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ON LIFTING OF STABLE RANGE ONE ELEMENTS

Meltem Altun-Özarslan and Ayşe Çiğdem Özcan

Abstract. Stable range of rings is a unifying concept for problems related to the substitution and cancellation of modules. The newly appeared element-wise setting for the simplest case of stable range one is tempting to study the lifting property modulo ideals. We study the lifting of elements having (idempotent) stable range one from a quotient of a ring R modulo a two-sided ideal I by providing several examples and investigating the relations with other lifting properties, including lifting idempotents, lifting units, and lifting of von Neumann regular elements. In the case where the ring R is a left or a right duo ring, we show that stable range one elements lift modulo every two-sided ideal if and only if R is a ring with stable range one. Under a mild assumption, we further prove that the lifting of elements having idempotent stable range one implies the lifting of von Neumann regular elements.

1. Introduction

Lifting of some special elements modulo an ideal of a ring is a quite substantial subject in ring theory. The structure of many classes of rings, including exchange, semiperfect, and semiregular rings is described in terms of lifting idempotents (for a detailed account on this, see for example [1] and [20]). On the other hand, a particular emphasis has been placed on lifting units by Menal and Moncasi [19] for certain types of self-injective rings; by Perera [22] for exchange rings and certain classes of C*-algebras with real rank zero; and by Šter [24] for clean rings. Recently, Khurana et al. [16], besides lifting of idempotents and units, have considered lifting of different types of elements; such as (von Neumann) regular elements, unit-regular elements, conjugate idempotents, etc. In this article, inspired by the work in [16], we present lifting of elements having (idempotent) stable range one modulo an ideal and investigate several properties and applications of such ideals. The concept of stable range was introduced

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by Bass [4] in the context of algebraic K-theory and many authors have worked on the simplest case of stable range one (see for example [5, 8, 10, 11, 13, 17, 26]) and obtained many important results. A new characterization of rings with stable range one is provided in this article. Before giving a more precise description of the results, we require some definitions and preliminary results that will be needed in the subsequent sections.

Throughout this paper, all rings are associative with identity. $\mathrm{U}(R)$ and $\mathrm{idem}(R)$ will denote the set of all unit (i.e., invertible) elements and all idempotent elements of a ring R, respectively. We write $I \lhd R$ for a two-sided ideal I of R. The Jacobson radical of a ring R will be denoted by J. We write $\mathrm{Soc}(R_R)$ and $\mathrm{Soc}({}_RR)$ for the right socle and the left socle of a ring R, respectively. In case they are equal, we may write $\mathrm{Soc}(R)$ for either socle.

Let R be any ring. An element a in R is called *regular* (resp., *unit-regular*) provided that there exists an element (resp., a unit element) $x \in R$ such that $a = axa$. Following [10], the ring R is called *regular* (resp., *unit-regular*) if every element in R is regular (respectively unit-regular). It is well known that for any $a \in R$, a is regular $\Leftrightarrow aR$ is a direct summand of $R_R \Leftrightarrow Ra$ is a direct summand of ${}_RR$.

An element $a \in R$ is said to have *stable range one* (written $\mathrm{sr}(a) = 1$) if, whenever $Ra + Rb = R$ for any $b \in R$, then there exists $x \in R$ such that $a + xb \in \mathrm{U}(R)$ (the right version is defined in [15]). A ring R has *stable range one* (written $\mathrm{sr}(R) = 1$) provided that each element has stable range one. Stable range one property of a ring is left-right symmetric due to Vaserstein [25]. Fuchs [8], Kaplansky [13], and Henriksen [11] proved independently that unit-regular rings have stable range one. They also proved that a regular ring is unit-regular if and only if it has stable range one. More generally, if an element $a \in R$ is regular, then a is unit-regular if and only if $\mathrm{sr}(a) = 1$ [15, Theorems 3.2 and 3.5] (see also Remark 2.14 below). It is well known that $\mathrm{sr}(R/J) = 1$ if and only if $\mathrm{sr}(R) = 1$. Hence, any semilocal ring R (i.e., a ring R with R/J semisimple Artinian) has stable range one (see [17, Corollary 2.10]).

Following [9], an element $a \in R$ is said to have *idempotent stable range one* (written $\mathrm{isr}(a) = 1$) if, whenever $Ra + Rb = R$ for any $b \in R$, then there exists $e \in \mathrm{idem}(R)$ such that $a + eb \in \mathrm{U}(R)$. A ring R is said to have *idempotent stable range one* (written $\mathrm{isr}(R) = 1$) provided that each element has idempotent stable range one [5]. This notion is also left-right symmetric for rings by [5]. Recently, Wang et al. proved that any unit-regular ring has idempotent stable range one (see [27, Corollary 3.4]).

Recall that a ring R is called *perspective* if $Ra+Rb = R$ for some $a, b \in R$ and $aR \oplus X = R$ for some right ideal X of R, then there exists $e \in \mathrm{idem}(R)$ such that $eR = X$ and $a + eb \in \mathrm{U}(R)$ [9, Theorem 4.2]. It was proved in [9, Theorem 3.3] that perspective property of rings is left-right symmetric. Abelian rings (i.e., rings with all idempotents central) are perspective. Hence, commutative rings, reduced rings, and all rings without non-trivial idempotents are perspective, all being examples of abelian rings. If $\mathrm{sr}(R) = 1$, then R is perspective. If

R is an exchange ring (that is, a ring in which idempotents lift modulo every one-sided ideal [20]), then R is perspective if and only if $\mathrm{sr}(R) = 1$ [9].

