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A NUMERICAL PROPERTY OF HILBERT FUNCTIONS AND LEX SEGMENT IDEALS

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Abstract. We introduce the *fractal expansions*, sequences of integers associated to a number. We show that these sequences characterize the O-sequences and encode some information about lex segment ideals. Moreover, we introduce numerical functions called *fractal functions*, and we use them to solve the open problem of the classification of the Hilbert functions of any bigraded algebra.

1. Introduction

In commutative algebra (and other fields of pure mathematics) it often happens that easy numerical conditions describe some deeper algebraic results. A significant example are the O-sequences.

Let $S := k[x_1, \dots, x_n]$ be the standard graded polynomial ring and let $I \subseteq S$ be a homogeneous ideal. The quotient ring S/I is called a *standard graded k-algebra.* The Hilbert function of S/I is defined as $H_{S/I} : \mathbb{N} \to \mathbb{N}$ such that

$$H_{S/I}(t) := \dim_k (S/I)_t = \dim_k S_t - \dim_k I_t.$$

A famous theorem, due to Macaulay (cf. [14]) and pointed out by Stanley (cf. [16]), characterizes the numerical functions that are Hilbert functions of a standard graded k-algebra, i.e., the functions H such that $H = H_{S/I}$ for some homogeneous ideal $I \subseteq S$. To introduce this fundamental result we need some preparatory material.

Let $h, i > 0$ be integers. We can uniquely write h as

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}, \tag{1.1}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expression is called the *i-binomial expansion* of the integer h.

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If $h > 0$ has i-binomial expansion as in (1.1), then we set

$$h^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_i}{i} + \cdots + \binom{m_j + 1}{j+1}.$$

We use the convention that $0^{\langle i \rangle} = 0$.

For example, since $7 = \binom{4}{3} + \binom{3}{2}$, the 3-binomial expansion of 7 is and $7^{\langle 3 \rangle} = \binom{5}{4} + \binom{4}{3} = 9$.

Definition 1. A sequence of non-negative integers $(h_0, h_1, h_2, \ldots)$ is called an O-sequence if

(i) $h_0 = 1$;
(ii) $h_{i+1} \leq h_i^{\langle i \rangle}$ for all $i > 0$.

An O-sequence is said to have maximal growth from degree i to degree $i+1$ if $h_{i+1} = h_i^{\langle i \rangle}$.

We are now ready to enunciate the Macaulay's theorem. It characterizes the Hilbert function of standard graded k-algebras bounding the growth from any degree to the next. The proof of this theorem and more details about O-sequences are also discussed in [3, Chapter 4]. We represent $H_{S/I}$ as a sequence of integers $(h_0, h_1, h_2, \ldots)$ where $h_t := H_{S/I}(t)$.

Theorem 1.1 (Macaulay, [14]). *Let $H := (h_0, h_1, h_2, \ldots)$ be a sequence of integers. Then the following are equivalent*:

(1) *H is the Hilbert function of a standard graded k-algebra*;
(2) *H is an O-sequence.*

It is interesting to find an extension of the above theorem to the multi-graded case. Multi-graded Hilbert functions and resolutions arise in many contexts, see for instance [7, 9, 11, 17, 18], and properties related to the Hilbert function of multi-graded algebras are currently studied, see [6, 8, 10, 15].

The generalization of Macaulay's theorem to multi-graded rings is an open problem. A first answer was given by the author in [5] where the Hilbert functions of a bigraded algebra in $k[x_1, x_2, y_1, y_2]$ are classified.

In this paper, see Theorem 4.8, we generalize the Macaulay's theorem to any bigraded algebra. First, in Section 2, we introduce $\Phi(n)$, a list of finite sequences called fractal expansion of n. Then we define a *coherent truncation* of these vectors and we show that these objects are strictly related to O-sequences. Indeed, in Section 3 we show that they also characterize the Hilbert function of standard graded k-algebras. Furthermore, we show that these sequences can be used to compute the graded Betti numbers of a lex ideal.

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2. The expansion of a fractal sequence

In this section we introduce a sequence of tuples, called *coherent fractal growth*, and we study its properties. In the main result of this section, Theorem 2.12, we prove that these sequences have the same behavior of the O-sequences.

Roughly speaking a numerical sequence σ is called fractal if once we delete the first occurrence of each number it remains identical to the original. Such property thus implies that we can repeat this process indefinitely, and σ contains infinitely many copy of itself. It has something like a *fractal* behavior. See [13] for a formal definition and further properties.

