\max_{1\leq i\leq n}\|\xi_\phi^i\|\cdot\max_{1\leq i\leq n+1}\|\xi_p^i\| + C\max_{2\leq i\leq n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|\cdot\max_{2\leq i\leq n-1}\|\xi_p^i\|,
\end{aligned}$$

where the last inequality is due to the assumption that $h \le 1$. Fourthly, we can also get

$$\begin{aligned}
\Lambda_{44} &= -\sum_{i=1}^{n}\int_I (p^{i+1}+p^{i-1})\eta_\phi^i(\xi_p^{i+1}-\xi_p^{i-1})dx \\
&= \int_I (p^2+p^0)\eta_\phi^1\xi_p^0 dx + \int_I (p^3+p^1)\eta_\phi^2\xi_p^1 dx - \int_I (p^n+p^{n-2})\eta_\phi^{n-1}\xi_p^n dx \\
&\quad + \sum_{i=2}^{n-1}\int_I \left((p^{i+2}+p^i)\eta_\phi^{i+1} - (p^i+p^{i-2})\eta_\phi^{i-1}\right)\xi_p^i dx - \int_I (p^{n+1}+p^{n-1})\eta_\phi^n\xi_p^{n+1}dx.
\end{aligned}$$

Note $(p^{i+2}+p^i)\eta_\phi^{i+1} - (p^i+p^{i-2})\eta_\phi^{i-1} = (p^{i+2}+p^i)(\eta_\phi^{i+1}-\eta_\phi^{i-1}) + \eta_\phi^{i-1}(p^{i+2}-p^{i-2})$. Thus, there is

$$\begin{aligned}
\Lambda_{44} &\le Ch^{k+1}(\|\xi_p^0\|+\|\xi_p^1\|) + Ch^{k+1}\max_{2\le i\le n-1}\|\xi_p^i\| + Ch^{k+1}(\|\xi_p^n\|+\|\xi_p^{n+1}\|) \\
&\le Ch^{k+1}\max_{0\le i\le n+1}\|\xi_p^i\|.
\end{aligned}$$

We now combine all the estimates for Λ_{41}, Λ_{42}, Λ_{43} and Λ_{44} to obtain

$$\begin{aligned}
\Lambda_4 &\le \left(C\max_{1\le i\le n}\|\phi_h^i\|_\infty + T\max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)\max_{1\le i\le n+1}\|\xi_p^i\|^2 \\
&\quad + Ch^{k+1}\max_{1\le i\le n}\|\phi_h^i\|_\infty\max_{1\le i\le n+1}\|\xi_p^i\| + Ch^{k+1}\max_{0\le i\le n+1}\|\xi_p^i\| \\
&\quad + Ch^{k+1}\max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\max_{2\le i\le n-1}\|\xi_p^i\| \\
&\quad + C\max_{1\le i\le n}\|\xi_\phi^i\|\max_{1\le i\le n+1}\|\xi_p^i\| + C\max_{2\le i\le n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|\max_{2\le i\le n-1}\|\xi_p^i\| \\
&\le C\left(1+\max_{1\le i\le n}\|\phi_h^i\|_\infty + \max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)\left(\max_{1\le i\le n+1}\|\xi_p^i\|^2 + h^{2k+2}\right) \\
&\quad + C\left(\max_{1\le i\le n}\|\xi_\phi^i\|^2 + \max_{2\le i\le n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|^2 + \max_{1\le i\le n+1}\|\xi_p^i\|^2\right).
\end{aligned}$$

Here we have used Cauchy-Schwartz inequality in the second inequality above. Note that Λ_5 can be obtained if we switch p and q in the formulation of Λ_4. Thus we can estimates $\Lambda_4+\Lambda_5$

as

$$\begin{aligned}\Lambda_4+\Lambda_5 &\leq C\left(\max_{1\leq i\leq n}\|\xi_\phi^i\|+\max_{2\leq i\leq n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|+\max_{1\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q^i\|)\right)^2\\ &\quad +C\left(1+\max_{1\leq i\leq n}\|\phi_h^i\|_\infty+\max_{2\leq i\leq n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)\times\\ &\quad \left(\max_{1\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q^i\|)+h^{k+1}\right)^2. \qquad (4.20)\end{aligned}$$