It is now useful to state the lifting properties in terms of some special classes together for having a complete interpretation. Let I be an ideal of a ring R and $\mathcal{C}(R)$ be a class of all elements having a property $\mathcal{C}$ in R. An element a in R is called $\mathcal{C}$ *lifting modulo* I if, whenever $a + I \in \mathcal{C}(R/I)$, then there exists $b \in \mathcal{C}(R)$ such that $a+I = b+I$. The ideal I is called $\mathcal{C}$*-lifting* if every element of R is $\mathcal{C}$ *lifting modulo* I. In this article, we will consider the following classes:

$$\mathrm{U}(R) = \{x \in R \,|\, x \text{ is a unit}\}, \qquad \mathrm{idem}(R) = \{x \in R \,|\, x \text{ is an idempotent}\},$$
$$\mathrm{reg}(R) = \{x \in R \,|\, x \text{ is regular}\}, \qquad \mathrm{ureg}(R) = \{x \in R \,|\, x \text{ is unit-regular}\},$$
$$\mathcal{SR}_1(R) = \{x \in R \,|\, \mathrm{sr}(x) = 1\}, \qquad \mathcal{ISR}_1(R) = \{x \in R \,|\, \mathrm{isr}(x) = 1\}.$$

For clarity, an ideal I is called an *idempotent lifting* in case I is $\mathrm{idem}(R)$-lifting; *unit lifting* if I is $\mathrm{U}(R)$-lifting; *regular lifting* if I is $\mathrm{reg}(R)$-lifting; *unit-regular lifting* if I is $\mathrm{Ureg}(R)$-lifting *stable range one lifting* if I is $\mathcal{SR}_1(R)$-lifting; and *idempotent stable range one lifting* if I is $\mathcal{ISR}_1(R)$-lifting.

Section 2 of this article is devoted to stable range one lifting ideals and Section 3 is concerned with idempotent stable range one lifting ideals. We obtain that these two lifting conditions properly imply lifting of units modulo an ideal. It is well-known that the Jacobson radical J of a ring R is always unit lifting. Further, we see that it is also stable range one lifting (Corollary 2.4). Moreover, if R is a regular ring, then an ideal I is unit lifting if and only if I is stable range one lifting (Proposition 2.8). If R is a left or a right duo ring, then every ideal is stable range one lifting if and only if $\mathrm{sr}(R) = 1$ (Theorem 2.10). Among other results, it is proved in Section 3 that if I is an idempotent stable range one lifting ideal such that R/I is perspective, then it is regular lifting (Theorem 3.11). This result yields a few useful corollaries. We characterize rings with idempotent stable range one, that is, we prove that $\mathrm{isr}(R) = 1$ if and only if $\mathrm{isr}(R/I) = 1$ and I is idempotent stable range one lifting for any ideal I contained in the Jacobson radical (Corollary 3.12). The Jacobson radical J is idempotent stable range one lifting in case it is idempotent lifting (Corollary 3.3). The converse of this statement is true if R is a left quasi-duo ring (Corollary 3.14). Last but not least, we prove that if R is a (right and left) duo ring, then every ideal is idempotent stable range one lifting if and only if every ideal is regular lifting if and only if $\mathrm{isr}(R) = 1$ if and only if R is exchange (Theorem 3.15).

2. Stable range one lifting

In this section, we introduce stable range one lifting ideals. First, recall that an element $a \in R$ is said to have *stable range one* (written $\mathrm{sr}(a) = 1$) if, whenever $Ra + Rb = R$ for any $b \in R$, then there exists $x \in R$ such that $a + xb \in \mathrm{U}(R)$. Clearly, every unit element and every element in the Jacobson radical of a ring R have stable range one.

Definition 2.1. Let I be an ideal of a ring R. I is called *a stable range one lifting ideal* if, for every $a \in R$ with $a+I \in \mathcal{SR}_1(R/I)$, there exists $b \in \mathcal{SR}_1(R)$ such that $a+I = b+I$.

Obviously, the trivial ideals of R are stable range one lifting. On the other hand, if $\mathrm{sr}(R) = 1$, then every ideal of R is stable range one lifting. Local rings, unit-regular rings, and semilocal rings are some examples of rings with stable range one. For more examples, see [26].

Example 2.2. Consider the ring $\mathbb{Z}$. We first note that the only elements of $\mathbb{Z}$ with stable range one are $0, 1, -1$, because if a is an integer different from $0, 1, -1$, then one can choose $b \in \mathbb{Z}$ such that $\gcd(a, b) = 1$, $b \nmid 1-a$, and $b \nmid 1+a$, so that $a\mathbb{Z}+b\mathbb{Z} = \mathbb{Z}$, but there do not exist $x \in \mathbb{Z}$ such that $a+xb = 1$ or $a+xb = -1$.

This fact gives immediately that the ideals $2\mathbb{Z}$ and $3\mathbb{Z}$ are stable range one lifting. But the ideal $4\mathbb{Z}$ is not stable range one lifting. To see this, consider the element $2+4\mathbb{Z}$. Clearly, $\mathrm{sr}(2+4\mathbb{Z}) = 1$, but there do not exist $b \in \mathbb{Z}$ with $2+4\mathbb{Z} = b+4\mathbb{Z}$ and $\mathrm{sr}(b) = 1$. Indeed, b cannot be $0, 1$, or -1.

More generally, if $n \geq 4$, then $n\mathbb{Z}$ is not stable range lifting. To show the last sentence, one can consider the unit-regular elements different from $0+n\mathbb{Z}, 1+n\mathbb{Z}$, and $-1+n\mathbb{Z}$ in the ring $\mathbb{Z}/n\mathbb{Z}$.