For instance, one can show that the sequence

$$\sigma = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots)$$

is fractal. Indeed, after removing the first occurrence of each number we get a sequence that is the same as the starting one

$$(\cancel{1}, 1, \cancel{2}, 1, 2, \cancel{3}, 1, 2, 3, \cancel{4}, 1, 2, 3, 4, \cancel{5}, \ldots) = \sigma.$$

We introduce some notation. Given a positive integer $a \in \mathbb{N}$ we denote by $[a] := (1, 2, \ldots, a) \in \mathbb{N}^a$ the tuple of length a consisting of all the positive integers less then or equal to a written in increasing order.

Given a (finite or infinite) sequence σ of positive integers we construct a new sequence, named the expansion of σ, denoted by $[\sigma]$. If $\sigma := (s_1, s_2, s_3, \ldots)$ we set $[\sigma] := [s_1] \,\|\, [s_2] \,\|\, [s_3] \,\|\, \cdots$, where the symbol "$\|$" denotes the associative operation of concatenation of two vectors. E.g. $(3, 5, 4) \,\|\, (2, 3) = (3, 5, 4, 2, 3)$.

This construction can be recursively applied. We denote by $[\sigma]^d := [[\sigma]^{d-1}]$ where we set $[\sigma]^0 := \sigma$. For a positive integer a, we also denote by $[a]^d := [[a]^{d-1}]$, where $[a]^0 := (a)$.

For instance we have

$$[3] = (1, 2, 3), \quad [3]^2 = (1, 1, 2, 1, 2, 3), \quad [3]^3 = (1, 1, 1, 2, 1, 1, 2, 1, 2, 3).$$

Lemma 2.1. *Let $\sigma := (s_1, s_2, s_3, \ldots)$ be a sequence of positive integers. Then $[\sigma]^d = [s_1]^d \,\|\, [s_2]^d \,\|\, [s_3]^d \,\|\, \cdots$.*

Proof. If $d = 0$, the statement is true. Assume $d > 0$. By definition $[\sigma]^d = [[\sigma]^{d-1}]$, and by the inductive hypothesis we have

$$[\sigma]^d = [[s_1]^{d-1}] \,\|\, [[s_2]^{d-1}] \,\|\, [[s_3]^{d-1}] \,\|\, \cdots = [s_1]^d \,\|\, [s_2]^d \,\|\, [s_3]^d \,\|\, \cdots . \qquad \square$$

Corollary 2.2. *Let $a \in \mathbb{N}$ be a positive integer. Then*

$$[a]^d = [1]^{d-1} \,\|\, [2]^{d-1} \,\|\, \cdots \,\|\, [a]^{d-1}.$$

Proof. The statement follows from Lemma 2.1 since $[a]^d = [(1, 2, \ldots, a)]^{d-1}$. $\square$

Remark 2.3. The sequence $[\mathbb{N}] := [1] \,\|\, [2] \,\|\, \cdots \,\|\, [n] \,\|\, \cdots$ is a fractal sequence.

Given a sequence σ, the symbols σ_i and $\sigma(i)$ both denote the i-th entry of σ. If σ is finite, then $|\sigma|$ denotes the number of entries and $\sum \sigma$ their sum. We use the convention that these values are "∞" for infinite sequences of positive integers. For instance, $[3]^3_4 = 2$, $\left|[3]^3\right| = 10$ and $\sum [3]^3 = 15$.

Throughout this paper we use the convention that, for a finite sequence σ, the notation σ_a or $\sigma(a)$ implies $a \leq |\sigma|$.

Remark 2.4. Note that, for a finite sequence of positive integers σ, the definition of $[\sigma]$ easily implies the equality $\sum \sigma = |[\sigma]|$.

Given a positive integer n, we define the *fractal expansion* of n as the list

$$\Phi(n) := ([n]^0, [n]^1, \ldots, [n]^d, \ldots).$$

Each element in $\Phi(n)$ is a finite sequence of positive integers. In the following lemma we compute the number of their entries.

Lemma 2.5. *Let n be a positive integer. Then $\left|[n]^d\right| = \binom{d+n-1}{d}$ and $\sum [n]^d = \binom{d+n}{d+1}$.*

Proof. By definition we have $\left|[n]^0\right| = \binom{n-1}{0} = 1$ and $\left|[n]^1\right| = \binom{n}{1} = n$. By Corollary 2.2 we have $[n]^d = [1]^{d-1} \parallel [2]^{d-1} \parallel \cdots \parallel [n]^{d-1}$, therefore

$$\left|[n]^d\right| = \sum_{j=1}^{n} \left|[j]^{d-1}\right| = \sum_{j=1}^{n} \binom{d+j-2}{d-1} = \binom{d+n-1}{d}.$$

Moreover, by Remark 2.4 we have $\sum [n]^d = \left|[n]^{d+1}\right| = \binom{d+n}{d+1}$. $\square$

The next lemma introduces a way to decompose a number as sum of binomial coefficients that is slight different to the Macaulay decomposition. We use the convention that $\binom{a}{b} = 0$ whenever $a < b$.