With the inequalities (4.16), (4.17), (4.18) and (4.20), we can get the upper bound of $\|\xi_u^{n+1}\|^2+\|\xi_z^{n+1}\|^2$ as

$$\begin{aligned}&\|\xi_u^{n+1}\|^2+\|\xi_z^{n+1}\|^2\\ \leq\ & Ch^{k+1}\max_{1\leq i\leq n+1}(\|\xi_u^i\|+\|\xi_z^i\|)+C(h^{k+1}+(\Delta t)^2)\max_{0\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q^i\|)\\ &+C\left(\max_{1\leq i\leq n}\|\xi_\phi^i\|+\max_{2\leq i\leq n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|+\max_{1\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q^i\|)\right)^2\\ &+C\left(1+\max_{1\leq i\leq n}\|\phi_h^i\|_\infty+\max_{2\leq i\leq n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)\times\\ &\left(\max_{1\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q\|)+h^{k+1}\right)^2.\end{aligned}$$

Since $\frac{1}{2}(\|\xi_u^{n+1}\|+\|\xi_z^{n+1}\|)^2\leq\|\xi_u^{n+1}\|^2+\|\xi_z^{n+1}\|^2$, we can replace the left side of the inequality above with $\frac{1}{2}(\|\xi_u^{n+1}\|+\|\xi_z^{n+1}\|)^2$. Also, because the inequality above is satisfied for all n, we can further apply Young's inequality to get

$$\begin{aligned}&\frac{1}{2}\left(\max_{1\leq i\leq n+1}(\|\xi_u^i\|+\|\xi_z^i\|)\right)^2\\ \leq\ & \frac{1}{4}\left(\max_{1\leq i\leq n+1}(\|\xi_u^i\|+\|\xi_z^i\|)\right)^2+Ch^{2k+2}+C(\Delta t)^4+C\left(\max_{0\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q^i\|)\right)^2\\ &+C\left(\max_{1\leq i\leq n}\|\xi_\phi^i\|+\max_{2\leq i\leq n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|+\max_{1\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q^i\|)\right)^2\\ &+C\left(1+\max_{1\leq i\leq n}\|\phi_h^i\|_\infty+\max_{2\leq i\leq n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)\times\\ &\left(\max_{1\leq i\leq n+1}(\|\xi_p^i\|+\|\xi_q\|)+h^{k+1}\right)^2.\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{4}\left(\max_{1\le i\le n+1}(\|\xi_u^i\|+\|\xi_z^i\|)\right)^2\\
\le\ & C\left(h^{k+1}+(\Delta t)^2+\max_{0\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|)+\max_{1\le i\le n}\|\xi_\phi^i\|+\max_{2\le i\le n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|\right)^2\\
+\ & C\left(1+\max_{1\le i\le n}\|\phi_h^i\|_\infty+\max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)\times\\
&\left(\max_{1\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|)+h^{k+1}\right)^2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\max_{1\le i\le n+1}(\|\xi_u^i\|+\|\xi_z^i\|)\\
\le\ & Ch^{k+1}+C(\Delta t)^2+C\max_{0\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|)+C\max_{1\le i\le n}\|\xi_\phi^i\|\\
&+C\max_{2\le i\le n-1}\left\|\frac{\xi_\phi^{i+1}-\xi_\phi^{i-1}}{\Delta t}\right\|+C\left(1+\max_{1\le i\le n}\|\phi_h^i\|_\infty+\max_{2\le i\le n-1}\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{\Delta t}\right\|_\infty\right)^{\frac{1}{2}}\times\\
&\left(\max_{1\le i\le n+1}(\|\xi_p^i\|+\|\xi_q^i\|)+h^{k+1}\right).
\end{aligned}
$$

It is easy to see that the inequality above implies the conclusion of the lemma.