Recall that a two-sided ideal I of a ring R is called a radical ideal if $1+x \in \mathrm{U}(R)$ for every $x \in I$. Obviously, every ring has a largest radical ideal, namely, the Jacobson radical of R (see [28]).

Proposition 2.3. *Let I be a proper ideal of a ring R. For any $a \in R$, if $\mathrm{sr}(a) = 1$ in R, then $\mathrm{sr}(a+I) = 1$ in R/I. The converse is true if I is a radical ideal.*

Proof. Let $a \in R$. Set $\overline{a} = a+I$ and $\overline{R} := R/I$. Assume that $\mathrm{sr}(a) = 1$. If $\overline{R}\overline{a} + \overline{R}\overline{b} = \overline{R}$, then $Ra + Rb + I = R$. Then we can find $r, s \in R$ and $y \in I$ such that $1 = ra + sb + y$. This implies that $Ra + R(sb+y) = R$. By assumption, there exists $x \in R$ such that $a + x(sb+y) \in \mathrm{U}(R)$. Hence $\overline{a} + \overline{x}(\overline{sb+y}) = \overline{a} + (\overline{xs})\overline{b} \in \mathrm{U}(\overline{R})$. Thus $\mathrm{sr}(a+I) = 1$.

For the converse, assume that I is a radical ideal and $\mathrm{sr}(a+I) = 1$. If $Ra+Rb = R$, then there exists $x \in R$ such that $\overline{a} + \overline{x}\overline{b} = \overline{u}$ where $u \in \mathrm{U}(R)$. This implies that $a + xb = u + j$ for some $j \in I$. Since I is a radical ideal, $1 + u^{-1}j$ is a unit. It follows that $a + xb = u + j = u(1 + u^{-1}j)$ is a unit, too. Thus, $\mathrm{sr}(a) = 1$. □

It is widely known that the Jacobson radical J of a ring R need not be idempotent lifting, but the following corollary of Proposition 2.3 shows an interesting property of J. Recall that $\mathrm{sr}(R) = 1$ if and only if $\mathrm{sr}(R/J) = 1$.

Corollary 2.4. *Any radical ideal of a ring R is stable range one lifting. In particular, J is stable range one lifting.*

Lemma 2.5. *Let $\varphi : R \longrightarrow S$ be a ring isomorphism. If* $\mathrm{sr}(a) = 1$ *in R, then* $\mathrm{sr}(\varphi(a)) = 1$ *in S.*

Example 2.6. Let $R = \{(x_1, \ldots, x_n, s, s, \ldots) \,|\, x_1, \ldots, x_n \in \mathbb{Q},\ s \in \mathbb{Z},\ n \geq 1\}$. Then R is a commutative ring with $J = 0$ and every regular element of R is unit-regular by [15, Remark 6.6]. Further, every regular element of R has idempotent stable range one by [9, Theorem 4.2]. Set $I := \{(x_1, \ldots, x_n, 0, 0, \ldots) \,|\, x_1, \ldots, x_n \in \mathbb{Q}, n \geq 1\}$. Then $R/I \cong \mathbb{Z}$ via the map $\varphi : (x_1, \ldots, x_n, s, s, \ldots) + I \mapsto s$. For any $a \in R$, we claim that

$$\mathrm{sr}(a) = 1 \iff a \text{ is unit-regular.}$$

Assume that $\mathrm{sr}(a) = 1$. Proposition 2.3 gives that $\mathrm{sr}(a + I) = 1$, and then $\mathrm{sr}(\varphi(a + I)) = 1$ in $\mathbb{Z}$ by Lemma 2.5. Hence $\varphi(a + I) = 0, 1$, or -1. This implies that $a = (x_1, \ldots, x_n, s, s, \ldots)$ where $s = 0, 1$, or -1, and so a is unit-regular. We note that any unit-regular element has stable range one and thus the claim follows.

Now we show that the ideal I is stable range one lifting. For this, assume that a is an element in R such that $\mathrm{sr}(a + I) = 1$. As in the above discussion, a is unit-regular, and hence $\mathrm{sr}(a) = 1$.

Lemma 2.7. *Any stable range one lifting ideal is unit lifting.*

Proof. Let I be a stable range one lifting ideal of a ring R. Take an invertible element $\overline{a} \in R/I$ with the inverse $\overline{b}$. Since $\overline{a}$ is unit-regular, $\mathrm{sr}(\overline{a}) = 1$. By hypothesis, there exists an $x \in R$ such that $\overline{a} = \overline{x}$ and $\mathrm{sr}(x) = 1$. Then $\overline{b}\overline{a} = \overline{b}\overline{x} = \overline{1}$, and so $c := 1 - bx \in I$. This gives us $Rx + Rc = R$. Since $\mathrm{sr}(x) = 1$, there exists $y \in R$ such that $x + yc = v$ is a unit in R, and hence $\overline{a} = \overline{x} = \overline{x + yc} = \overline{v}$. Thus v is the required element. □

The converse of Lemma 2.7 is not true in general. For instance, take the ideal $4\mathbb{Z}$ of $\mathbb{Z}$. This ideal is clearly unit lifting ($\overline{1}$ and $\overline{3} = \overline{-1}$ in $\mathbb{Z}/4\mathbb{Z}$ lift to units), but it is not stable range one lifting by Example 2.2. In the following result, we show that the converse of Lemma 2.7 is true when R is a regular ring.