Lemma 2.6. *Let d be a positive integer. Any $a \in \mathbb{N}$ can be written uniquely in the form*

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} + \binom{k_1}{1}, \tag{2.1}$$

where $k_d > k_{d-1} > \cdots > k_2 \geq k_1 \geq 1$.

Proof. In order to prove the existence, we choose k_d maximal such that $\binom{k_d}{d} < a$. If $d = 1$, the statement is trivial. If $d = 2$, then let k_2 be the maximal integer such that $a - \binom{k_2}{2} > 0$. Then set $k_1 := a - \binom{k_2}{2}$. Since $a \leq \binom{k_2+1}{2}$ we have $k_1 = a - \binom{k_2}{2} \leq \binom{k_2+1}{2} - \binom{k_2}{2} = k_2$. Assume $d > 2$, and let k_d be the maximum integer such that $a - \binom{k_d}{d} > 0$. By induction, $a - \binom{k_d}{d}$ can be written as $a - \binom{k_d}{d} = \sum_{i=1}^{d-1} \binom{k_i}{i}$, where $k_{d-1} > \cdots > k_2 \geq k_1 \geq 1$. Since $k_1 \geq 1$ we have $a - \binom{k_d}{d} > \binom{k_{d-1}}{d-1}$. Moreover, since $\binom{k_d+1}{d} \geq a$, it follows that

$$\binom{k_d}{d-1} = \binom{k_d+1}{d} - \binom{k_d}{d} \geq a - \binom{k_d}{d} > \binom{k_{d-1}}{d-1}.$$

Hence $k_d > k_{d-1}$. The uniqueness follows by induction on d. If $d = 1$, it is trivial. Now let $d > 1$ and assume for each $b \in \mathbb{N}$ the uniqueness of the decomposition

$$b = \binom{k'_{d-1}}{d-1} + \cdots + \binom{k'_2}{2} + \binom{k'_1}{1}, \tag{2.2}$$

where $k'_{d-1} > \cdots > k'_2 \geq k'_1 \geq 1$. Let $a \in \mathbb{N}$ and let

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} + \binom{k_1}{1} \tag{2.3}$$

be a decomposition of a. Then we claim that k_d is the maximal integer such that $\binom{k_d}{d} < a$. Indeed, if $\binom{k_d+1}{d} < a$, we get $a > \binom{k_d+1}{d} \geq \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} + \binom{k_1}{1} = a$. Consider now the number $a - \binom{k_d}{d}$. From Equation (2.3) we have $a - \binom{k_d}{d} = \sum_{i=1}^{d-1} \binom{k_i}{i}$ where $k_{d-1} > \cdots > k_2 \geq k_1 \geq 1$. The uniqueness of this decomposition follows from inductive hypothesis. This also implies the uniqueness of Equation (2.3). $\square$

Remark 2.7. The decomposition in Lemma 2.6 is different from the Macaulay decomposition since it is always required that $k_1 \geq 1$. Moreover, for any $j \geq 2$ we only have that $k_j \geq j - 1$, thus some binomial coefficient could be zero. For instance, we have $1 = \binom{d-1}{d} + \binom{d-2}{d-1} + \cdots + \binom{1}{2} + \binom{1}{1}$ where the first $d - 1$ binomial coefficients in the sum are equal to zero.

Definition 2. We refer to Equation (2.1) as the *d-fractal decomposition* of a. We denote by

$$[a]^{(d)} := (k_d - d + 2, k_{d-1} - d + 3, \ldots, k_4 - 2, k_3 - 1, k_2, k_1) \in \mathbb{N}^d.$$

We call these numbers the d-fractal coefficients of a.

$[a]^{(d)}$ is a not increasing sequence of positive integers. Indeed, $[a]^{(d)}_d > 0$ and by construction $k_1 \leq k_2$. Moreover, for $j > 1$, since $k_j > k_{j-1}$, we have $k_j - j + 2 \geq k_{j-1} - j + 1$.

The next result explains the name "d-fractal decomposition". We show that k_1 is the a-th entry in $[n]^d$, i.e., in $[\mathbb{N}]^{d-1}$. We use the convention that $[0]^d$ is the empty sequence and $\left|[0]^d\right| = 0$.