4.4. **Proof of Lemma 3.6.** Let $w_5 = \xi_\phi^{n+1} - \xi_\phi^{n-1}$ in (3.12) and $w_6 = \frac{1}{2}(\xi_s^{n+1} + \xi_s^{n-1})$ in (3.13). We then add the resulting equations to get the energy equation for the Klein-Gordon equation part:

$$
\begin{aligned}
&\left\|\frac{\xi_\phi^{n+1}-\xi_\phi^n}{\Delta t}\right\|^2-\left\|\frac{\xi_\phi^n-\xi_\phi^{n-1}}{\Delta t}\right\|^2+\frac{1}{2}\|\xi_s^{n+1}\|^2-\frac{1}{2}\|\xi_s^{n-1}\|^2+\frac{1}{2}\|\xi_\phi^{n+1}\|^2-\frac{1}{2}\|\xi_\phi^{n-1}\|^2\\
=\ & \Lambda_1+\Lambda_2+\Lambda_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_1 =\ & -\frac{1}{2}\left(\int_I(\xi_s^{n+1}+\xi_s^{n-1})(\xi_\phi^{n+1}-\xi_\phi^{n-1})_x dx+\int_I(\xi_\phi^{n+1}-\xi_\phi^{n-1})(\xi_s^{n+1}+\xi_s^{n-1})_x dx\right)\\
&-\frac{1}{2}\sum_{j=1}^{N}\left(\left(\xi_s^{n+1}+\xi_s^{n-1}\right)[\xi_\phi^{n+1}-\xi_\phi^{n-1}]+\left(\xi_\phi^{n+1}+\xi_\phi^{n-1}\right)[\xi_s^{n+1}+\xi_s^{n-1}]\right)_{j+\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2 &= \int_I \frac{\eta_\phi^{n+1} - 2\eta_\phi^n + \eta_\phi^{n-1}}{(\Delta t)^2}(\xi_\phi^{n+1} - \xi_\phi^{n-1})dx + \frac{1}{2}\int_I (\eta_\phi^{n+1} + \eta_\phi^{n-1})(\xi_\phi^{n+1} - \xi_\phi^{n-1})dx \\
&\quad - \int_I T_\phi^n(\xi_\phi^{n+1} - \xi_\phi^{n-1})dx + \frac{1}{2}\int_I (\eta_s^{n+1} - \eta_s^{n-1})(\xi_s^{n+1} + \xi_s^{n-1})dx,
\end{aligned}
$$

and

$$\Lambda_3 = \int_I \left((p_h^n)^2 + (q_h^n)^2 - (p^n)^2 - (q^n)^2\right)(\xi_\phi^{n+1} - \xi_\phi^{n-1})dx.$$

It is easy to show $\Lambda_1 = 0$ using property of jump (2.1). We then apply Lemma 3.1 and 3.2 to get

$$
\begin{aligned}
\Lambda_2 &\leq Ch^{k+1}\|\xi_\phi^{n+1} - \xi_\phi^{n-1}\| + C(\Delta t)^2\|\xi_\phi^{n+1} - \xi_\phi^{n-1}\| + C(\Delta t)h^{k+1}(\|\xi_s^{n+1}\| + \|\xi_s^{n-1}\|) \\
&\leq C(\Delta t)\left(h^{2k+2} + (\Delta t)^4 + \left\|\frac{\xi_\phi^{n+1} - \xi_\phi^{n-1}}{\Delta t}\right\|^2 + \|\xi_s^{n+1}\|^2 + \|\xi_s^{n-1}\|^2\right).
\end{aligned}
$$