Proposition 2.8. *If R is a regular ring, then any ideal I of a ring R is unit lifting if and only if it is stable range one lifting.*

Proof. Assume that I is unit lifting and let $\overline{a} \in \mathcal{SR}_1(R/I)$. Then the regular element $\overline{a}$ is unit-regular by the fact that a regular element has stable range one if and only if it is unit-regular [15, Theorems 3.2 and 3.5]. Since any regular ring is exchange, every ideal is idempotent lifting by [20, Corollary 1.3]. Hence I is both unit lifting and idempotent lifting. This is equivalent to saying that I is unit-regular lifting by [16, Theorem 6.2]. Thus there exists a unit-regular element $b \in R$ such that $\overline{a} = \overline{b}$. Since b is unit-regular, $\mathrm{sr}(b) = 1$, and hence b is the desired element. □

Bacella showed in [2, Lemma 3.5] that a regular ring R with an ideal I is unit-regular if and only if eRe is unit-regular for every idempotent $e \in I$, R/I is unit-regular, and I is unit lifting. Hence Proposition 2.8 implies that "unit lifting" can be interchanged with "stable range one lifting" in Bacella's result.

In [16, Theorem 6.2], the authors proved that an ideal is unit-regular lifting if and only if it is both unit lifting and idempotent lifting. It is tempting to suspect that stable range one lifting ideals need not be unit-regular lifting. For this, it suffices to consider a ring R with Jacobson radical J such that J is not idempotent lifting (see Example 3.16), and thus J is the required ideal by Corollary 2.4. The following corollary is immediate by the fact that over a regular (or an exchange) ring every ideal is idempotent lifting by [20, Corollary 1.3].

Corollary 2.9. *If R is a regular ring, then any ideal I of R is stable range one lifting if and only if it is unit-regular lifting.*

Let R be a commutative ring. Then $\mathrm{sr}(R) = 1$ if and only if the natural map $\mathrm{U}(R) \to \mathrm{U}(R/I)$ is a group epimorphism for every ideal I of R by [6, Lemma 6.1]. The latter condition actually means that every ideal I is unit lifting. More generally, Siddique proved in [23, Theorem 3] that $\mathrm{sr}(R) = 1$ if and only if every left unit lifts modulo every principal left ideal, i.e., if $ba - 1 \in Rc$ for some $a, b, c \in R$, there exists a left unit $u \in R$ such that $a - u \in Rc$. Now we will give a description of rings with stable range one over a left or a right duo ring.

Recall that a ring R is called *left duo* if every left ideal is a right ideal; equivalently $aR \subseteq Ra$ for every $a \in R$. If R is a left and right duo ring, then we say that R is *a duo ring*. Note also that a ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for all $a, b \in R$. Any left duo ring is directly finite (see, for example [21, Corollary 1.11]).

Theorem 2.10. *If R is a left or a right duo ring, then the following are equivalent*:

(1) *Every ideal of R is stable range one lifting*;
(2) *Every ideal of R is unit lifting*;
(3) $\mathrm{sr}(R) = 1$.

Proof. (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (3) Assume that R is left duo. It is enough to show that every left unit lifts modulo every principal ideal left ideal by [23, Theorem 3]. Let $ba-1 \in Rc$ for some $a, b, c \in R$. Then Rc is an ideal of R and hence R/Rc is a left duo ring. It follows that R/Rc is directly finite. Hence, $a + Rc$ is a unit. The hypothesis implies that there exists a unit $u \in R$ such that $a - u \in Rc$. Thus, $\mathrm{sr}(R) = 1$. By the left-right symmetry of stable range one condition for rings, the right duo case has a similar proof. □

Proposition 2.11. *Let I and K be ideals of a ring R with $I \subseteq K$. If K is stable range one lifting, then K/I is stable range one lifting. The converse is true if, in addition, I is stable range one lifting.*

Proof. Assume that K is stable range one lifting. Let $a \in R$ with $\mathrm{sr}(a + I + K/I) = 1$. The mapping $\varphi : \frac{R/I}{K/I} \to R/K$, defined by $\varphi(r + I + K/I) = r + K$ for every $r \in R$, is a ring isomorphism, so that $\mathrm{sr}(a + K) = 1$ by Lemma 2.5. By hypothesis, we can find an element $b \in \mathcal{SR}_1(R)$ such that $a + K = b + K$. Further, $b+I \in \mathcal{SR}_1(R/I)$ by Proposition 2.3. Thus $a+I+K/I = b+I+K/I$ and $\mathrm{sr}(b + I) = 1$.

Conversely, assume that I and K/I are stable range one lifting. Let $a \in R$ with $\mathrm{sr}(a + K) = 1$. By the above isomorphism and Lemma 2.5, $\mathrm{sr}(a + I + K/I) = 1$. Since K/I is stable range one lifting, there exists $b+I \in \mathcal{SR}_1(R/I)$ such that $a + I + K/I = b + I + K/I$. Hence $a - b \in K$. Since I is stable range one lifting, there exists $c \in \mathcal{SR}_1(R)$ such that $b+I = c+I$. Now $b-c \in I \subseteq K$ implies that $a - c \in K$. Thus $a + K = c + K$ and $c \in \mathcal{SR}_1(R)$. □

If $I \subseteq K$ and K is stable range one lifting, then I need not be stable range one lifting. For example, take $I = 4\mathbb{Z}$ and $K = 2\mathbb{Z}$ in $\mathbb{Z}$.