Theorem 2.8. $[\mathbb{N}]^{d-1}_a = [a]^{(d)}_d$.

Proof. Assume n is large enough. We will show that $[n]^d_a = [a]^{(d)}_d$. If $d = 1$, then $[n]^1_a = a$ and $a = \binom{a}{1}$, thus $[a]^{(1)}_1 = a = [n]^1_a$. We now assume $d > 1$. Let $a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} + \binom{k_1}{1}$ be the d-fractal decomposition of a. Note that, since $\binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} + \binom{k_1}{1}$ is the $(d-1)$ fractal decomposition of $a - \binom{k_d}{d}$, by the inductive hypothesis, we have $k_1 = [n]^{d-1}_{a-\binom{k_d}{d}}$. Therefore, we only have to show that $[n]^d_a = [n]^{d-1}_{a-\binom{k_d}{d}}$. Since $a \leq \binom{k_d+1}{d}$, by Lemma 2.5, we have $\left|[k_d - d + 2]^d\right| \geq a > \left|[k_d - d + 1]^d\right|$. Since from Corollary 2.2 we have the equality $[n]^d = [1]^{d-1} \parallel [2]^{d-1} \parallel \cdots \parallel [k_d - d + 1]^{d-1} \parallel [k_d - d + 2]^{d-1} \parallel \cdots$, it follows that the a-th entry in $[n]^d$ is the $\left(a - \binom{k_d}{d}\right)$-th in $[k_d - d + 2]^{d-1}$, i.e., $[n]^d_a = [k_d - d + 2]^{d-1}_{a-\binom{k_d}{d}} = [n]^{d-1}_{a-\binom{k_d}{d}}$. □

We introduce the lexicographic order for elements in $\mathbb{N}^d$. Given $\alpha, \beta \in \mathbb{N}^d$ then $\alpha <_{lex} \beta$ if and only if for some $i \leq d$ we have $\alpha_j = \beta_j$ for $j < i$ and $\alpha_i < \beta_i$. The following lemma is crucial for our intent. We prove that d-fractal coefficients have a good behavior with respect to the lex order.

Lemma 2.9. *$[a]^{(d)} <_{lex} [b]^{(d)}$ if and only if $a < b$.*

Proof. If $d = 1$, the assertion is trivial. Let $d > 1$ and let a, b be two integers with fractal decomposition

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_2}{2} + \binom{a_1}{1}, \quad a_d > \cdots > a_2 \geq a_1 \geq 1$$

and

$$b = \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_2}{2} + \binom{b_1}{1}, \quad b_d > \cdots > b_2 \geq b_1 \geq 1.$$

If $[a]^{(d)} <_{lex} [b]^{(d)}$, then there is an index j such that $[a]^{(d)}_i = [b]^{(d)}_i$ for any $i < j$ and $[a]^{(d)}_j < [b]^{(d)}_j$. Hence, $\binom{a_i}{i} = \binom{b_i}{i}$ for any $i < j$ and $\binom{a_j}{j} < \binom{b_j}{j}$. If $j = d$, then easily $b > a$, otherwise we have

$$b > \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_j}{j} \geq \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_j + 1}{j} \geq a.$$

Vice versa let $b > a$. We claim that $b_d \geq a_d$. Indeed, if $b_d < a_d$, we get $a > \binom{a_d}{d} \geq \binom{b_d+1}{d} \geq b$ contradicting $b > a$. So if $b_d > a_d$, we are done otherwise the statement follows by induction. □

Given two sequences τ and σ we say that τ is a truncation of σ if $|\tau| \leq |\sigma|$ and $\tau(j) = \sigma(j)$ for any $j \leq |\tau|$. For instance $(1, 1, 2, 1, 2)$ is a truncation of $[3]^2$.

Next definition introduces the main tool of the paper. *Coherent fractal growths* are suitable truncations of the elements in the fractal expansion of n.

Definition 3. We say that $T := (\tau_0, \tau_1, \tau_2, \ldots)$ is a coherent fractal growth if $\tau_0 := (n)$ and τ_j is a truncation of $[\tau_{j-1}]$ for each $j \geq 1$.

For instance $((3), (1, 2, 3), (1, 1, 2, 1, 2), (1, 1, 1, 2, 1, 1), (1, 1, 1, 1))$ is a coherent fractal growth. Indeed one can check that each elements is truncation of the expansion of the previous one. On the other hand, for instance, $((3), (1, 2, 3), (1, 1, 2), (1, 1, 1, 2, 1))$ is not a coherent fractal growth. Indeed, $(1, 1, 1, 2, 1)$ is not a truncation of $[(1, 1, 2)] = (1, 1, 1, 2)$.