For Λ_3, there is

$$
\begin{aligned}
\Lambda_3 &= \int_I (p_h^n + p^n)(\xi_p^n - \eta_p^n)(\xi_\phi^{n+1} - \xi_\phi^{n-1})dx + \int_I (q_h^n + q^n)(\xi_q^n - \eta_q^n)(\xi_\phi^{n+1} - \xi_\phi^{n-1})dx \\
&\leq C(1 + \|p_h^n\|_\infty)(\|\xi_p^n\| + h^{k+1})\|\xi_\phi^{n+1} - \xi_\phi^{n-1}\| \\
&\quad + C(1 + \|q_h^n\|_\infty)(\|\xi_q^n\| + h^{k+1})\|\xi_\phi^{n+1} - \xi_\phi^{n-1}\| \\
&\leq C(1 + \|p_h^n\|_\infty + \|q_h^n\|_\infty)(\|\xi_p^n\| + \|\xi_q^n\| + h^{k+1})\|\xi_\phi^{n+1} - \xi_\phi^{n-1}\| \\
&\leq C(\Delta t)(1 + \|p_h^n\|_\infty + \|q_h^n\|_\infty)\left(h^{2k+2} + \|\xi_p^n\|^2 + \|\xi_q^n\|^2 + \left\|\frac{\xi_\phi^{n+1} - \xi_\phi^{n-1}}{\Delta t}\right\|^2\right).
\end{aligned}
$$

Finally, we combine the estimates of Λ_1, Λ_2 and Λ_3 to conclude the lemma.

5. Numerical Tests

In this section, we test the numerical performance of our proposed decoupled conservative local discontinuous Galerkin method, including the convergence rate, conservation property of energy and Hamiltonian, dispersion and dissipation errors. We then compare the performance of the decoupled LDG method with the undecoupled LDG method in [16]. We perform the numerical experiments using the following solitary wave solutions:

$$
\begin{cases}
p(x,t) = 3\,\mathrm{sech}^2(x + \frac{\sqrt{3}}{2}t)\cos(-\frac{\sqrt{3}}{4}x + \frac{61}{16}t), \\
q(x,t) = 3\,\mathrm{sech}^2(x + \frac{\sqrt{3}}{2}t)\sin(-\frac{\sqrt{3}}{4}x + \frac{61}{16}t), \\
\phi(x,t) = 6\,\mathrm{sech}^2(x + \frac{\sqrt{3}}{2}t).
\end{cases}
\tag{5.1}
$$

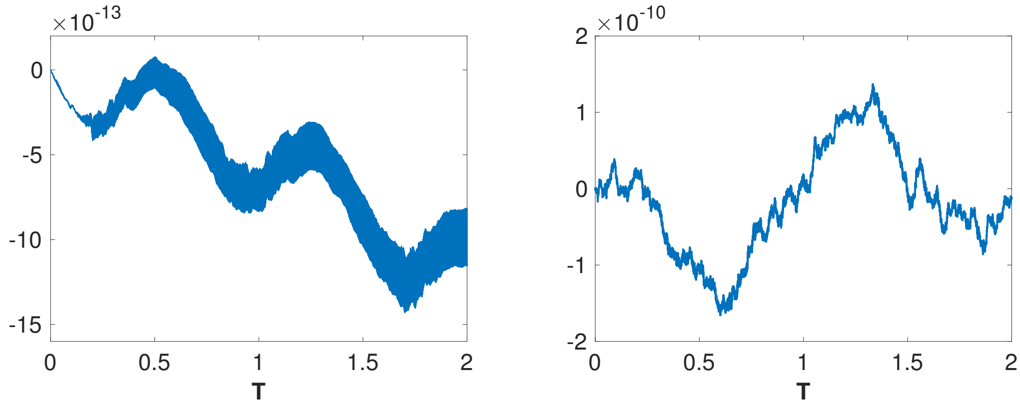
To test the convergence rate of our proposed scheme, we take $p(x,0)$, $q(x,0)$ and $\phi(x,0)$ in (5.1) as initial conditions, use $I = [-32, 32]$ as the computational domain, run the simulations up to $T = 0.1$ and compute the L^2 errors. In order to make the spatial errors dominate, we choose very small time step size, i.e., $\Delta t = 10^{-4}$. As far as the spatial mesh size h is concerned, we start with 64 elements so that $h = 1$, and then refine the mesh uniformly into 128 elements and compute the L^2 errors. We process this way to the cases of 256 and 512 elements. Suppose we fix a k, and the computed L^2 errors corresponding to the mesh size h_1 and h_2, are E_1 and E_2, respectively, then we can estimate the convergence rate using $\log(E_1/E_2)/\log(h_1/h_2)$. The L^2 errors at $T = 0.1$ and convergence rates of the decoupled LDG method are presented in the following table:

From Table 1, we can observe the $(k+1)^{th}$ ($k = 1, 2$ or 3) order convergence rate when p_h, q_h, $\phi_h \in V_h^k$ is used. If we compare the L^2 errors and convergence rates in Table 1 with that of the un-decoupled LDG method in Table 7.1 of [16], we can see that both methods lead to quite similar errors and convergence rates. In other words, they both produce the optimal convergence order and comparable L^2 errors.