Following [30], we denote by $\delta(R_R)$ the ideal which is the intersection of all essential maximal right ideals of a ring R. Clearly, $J \subseteq \delta(R_R)$ and $\mathrm{Soc}(R_R) \subseteq \delta(R_R)$ (see also [30, Lemma 1.9]). In view of [30, Corollary 1.7], $J(R/\mathrm{Soc}(R_R))$ $= \delta(R_R)/\mathrm{Soc}(R_R)$; in particular, R is semisimple if and only if $\delta(R_R) = R$. Now as a consequence of Proposition 2.11 and Corollary 2.4, one has the following result.

Corollary 2.12. *Let R be a ring. Then the following hold.*

(a) *$\delta(R_R)$ is stable range one lifting if and only if $\delta(R_R)/J$ is stable range one lifting in R/J.*

(b) *If $\mathrm{Soc}(R_R)$ is stable range one lifting, then $\delta(R_R)$ is stable range one lifting.*

Proof. Take $I = J$ and $K = \delta(R_R)$ for (a) and $I = \mathrm{Soc}(R_R)$ and $K = \delta(R_R)$ for (b) in Proposition 2.11. □

Note that, for any ring R, $\mathrm{Soc}(R_R)$ is always idempotent lifting by [3], but it need not be stable range one lifting as the following example shows.

Example 2.13 ([18, Example 1])**.** Let F be a field, V_F a countably infinite-dimensional vector space, and $Q = \mathrm{End}_F(V)$. Then there exists a regular directly infinite subring R of Q such that $\mathrm{Soc}(Q) = \mathrm{Soc}(R) \subseteq R$ and $R/\mathrm{Soc}(Q)$ is a field. On the other hand, Baccella proved in [2, Lemma 3.4] that for a subring T of Q with $\mathrm{Soc}(Q) \subseteq T$, T is directly finite if and only if $T/\mathrm{Soc}(T)$ is directly finite and $\mathrm{Soc}(T)$ is unit lifting. From this result we deduce that $\mathrm{Soc}(Q)$ is not unit lifting. Hence $\mathrm{Soc}(Q)$ is not stable range one lifting by Proposition 2.7. We further note that $\mathrm{Soc}(Q) = \mathrm{Soc}(R) = \delta({}_RR) = \delta(R_R)$.

At the end of this section, we discuss about the symmetry of stable range one lifting ideals.

Remark 2.14. In this article, we define stable range one elements by considering principal left ideals. There is a symmetric right version: an element $a \in R$ is said to have *right stable range one* if, for any $b \in R$, $aR + bR = R$ implies that $a + bx \in \mathrm{U}(R)$ for some $x \in R$. To avoid ambiguity about left and right versions of this definition, we will subscript the notation by l or r. It is not known yet whether the left version of element-wise stable range one condition is equivalent to that of right one. The best result in this direction is that if a is a regular element, then $\mathrm{sr}_l(a) = 1$ if and only if a is unit-regular if and only if $\mathrm{sr}_r(a) = 1$ in [15].

There is a more natural situation in which the left-right symmetry of stable range one element and stable range one lifting ideal can occur.

Lemma 2.15. *Let R be a duo ring. Then* $\mathrm{sr}_l(a) = 1$ *if and only if* $\mathrm{sr}_r(a) = 1$ *for any $a \in R$.*

Proof. Assume that $\mathrm{sr}_l(a) = 1$. Write $aR + bR = R$. Since R is left duo, $Ra+Rb = R$. Then there exists an element $x \in R$ such that $a+xb = u \in \mathrm{U}(R)$. Since $Rb \subseteq bR$, $xb = by$ for some $y \in R$. Hence $a + by = u$, i.e., $\mathrm{sr}_r(a) = 1$. □

Corollary 2.16. *Let R be a duo ring and $I \lhd R$. Then the following are equivalent*:

(a) *I is right stable range one lifting*;
(b) *I is left stable range one lifting.*

Proof. We just note that if R is a duo ring, then R/I is a duo ring by [21, Proposition 1.4]. □

3. Idempotent stable range one lifting

As we mentioned earlier, an element $a \in R$ is said to have *idempotent stable range one* if, whenever $Ra+Rb = R$ for any $b \in R$, then there exists $e \in \mathrm{idem}(R)$ such that $a+eb \in \mathrm{U}(R)$. Obviously, if u is a unit in R, then $\mathrm{isr}(u) = 1$. Further, if R is a directly finite ring, then $\mathrm{isr}(0) = 1$.

Recently, Wang et al. discovered that every regular element has idempotent stable range one over a ring R with $\mathrm{sr}(R) = 1$ [27, Theorem 3.3]. As a consequence of this, they showed that any unit regular ring has idempotent stable range one [27, Corollary 3.4]. However, by [14], this result does not hold on the element-wise level, i.e., unit-regular elements need not have idempotent stable range one. This was shown by finding a unit-regular element which is not clean, i.e., not a sum of an idempotent and a unit. Indeed, any element $a \in R$ with $\mathrm{isr}(a) = 1$ is clean. This can easily be seen by considering the equality $Ra + R(-1) = R$.

Definition 3.1. Let I be an ideal of a ring R. If for any element $a \in R$ with $a + I \in \mathcal{ISR}_1(R/I)$, there exists $b \in \mathcal{ISR}_1(R)$ such that $a + I = b + I$, then I is called *an idempotent stable range one lifting ideal.*

If $\mathrm{isr}(R) = 1$, then every ideal of R is idempotent stable range one lifting. Obviously, if $\mathrm{isr}(R) = 1$, then $\mathrm{sr}(R) = 1$. In this vein, one can ask the following question:

Is any idempotent stable range one lifting ideal stable range one lifting?