Remark 2.10. Note that, in a coherent fractal growth, τ_d consists of the first $|\tau_d|$ elements in $[n]^d$. Moreover, the length of the elements in a coherent fractal growth $T := (\tau_0, \tau_1, \tau_2, \ldots)$ is bounded for any d. Indeed, by Remark 2.4 we have

$$|\tau_d| \leq \sum \tau_{d-1} \tag{2.4}$$

for each $d \geq 1$.

In the last part of this section we prove that the bound in Remark 2.10 is equivalent to the binomial expansion for a O-sequence. In order to relate the coherent fractal growth with O-sequences we need the following lemma.

Lemma 2.11. *Let* $[a]^{(d)} = (c_d, c_{d-1}, \ldots, c_1)$ *be the* d*-fractal coefficients of* a*. Then the* $(d+1)$*-fractal coefficients of* $a^{\langle d\rangle}$ *are*

$$[a^{\langle d\rangle}]^{(d+1)} = (c_d, c_{d-1}, \ldots, c_2, c_1, c_1).$$

Proof. The d-fractal decomposition of a is, by definition,

$$a = \binom{c_d + d - 2}{d} + \binom{c_{d-1} + d - 3}{d-1} + \cdots + \binom{c_3 + 1}{3} + \binom{c_2}{2} + \binom{c_1}{1}.$$

If $c_1 < c_2$, we get the Macaulay decomposition of a by removing the binomials $\binom{j}{i}$ equal to 0. Since $\binom{j}{i} = 0$ implies $\binom{j+1}{i+1} = 0$ we have

$$a^{\langle d\rangle} = \binom{c_d + d - 1}{d+1} + \binom{c_d + d - 2}{d} + \cdots + \binom{c_3 + 2}{4} + \binom{c_2 + 1}{3} + \binom{c_1 + 1}{2}.$$

Since $\binom{c_1+1}{2} = \binom{c_1}{2} + \binom{c_1}{1}$ we are done. Now we consider the case $c_1 = c_2$. Then we get the following decomposition of a

$$a = \binom{c_d + d - 2}{d} + \binom{c_{d-1} + d - 3}{d-1} + \cdots + \binom{c_3 + 1}{3} + \binom{c_2 + 1}{2}.$$

Thus, if $c_3 > c_2$, this representation is the Macaulay decomposition of a once we remove the binomials $\binom{j}{i}$ equal to 0.

$$\begin{aligned}
a^{\langle d\rangle} &= \binom{c_d + d - 1}{d+1} + \binom{c_{d-1} + d - 2}{d} + \cdots + \binom{c_3 + 2}{4} + \binom{c_2 + 2}{3} \\
&= \binom{c_d + d - 1}{d+1} + \binom{c_{d-1} + d - 2}{d} + \cdots + \binom{c_3 + 2}{4} + \binom{c_2 + 1}{3} \\
&\quad + \binom{c_1 + 1}{2} \\
&= \binom{c_d + d - 1}{d+1} + \binom{c_{d-1} + d - 2}{d} + \cdots + \binom{c_3 + 2}{4} + \binom{c_2 + 1}{3} \\
&\quad + \binom{c_1}{2} + \binom{c_1}{1}.
\end{aligned}$$

The proof follows in a finite number of steps by iterating this argument. ∎

The following theorem is the main result of this section. We show that the length of the elements in a coherent fractal growth is an O-sequence.

Theorem 2.12. *Let* $T = (\tau_0, \tau_1, \tau_2 \dots)$ *be a list of truncations of* $\Phi(n) = ((n), [n], [n]^2, \dots)$. *Then the following are equivalent*:

(i) $T = (\tau_0, \tau_1, \tau_2 \dots)$ *is a coherent fractal growth*;
(ii) $(|\tau_0|, |\tau_1|, |\tau_2|, \dots)$ *is an O-sequence.*

Proof. In order to prove (i)⇒(ii) we need to show $|\tau_{d+1}| \leq |\tau_d|^{\langle d \rangle}$ for each $d \geq 0$. Set $a = |\tau_d|$ and take the d-fractal decomposition of a

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_3}{3} + \binom{k_2}{2} + \binom{k_1}{1}.$$