Next we consider the conservation properties of the decoupled LDG method. We run the simulations using V_h^2 space, 512 elements and time step size $\Delta t = 10^{-4}$. Due to the temporal derivative in the formulation of Hamiltonian (1.4), here we use $\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\Delta t}$ to approximate $\phi_{h,t}$ at t^n, so that the discrete Hamiltonian is given by $H(t^n) = \frac{1}{2}\|\phi_h^n\|^2 + \frac{1}{2}\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\Delta t}\right\|^2 + \frac{1}{2}\|s_h^n\|^2 + \|u_h^n\|^2 + \|z_h^n\|^2 - \int_I((p_h^n)^2 + (q_h^n)^2)\phi_h^n dx$. The total energy at t^n is obtained by

TABLE 1. L^2 errors at $T = 0.1$ and convergence rates of the decoupled local discontinuous Galerkin method. p_h, q_h, $\phi_h \in V_h^k$ with $k = 1, 2$ and 3. $h = 1$.

		p		q		ϕ	
$k=1$	h	3.763E-1	-	3.001E-1	-	4.491E-1	-
	$\frac{h}{2}$	8.321E-2	2.177	5.508E-2	2.446	1.626E-1	1.466
	$\frac{h}{4}$	2.025E-2	2.039	1.218E-2	2.177	4.152E-2	1.969
	$\frac{h}{8}$	5.037E-3	2.007	3.040E-3	2.002	9.977E-3	2.057
$k=2$	h	8.230E-2	-	3.124E-2	-	1.430E-1	-
	$\frac{h}{2}$	8.112E-3	3.343	5.358E-3	2.544	1.447E-2	3.305
	$\frac{h}{4}$	9.864E-4	3.040	6.223E-4	3.106	1.927E-3	2.909
	$\frac{h}{8}$	1.231E-4	3.002	7.833E-5	2.990	2.366E-4	3.026
$k=3$	h	3.817E-3	-	8.726E-3	-	7.642E-3	-
	$\frac{h}{2}$	7.145E-4	2.417	5.218E-4	4.064	1.365E-3	2.485
	$\frac{h}{4}$	4.652E-5	3.941	3.036E-5	4.103	8.790E-5	3.957
	$\frac{h}{8}$	2.930E-6	3.989	1.916E-6	3.986	5.501E-6	3.998



(a) Relative error in the total energy (b) Relative error in the Hamiltonian

FIGURE 1. Relative error in the total energy and Hamiltonian when V_h^2 with spatial mesh size $h = 1/8$ and temporal mesh size $\Delta t = 10^{-4}$ are used. Left: the relative error in the total energy. Right: the relative error in the Hamiltonian.

computing $E(t^n) = \int_I \left((q_h^n)^2 + (p_h^n)^2\right) dx$. The relative errors of the total energy and the Hamiltonian are shown in Figure 1.

Here we use the approximation space V_h^2 for p_h, q_h and ϕ_h, the spatial mesh size $h = 1/8$ and the temporal mesh size $\Delta t = 10^{-4}$. Figure 1(a) shows that the relative error of the total energy is at the magnitude of 10^{-13}. That is, the total energy is conserved up to machine epsilon. This is consistent with the conclusion from Theorem 2.1. Figure 1(b) shows that the relative error of the Hamiltonian is conserved up to the magnitude of 10^{-10}. According to Theorem 2.16 and Remark 2.1, we can see that $H(t^n)$ is not exactly conserved, but its conservation depends on Δt and the accuracy of numerical solutions. We then compare the conservation properties of the decoupled LDG method and un-decoupled LDG method in [16], and find that the two methods lead to similar relative errors. To investigate the subtle difference between these two methods, we further compute the relative errors of total energy and Hamiltonian at several discrete time, i.e., $t = 0.5$, 1, 1.5 and 2. The results are given in Table 2 and 3.