We do not have an answer to this question, but the converse of this is answered in the negative in Example 3.16.

Proposition 3.2. *Let I be a radical ideal of a ring R. If $\mathrm{isr}(a) = 1$, then $\mathrm{isr}(a + I) = 1$ for any $a \in R$. The converse is true if, in addition, I is idempotent lifting.*

Proof. Let I be a radical ideal and $a \in R$ with $\mathrm{isr}(a) = 1$. Set $\overline{a} = a + I$ and $\overline{R} := R/I$. Assume that $\overline{R}\overline{a} + \overline{R}\overline{b} = \overline{R}$. Then $Ra + Rb + I = R$. Since I is a radical ideal, it is a small left ideal of R, and hence $Ra + Rb = R$. By assumption, there exists $e^2 = e \in R$ such that $a + eb$ is a unit in R. Thus $\overline{a} + \overline{e}\overline{b}$ is a unit in R/I.

For the converse, assume in addition that I is idempotent lifting. Let $a \in R$ with $\mathrm{isr}(\overline{a}) = 1$, and let $Ra + Rb = R$. Since I is idempotent lifting, there exist $e \in \mathrm{idem}(R)$ and $\overline{u} \in \mathrm{U}(R/I)$ such that $\overline{a} + \overline{e}\overline{b} = \overline{u}$. Let $\overline{v}$ be the inverse of $\overline{u}$. Multiplying the last equality by $\overline{v}$ on the left, we obtain $\overline{va} + \overline{veb} = \overline{1}$. Since $v(a + eb) - 1 \in I$ and I is a radical ideal, $v(a + eb)$ is invertible in R. This implies that $a + eb$ is left invertible. Similarly, the multiplication by v on the right will give that $a + eb$ is right invertible. Hence $\mathrm{isr}(a) = 1$. □

Corollary 3.3. *Any idempotent lifting radical ideal I of a ring R is idempotent stable range one lifting.*

Remark 3.4. Idempotent lifting condition in Corollary 3.3 is not superfluous: There exists a ring R such that J is neither idempotent lifting nor idempotent stable range one lifting by Example 3.16.

Remark 3.5. The radial ideal condition on I in Corollary 3.3 is not superfluous: For example, the ideal $I = 4\mathbb{Z}$ in the ring $R = \mathbb{Z}$ is idempotent lifting but it is not idempotent stable range one lifting. To see this, consider $2 + 4\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}$. Clearly, $\mathrm{isr}(2 + 4\mathbb{Z}) = 1$, but $2 + 4\mathbb{Z}$ does not lift to an idempotent stable range one element in $\mathbb{Z}$ by Example 2.2. Observe that $I \nsubseteq J = 0$.

Example 3.6. Let $R = \{(x_1, \ldots, x_n, s, s, \ldots) \mid x_1, \ldots, x_n \in \mathbb{Q}, s \in \mathbb{Z}, n \geq 1\}$. By Example 2.6, being a unit-regular element is equivalent to being an element with stable range one in R. Since every regular element of R has idempotent stable range one, we get that, for any $a \in R$,

$$\mathrm{sr}(a) = 1 \iff a \text{ is unit-regular} \iff \mathrm{isr}(a) = 1.$$

Further, the ideal I in Example 2.6 is idempotent stable range one lifting because $R/I \cong \mathbb{Z}$ and the only elements with idempotent stable range one of the ring $\mathbb{Z}$ are $0, 1$ and -1.

Lemma 3.7. *Any idempotent stable range one lifting ideal is unit lifting.*

Proof. We proceed with the same argument as in the proof of Lemma 2.7. Let I be an idempotent stable range one lifting ideal of a ring R. Take an invertible element $\overline{a} \in R/I$ with the inverse $\overline{b}$. Since $\overline{a}$ is unit, $\operatorname{isr}(\overline{a}) = 1$. By hypothesis, we can find an element $x \in R$ such that $\overline{a} = \overline{x}$ and $\operatorname{isr}(x) = 1$. Then $\overline{b}\overline{a} = \overline{b}\overline{x} = \overline{1}$, and so $c := 1 - bx \in I$. This implies that $Rx + Rc = R$. Since $\operatorname{isr}(x) = 1$, there exists $e^2 = e \in R$ such that $x + ec = v$ is a unit in R, and hence $\overline{a} = \overline{x} = \overline{x + ec} = \overline{v}$. Thus v is the required element. □

The converse of Lemma 3.7 need not be true. For example, the ideal $I = 4\mathbb{Z}$ in the ring $R = \mathbb{Z}$ is unit lifting, but as we have pointed out before, it is not idempotent stable range one lifting.

Following Nicholson [20], an element x in a ring R is called *suitable* if there exists an idempotent $e \in R$ such that $e - x \in R(x - x^2)$. He proved that a ring R is an exchange ring if and only if every element of R is suitable. He further proved that any clean element is suitable.

In the literature, there are some natural equivalence relations on idempotents: First, two idempotents e and f in a ring R are said to be *isomorphic* if $eR \cong fR$ as right R-modules, and second, they are called *conjugate* if $f = u^{-1}eu$ for some unit $u \in \mathrm{U}(R)$. Close attention to the lifting of isomorphic idempotents and conjugate idempotents has been paid recently in [16]. Now we preface Theorem 3.11 with three lemmas from [16] needed for its proof.