If $a = \binom{n+d-1}{d} = \big|[n]^d\big|$ the statement follows by Lemma 2.5, indeed $|\tau^{d+1}| \leq |[n]^{d+1}| = \binom{n+d}{d} = a^{\langle d \rangle}$. Assume now $a < \binom{n+d-1}{d}$. Since τ_d is a truncation of $[n]^d = [1]^{d-1} \parallel [2]^{d-1} \parallel \cdots \parallel [n]^{d-1}$ and, by Lemma 2.5,

$$\big|[k_d - d + 1]^d\big| = \binom{k_d}{d} < a \leq \binom{k_d + 1}{d} = \big|[k_d - d + 2]^d\big|$$

we get (denoted by $[0]$ the empty sequence)

$$\tau_d = [k_d - d + 1]^d \parallel \tau_d',$$

where τ_d' is a truncation of $[k_d - d + 2]^{d-1}$. Therefore, iterating this argument, we have

$$\tau_d = [k_d - d + 1]^d \parallel [k_{d-1} - d + 2]^{d-1} \parallel \cdots \parallel [k_3 - 1]^3 \parallel [k_2 - 1]^2 \parallel [k_1].$$

By (2.4) in Remark 2.10, we have

$$\begin{aligned} |\tau_{d+1}| \leq \sum \tau_d &= \sum_{i=1}^{d} \binom{k_i + 1}{d+1} \\ &= \binom{k_d + 1}{d+1} + \binom{k_{d-1} + 1}{d} + \dots + \binom{k_2 + 1}{3} + \binom{k_1}{2} + \binom{k_1}{1}. \end{aligned}$$

This sum, by Lemma 2.11, is equal to $a^{\langle d \rangle}$.

Vice versa, to prove (ii)⇒(i), we have to show that, for each $d \geq 0$, the sequence τ_{d+1} is a truncation of $[\tau_d]$, i.e., $|\tau_{d+1}| \leq |[\tau_d]|$. By using the same argument as above, we get $|\tau_d|^{\langle d \rangle} = \sum \tau_d$. Then, the statement follows since $|\tau_{d+1}| \leq |\tau_d|^{\langle d \rangle}$, by hypothesis, and $|[\tau_d]| = \sum \tau_d$, by Remark 2.4. □

Let's check, for instance, that $H := (1, 3, 3, 4)$ is an O-sequence. We write a sequence of truncations of $\Phi(3)$ of length $1, 3, 3, 4$ respectively. We get

$$T := ((3), (1, 2, 3), (1, 1, 2), (1, 1, 1, 2)).$$

It is a coherent fractal growth. Indeed, by definition each sequence is a truncation of the bracket of the previous one, e.g. $[(1, 2, 3)] = (1, 1, 2, 1, 2, 3)$ and

$[(1,1,2)] = (1,1,1,2)$. Now, we check that $H := (1,3,5,8)$ is not an O-sequence. Indeed, take a coherent fractal growth

$$((3),(1,2,3),(1,1,2,1,2),\tau_3),$$

where the first three sequences are truncations of length $1,3,5$ of the elements in $\Phi(3)$. Then τ_3 should be a truncation of $[(1,1,2,1,2)] = (1,1,1,2,1,1,2)$ that has length 7. Thus $|\tau_3| \leq 7$ and $(1,3,5,8)$ is not an O-sequence.

3. Fractal expansions and homological invariants

In Section 2 we introduced a novel approach to describe the O-sequences. In this section we show the "algebraic" meaning of a coherent fractal growth. We directly relate these sequences to lex segment ideals and their homological invariants. In particular, the Eliahou-Kervaire formula is naturally applied to our case. Therefore, the fractal expansion of n is used in Proposition 3.4 to compute the graded Betti numbers of a lex segment ideal. See Section 2.1.2 in [12] for a complete discussion on monomial orders, including the lexicographic order.

Let a and d be positive integers. Let $[a]^{(d)} := (c_d, c_{d-1}, \ldots, c_2, c_1)$ be the d-fractal coefficient of a, see Lemma 2.6 and Definition 2. We associate to a and d a monomial $X_a^{(d)}$ of degree d in the variables $X := \{x_1, \ldots, x_n\}$ in such a way

$$X_a^{(d)} := x_{c_1} x_{c_2} \cdots x_{c_{d-1}} x_{c_d}.$$

Vice versa, a monomial $T = x_{c_1} x_{c_2} \cdots x_{c_{d-1}} x_{c_d}$ of degree d in the variables in X identifies a d-tuple $(c_d, c_{d-1}, \ldots, c_1)$ such that $c_i \leq c_{i+1}$ for each i.