As we can see in Table 2, even though the relative errors of our decoupled LDG method is slightly larger then that of the un-decoupled LDG method, both methods lead to satisfying

TABLE 2. Relative errors of total energy for our proposed decoupled LDG (D-LDG) method and un-decoupled LDG (U-LDG) method for various T.

	$T = 0.5$	$T = 1$	$T = 1.5$	$T = 2$
D-LDG	-7.439E-14	-5.931E-13	-7.376E-13	-8.190E-13
U-LDG	2.220E-16	-1.288E-14	-4.552E-15	-2.354E-14

energy conservation property. The results in Table 3 show that the Hamiltonian conservation property for both methods are also similar. At $T = 1, 1.5$ and 2, the relative error of the Hamiltonian of the decoupled LDG method is even smaller than that of the un-decoupled method. We then compute the ratio of the relative errors for both methods at $T = 0.5, 1, 1.5$ and 2. The results are presented in the third row of Table 3. It is easy to see that the ratio is decreasing, which means Hamiltonian conservation of our proposed decoupled LDG method is better. Based on the observation from Table 2 and 3, we can draw the conclusion that overall speaking, the decoupled LDG method leads to slightly larger error in the energy conservation (even though it is still at the magnitude of machine epsilon), but slightly smaller error in the Hamiltonian conservation. If we use V_h^2 basis for p_h, q_h and ϕ_h, the spatial mesh size $h = 1/8$, the temporal mesh size $\Delta t = 0.01$ and run the code for $T = 10$, then the relative error in energy is at the magnitude of 10^{-14}, and the relative error in Hamiltonian is at the magnitude of 10^{-6}. If we further refine the time step Δt, then the relative error in Hamiltonian will decrease to the magnitude of 10^{-9}, which indicates the good conservation property of our proposed scheme.

In addition, we investigate the dispersion and dissipation behaviors of the decoupled LDG method. The ability of preserving the dispersion relation is an important criterion of judging the performance of numerical methods for the wave simulations. The authors in [26] have demonstrated the dispersion and dissipation behaviors of the two types of discontinuous Galerkin methods for linear wave equations. Here we compute ϕ and $|\psi|^2 = p^2 + q^2$ at $T = 2$ and investigate the dispersion and dissipation errors of the decoupled LDG method when V_h^2 space is used. The exact and numerical values of ϕ and $|\psi|^2$ are plotted in Figure 2. We observe that the numerical solution can capture the magnitude and the phase of the solitary wave accurately. To quantify such dispersion and dissipation behaviors, we need to find the location of the peaks for ϕ_h and $|\psi_h|^2$, and the magnitude of the peaks. When we use 512 elements and $\Delta t = 10^{-4}$, we can find that ϕ_h achieves its maximum in the 243^{th} element. In this element, we can represent $\phi_h = a + b\frac{x-x_j}{h_j} + c\left(\frac{x-x_j}{h_j}\right)^2$, where $j = 243$ and x_j is the center of this element. Therefore, the maximum of ϕ_h is at $x = x_j - h_j b/(2c) \approx -1.732054284140232$. Since the peak of the exact solution ϕ is at $x = -\sqrt{3}$, we can now compute that the absolute and relative dispersion errors are approximately 3.477×10^{-6} and 2.007×10^{-6}, respectively. In addition, we can compute the maximum of ϕ_h to be 5.999989137826894. Thus the absolute and relative dissipation errors are approximately 1.086×10^{-5} and 1.810×10^{-6}, respectively.

TABLE 3. Relative errors of the Hamiltonian for our proposed decoupled LDG (D-LDG) method and the un-decoupled LDG (U-LDG) method for various T.