Lemma 3.8 ([16, Theorem 5.2])**.** *Let R be a ring, $I \lhd R$, and let $x \in R$ be an idempotent modulo I. Then x lifts to an idempotent modulo I if and only if x lifts to a suitable element modulo I.*

Lemma 3.9 ([16, Proposition 3.11])**.** *Let R be a ring and let $I \lhd R$. If R/I is perspective, and I is idempotent lifting, then I is isomorphic idempotent lifting and conjugate idempotent lifting.*

Lemma 3.10 ([16, Proposition 5.20])**.** *If units and isomorphic idempotents lift modulo $I \lhd R$, then regular elements lift.*

Theorem 3.11. *Let R be a ring and $I \lhd R$ such that R/I is perspective. If I is idempotent stable range one lifting, then it is regular lifting* (*hence it is idempotent lifting*)*.*

Proof. First we claim that I is idempotent lifting. Let $\overline{a}$ be an idempotent in R/I. Since R/I is perspective, every regular element has idempotent stable range one by [9, Theorem 4.2]. Hence $\operatorname{isr}(\overline{a}) = 1$. Since I is idempotent stable range one lifting, there exists $b \in \mathcal{ISR}_1(R)$ such that $a - b \in I$. Now $\operatorname{isr}(b) = 1$ and the equality $Rb + R(-1) = R$ implies that b is clean. Then b is suitable.

This means that $\overline{a}$ lifts to a suitable element of R. By Lemma 3.8, this is equivalent to saying that $\overline{a}$ lifts to an idempotent of R. Hence I is idempotent lifting.

Now by Lemma 3.9, I is isomorphic idempotent lifting. Finally, Lemmas 2.7 and 3.10 yield that I is regular lifting, as desired. □

The converse of Theorem 3.11 is not true in general. For example, let $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$. Then I is regular lifting, because all regular elements of R/I are $\overline{0}, \overline{1}$ and $\overline{-1}$ and they are lifted to regular elements of $\mathbb{Z}$. But we have pointed out before that I is not idempotent stable range one lifting. Note that regular lifting ideals are idempotent lifting in general and they are equivalent for ideals contained in the Jacobson radical by [16, Theorem 5.24].

Before stating the next corollary, we recall that a ring R has idempotent stable range one if each element has idempotent stable range one. A characterization of this class of rings was obtained by Hiremath and Hedge in [12, Proposition 2.18] as follows:

If I is an ideal contained in the Jacobson radical of the ring R, then

$$\operatorname{isr}(R) = 1 \text{ if and only if } \operatorname{isr}(R/I) = 1 \text{ and } I \text{ is idempotent lifting.}$$

Note that this result was first proved for $I = J$ in [5, Theorem 9].

Corollary 3.12. *Let R be a ring and $I \lhd R$ such that $I \subseteq J$. Then* $\operatorname{isr}(R) = 1$ *if and only if* $\operatorname{isr}(R/I) = 1$ *and I is idempotent stable range one lifting.*

Proof. ($\Rightarrow$) If $\operatorname{isr}(R) = 1$, then $\operatorname{isr}(R/I) = 1$ by [12, Proposition 2.18] and clearly I is idempotent stable range one lifting.

($\Leftarrow$) Assume that $\operatorname{isr}(R/I) = 1$ and I is idempotent stable range one lifting. Since any ring with stable range one is perspective, R/I is perspective. It follows that I is idempotent lifting by Theorem 3.11. Hence $\operatorname{isr}(R) = 1$ by [12, Proposition 2.18].

There is also an alternative (direct) way to get that I is idempotent lifting. Let $\overline{a}$ be an idempotent in R/I. Since $\operatorname{isr}(\overline{1-a}) = 1$, there exists a $c \in R$ such that $\overline{1-a} = \overline{c}$ and $\operatorname{isr}(c) = 1$. Since c is clean, $c = e + u$ for some idempotent e and a unit u in R. Then $\overline{a} = \overline{1-c} = \overline{1-e} - \overline{u} = (\overline{1-e} - \overline{u})^2$ gives that $\overline{eu - u + ue + u^2} = \overline{0}$, and multiplying by $\overline{u^{-1}}$ from the left we have $\overline{a} = \overline{u^{-1}eu}$ where $u^{-1}eu$ is an idempotent in R. □

Corollary 3.13. *Let R be a ring and $I \lhd R$ such that $I \subseteq J$. Then R is perspective and I is idempotent lifting if and only if R/I is perspective and I is idempotent stable range one lifting.*

Proof. It follows from [9, Proposition 5.7], Corollary 3.3 and Theorem 3.11. □

Last but not least, we have the following corollary of Theorem 3.11, but first recall that a ring R is *left quasi-duo* if every maximal left ideal is a two-sided ideal [29].

Corollary 3.14. *If R is a left quasi-duo ring and I is an idempotent stable range one lifting ideal, then I is regular lifting. In particular, J is idempotent stable range one lifting if and only if J is idempotent lifting.*

Proof. Since R/I is left quasi-duo, it is perspective by [9, Corollary 4.8]. Hence I is regular lifting. The last assertion follows from Corollary 3.3. □

There is a close relationship between the class of exchange rings and the lifting property of regular elements modulo left ideals. A ring R is an exchange ring if and only if every left ideal is regular lifting, i.e., if L is a left ideal and $a - aba \in L$, then there exists a regular element $c \in R$ such that $a - c \in L$ [7, Corollary 5].

Rings with idempotent stable range one were characterized in [5, Theorem 12] over abelian rings. Here we provide a characterization over duo rings. Note that a ring is abelian if and only if every direct summand left ideal is fully invariant (see [21, p. 536]). Hence any left duo ring is abelian.