Let $S := k[X] = k[x_1, \ldots, x_n]$ be the standard graded polynomial ring. We introduce in S the lexicographic order induced by $x_1 < x_2 < \cdots < x_n$. Note that this order of the variables is different from the one used in some literature (included [4] and [12]) where $x_1 > x_2 > \cdots > x_n$. Of course, it can be seen as a relabeling of the unknowns, but we will need to slight reformulate the results we quote from the references. The next remark shows that with respect the lexicographic order induced by $x_1 < x_2 < \cdots < x_n$ one can directly compute the a-th smallest monomial in S_d.

Remark 3.1. An immediate consequence of Lemma 2.9 is that $X_a^{(d)} >_{lex} X_b^{(d)}$ if and only if $a > b$. Thus, in the ordered list of the monomials in $k[X]_d$, from the smallest x_1^d to the biggest x_n^d, the monomial $X_a^{(d)}$ occupies the a-th position.

We set $\mathcal{G}(X)_{\leq t}^{(d)} := \{X_a^{(d)} \mid a \leq t\}$ and $\mathcal{G}(X)_{>t}^{(d)} := \{X_a^{(d)} \mid a > t\}$. Thus, S_d is spanned by the monomials in $\mathcal{G}(X)_{\leq t}^{(d)} \cup \mathcal{G}(X)_{>t}^{(d)}$.

By Remark 3.1 $G(X)_{>t}^{(d)}$ is a lex set of monomials of degree d.

Given $T := \{\tau_0, \tau_1, \tau_2, \ldots\}$ a coherent fractal growth, we set $I(T)_d := \langle \mathcal{G}(X)_{>|\tau_d|}^{(d)} \rangle_k$ the k-vector space spanned by the monomial in $\mathcal{G}(X)_{>|\tau_d|}^{(d)}$. Then, by Theorem 2.12 and Theorem 1.1, the following result holds.

Proposition 3.2. *Let T be a coherent fractal growth. Then $I(T) := \oplus_d I(T)_d \subseteq S$ is a lex segment ideal and $H_{S/I}(d) = |\tau_d|$.*

Given a minimal free resolution of a lex segment ideal

$$0 \to \bigoplus_j S(-j)^{\beta_{p,p+j}(I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,1+j}(I)} \to \bigoplus_j S(-j)^{\beta_{0,j}(I)} \to I \to 0$$

the graded Betti numbers of I can be computed by the Eliahou-Kervaire formula, see [4].

Theorem 3.3 (Eliahou-Kervaire formula)**.** *Let I be a lex segment ideal. For $u \in \mathcal{G}(I)$, a monomial minimal generator of I, let $m'(u)$ denotes the smallest integer a such that x_a divides u. Let m'_{js} be the number of monomials $u \in \mathcal{G}(I)_j$ with $m'(u) = s$. Then*

$$\beta_{i,i+j}(I) = \sum_{u \in \mathcal{G}(I)_j} \binom{n - m'(u)}{i} = \sum_{s=1}^{n} \binom{n-s}{i} m'_{js},$$

where we use the convention that $\binom{a}{0} = 1$ for $a \geq 0$.

This result can be written in terms of coherent fractal growth.

Proposition 3.4. *Given $T := \{\tau_0, \tau_1, \tau_2, \ldots\}$ a coherent fractal growth. Let τ'_j be the string such that $[\tau_{j-1}] = \tau_j \parallel \tau'_j$. Let w_{js} denotes the number of occurrence of the integer "s" in τ'_j, then*

$$\beta_{i,i+j}(I(T)) = \sum_{a=|\tau_j|+1}^{|[\tau_{j-1}]|} \binom{n - \tau_j(a)}{i} = \sum_{s=1}^{n} \binom{n-s}{i} w_{js},$$

where we use the convention that $\binom{a}{0} = 1$ for $a \geq 0$.

Proof. It is an immediate consequence of Theorem 3.3 and Theorem 2.8, since $\tau_j(a) = [a]_j^{(j)}$. □

Example 3.5. Let $S := k[x_1, x_2, x_3, x_4]$ be the polynomial ring in four variables. We introduce in S the lexicographic order induced by $x_1 < x_2 < x_3 < x_4$. Let I be the ideal of S generated by the eight largest monomials in S_3. Since $\dim_k S_3 = 20$, form Remark 3.1, these monomials can be constructed from a 3-fractal decomposition of the integers from 13 to 20. We get

a	13	14	15	16	17	18	19	20
$[a]^{(3)}$	$(4,2,2)$	$(4,3,1)$	$(4,3,2)$	$(4,3,3)$	$(4,4,1)$	$(4,4,2)$	$(4,4,3)$	$(4,4,4)$
$X_a^{(3)}$	$x_4x_2^2$	$x_4x_3x_1$	$x_4x_3x_2$	$x_4x_3^2$	$x_4^2x_1$	$x_4^2x_2$	$x_4^2x_3$	x_4^3