	$T = 0.5$	$T = 1$	$T = 1.5$	$T = 2$
D-LDG	-1.042E-10	-9.052E-12	-7.000E-12	-1.295E-11
U-LDG	-8.328E-11	-1.104E-11	-2.170E-11	-8.997E-11
Ratio	1.251	0.820	0.323	0.144

We then compute the dispersion and dissipation errors of $|\psi|^2$. Since p_h, $q_h \in V_h^2$, we know that $|\psi_h|^2$ is a quartic polynomial at each element, and it obtains its maximum at the 243^{th} element. We thus solve the maximization problem to find that $|\psi|^2$ obtains its maximum, 8.999959738979777 at $x \approx -1.732053375$. The absolute and relative dissipation errors of $|\psi|^2$ are approximately 4.026×10^{-5} and 4.473×10^{-6}, respectively. The absolute and relative dispersion errors of $|\psi|^2$ are approximately 2.567×10^{-6} and 1.482×10^{-6}, respectively. To summarize, all of the numerical results above, including the accuracy tests, time history of energy and Hamiltonian conservation, and the dispersion and dissipation errors indicate that our proposed decoupled LDG method leads to high-order accurate, energy- and Hamiltonian-conserving numerical solutions which preserve the dispersion and dissipation properties. Overall speaking, the numerical performance of the decoupled and the un-decoupled LDG methods are quite similar. Since the decoupled LDG method is more suitable for parallel computing, this method is a good option for solving the Klein-Gordon-Schrödinger equations.

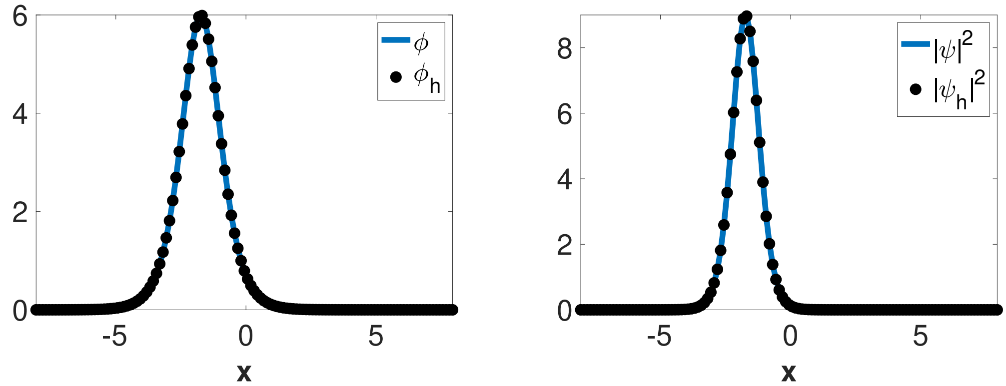


FIGURE 2. The exact and numerical values of ϕ and $|\psi|^2$ at $T = 2$. In both figures, the solid line and the dotted line represent the exact and numerical values, respectively. Left: ϕ. Right: $|\psi|^2$.

6. CONCLUDING REMARKS

In this paper, we propose a decoupled conservative local discontinuous Galerkin method for the Klein-Gordon-Schrödinger equations. We prove the optimal error estimate of the fully discrete scheme. A key component of the proof is the L^∞-assumption of the numerical solutions up to the n^{th} time level. With such an assumption, we can show the optimal convergence rate of the numerical solutions at $(n+1)^{th}$ time level. The assumption can be further proved by mathematical induction. Our proposed LDG method also has provable energy- and Hamiltonian conservation properties. Both the mathematical proof and numerical tests show that the total energy is conserved up to machine accuracy. However, the Hamiltonian has be computed using finite difference approximation in time. Numerical results show that the conservation of Hamiltonian depends on the time step size Δt and the spatial mesh size h. Compared with the un-decoupled LDG method, our decoupled LDG method leads to comparable accuracy, dispersion and dissipation behavior, and slightly better Hamiltonian conservation. Unlike in the un-decoupled LDG method, in our proposed method we can update ϕ_h^n and (p_h^n, q_h^n) simultaneously at each time level n for $n \geq 2$. Therefore, our method has the potential to improve the computational efficiency if parallel computing in time is used, which is left for future work.

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