Theorem 3.15. *If R is a duo ring, then the following are equivalent*:

(1) *Every ideal is idempotent stable range one lifting*;
(2) *Every ideal is regular lifting*;
(3) *R is exchange*;
(4) $\mathrm{isr}(R) = 1$;
(5) *R is clean.*

Proof. First note that (3)-(5) are equivalent for any abelian ring by [5, Theorem 12]. The equivalence of (2) and (3) is by [7, Corollary 5].

(1) ⇒ (2) Since R is a duo ring, R/I is a duo ring for any ideal I of R by [21, Proposition 1.4], and so it is perspective by [9]. Now Theorem 3.11 implies that I is regular lifting.

(4) ⇒ (1) It is obvious. □

Hence Theorem 2.10 and Theorem 3.15 together imply that, over a commutative ring, if every ideal is idempotent stable range one lifting, then every ideal is stable range one lifting.

Now we can present an example of a stable range one lifting ideal which is not idempotent stable range one lifting.

Example 3.16. There exists an ideal I of a ring R such that I is stable range one lifting but it is neither idempotent stable range one lifting nor idempotent lifting:
Consider a semilocal commutative domain with two maximal ideals M_1 and M_2 (for example, take $R = \{\frac{m}{n} \in \mathbb{Q} \,|\, 2 \nmid n, 3 \nmid n\}$). Then $J = M_1 \cap M_2$ and $R/J \cong R/M_1 \times R/M_2$. The factor ring R/J has two non-trivial idempotents which do not lift to idempotents in R, because R has no non-trivial idempotents. Hence J is not idempotent lifting. However, it is stable range one lifting by Corollary 2.4. Moreover, since any commutative ring is perspective, R/J is perspective. Thus, J is not idempotent stable range one lifting by Theorem 3.11.

Finally, we investigate some extensions of idempotent stable range one lifting ideals.

Lemma 3.17. *Let* $\varphi : R \longrightarrow S$ *be a ring isomorphism with* $\varphi(1_R) = 1_S$*. If* $\operatorname{isr}(a) = 1$ *in* R*, then* $\operatorname{isr}(\varphi(a)) = 1$ *in* S*.*

Proposition 3.18. *Let* I *and* K *be ideals of a ring* R *with* $I \subseteq J \cap K$*. If* K *is idempotent stable range one lifting, then* K/I *is idempotent stable range one lifting. The converse is true if, in addition,* I *is idempotent stable range one lifting.*

Proof. The proof is similar to that of Proposition 2.11. Assume that K is idempotent stable range one lifting. Let $a \in R$ with $\operatorname{isr}(a + I + K/I) = 1$. The mapping $\varphi : \frac{R/I}{K/I} \to R/K$, defined by $\varphi(r + I + K/I) = r + K$ for every $r \in R$, is a ring isomorphism, so that $\operatorname{isr}(a + K) = 1$ by Lemma 3.17. By hypothesis, there exists $b \in R$ such that $a + K = b + K$ and $\operatorname{isr}(b) = 1$. On the other hand, $\operatorname{isr}(b + I) = 1$ by Proposition 3.2. Thus $a + I + K/I = b + I + K/I$ and $\operatorname{isr}(b + I) = 1$.

Conversely, assume that I and K/I are idempotent stable range one lifting. Let $a \in R$ with $\operatorname{isr}(a + K) = 1$. The above mentioned isomorphism and Lemma 3.17 implies that $\operatorname{isr}(a+I+K/I) = 1$. Since K/I is idempotent stable range one lifting, there exists $b+I \in R/I$ such that $a+I+K/I = b+I+K/I$ and $\operatorname{isr}(b + I) = 1$. Then $a - b \in K$. Since I is idempotent stable range one lifting, there exists $c \in R$ such that $b + I = c + I$ and $\operatorname{isr}(c) = 1$. Now $b - c \in I \subseteq K$ gives that $a - c \in K$. Hence $a + K = c + K$ and $\operatorname{isr}(c) = 1$. □

In particular, taking $K = J$ and $I = J$ in Proposition 3.18 respectively yields the following corollary.

Corollary 3.19. (a) *Let* I *be an ideal of a ring* R *with* $I \subseteq J$*. If* J *is idempotent stable range one lifting, then* J/I *is idempotent stable range one lifting. The converse is true if* I *is idempotent stable range one lifting.*

(b) *Let* K *be an ideal of a ring* R *with* $J \subseteq K$*. If* K *is idempotent stable range one lifting, then* K/J *is idempotent stable range one lifting. The converse is true if* J *is idempotent stable range one lifting.*

As an example, consider any ring R. If $\delta(R_R)$ is idempotent stable range one lifting, then $\delta(R_R)/J$ is idempotent stable range one lifting. The converse is true if J is idempotent stable range one lifting.

We end this article with a number of related questions that we were unable to answer.

Question 3.20. Is any idempotent stable range one lifting ideal stable range one lifting? Since the Jacobson radical is always stable range one lifting, it is necessary to consider an ideal different from the Jacobson radical.

Question 3.21. Is the converse of Corollary 3.3 true? This is equivalent to asking that whether there is a (non-quasi-duo) ring R such that the Jacobson radical is idempotent stable range one lifting but not idempotent lifting?

Question 3.22. Is the element-wise definition of (idempotent) stable range one is left-right symmetric?

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Meltem Altun-Özarslan
Department of Mathematics
Hacettepe University
06800 Beytepe Ankara, Turkey
Email address: meltemaltun@hacettepe.edu.tr

Ayşe Çiğdem Özcan
Department of Mathematics
Hacettepe University
06800 Beytepe Ankara, Turkey
Email address: ozcan@hacettepe.edu.tr