Using the notation as in Proposition 3.4 we have

$$\tau'_3 = ([a]_3^{(3)} \mid a = 13, \ldots, 20) = (2, 1, 2, 3, 1, 2, 3, 4)$$

and

$$w_{31} = 2, \quad w_{32} = 3, \quad w_{33} = 2, \quad w_{34} = 1.$$

So, the graded Betti numbers of I are

$$\beta_{0,3}(I) = \sum_{s=1}^{4} \binom{4-s}{0} w_{3s} = w_{31} + w_{32} + w_{33} + w_{34} = 8;$$

$$\beta_{1,4}(I) = \sum_{s=1}^{4} \binom{4-s}{1} w_{3s} = 3w_{31} + 2w_{32} + w_{33} = 14;$$

$$\beta_{2,5}(I) = \sum_{s=1}^{4} \binom{4-s}{2} w_{3s} = 3w_{31} + w_{32} = 9;$$

$$\beta_{3,6}(I) = \sum_{s=1}^{4} \binom{4-s}{3} w_{3s} = w_{31} = 2.$$

4. The Hilbert function of a bigraded algebra and the fractal functions

Let k be an infinite field, and let $R := k[x_1, \dots, x_n, y_1, \dots, y_m]$ be the polynomial ring in $n+m$ indeterminates with the grading defined by $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$. Then $R = \oplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$, where $R_{(i,j)}$ denotes the set of all homogeneous elements in R of degree (i,j). Moreover, $R_{(i,j)}$ is generated, as a k-vector space, by the monomials $x_1^{i_1} \cdots x_n^{i_n} y_1^{j_1} \cdots y_m^{j_m}$ such that $i_1 + \cdots + i_n = i$ and $j_1 + \cdots + j_m = j$. An ideal $I \subseteq R$ is called a *bigraded* ideal if it is generated by homogeneous elements with respect to this grading. A bigraded algebra R/I is the quotient of R with a bigraded ideal I. The Hilbert function of a bigraded algebra R/I is defined such that $H_{R/I} : \mathbb{N}^2 \to \mathbb{N}$ and $H_{R/I}(i,j) := \dim_k (R/I)_{(i,j)} = \dim_k R_{(i,j)} - \dim_k I_{(i,j)}$, where $I_{(i,j)} = I \cap R_{(i,j)}$ is the set of the bihomogeneous elements of degree (i,j) in I.

From now on, we will work with the degree lexicographical order on R induced by $x_n > \cdots > x_1 > y_m > \cdots > y_1$. With this ordering, we recall the definition of bilex ideal, introduced and studied in [2]. We refer to [2] for all preliminaries and for further results on bilex ideals.

Definition 4 ([2, Definition 4.4]). A set of monomials $L \subseteq R_{(i,j)}$ is called *bilex* if for every monomial $uv \in L$, where $u \in R_{(i,0)}$ and $v \in R_{(0,j)}$, the following conditions are satisfied:

- if $u' \in R_{(i,0)}$ and $u' > u$, then $u'v \in L$;
- if $v' \in R_{(0,j)}$ and $v' > v$, then $uv' \in L$.

A monomial ideal $I \subseteq R$ is called a *bilex ideal* if $I_{(i,j)}$ is generated as k-vector space by a bilex set of monomials, for every $i, j \geq 0$.

Bilex ideals play a crucial role in the study of the Hilbert function of bigraded algebras.

Theorem 4.1 ([2, Theorem 4.14]). *Let $J \subseteq R$ be a bigraded ideal. Then there exists a bilex ideal I such that $H_{R/I} = H_{R/J}$.*

In [5] it was solved the question of characterize the Hilbert functions of bigraded algebras of $k[x_1, x_2, y_1, y_2]$ by introducing the Ferrers functions. In this section we generalize these functions by introducing the fractal functions, see Definition 7. We prove, Theorem 4.8, that these classify the Hilbert functions of bigraded algebras.

We need some preparatory material. We denote by $\mathcal{M}^{a\times b}(U)$ the set of all the matrices with size (a, b) - a rows and b columns - and entries in a set $U \subseteq \mathbb{N}$. Given a matrix $M = (m_{ij}) \in \mathcal{M}^{a\times b}(U)$ we denote by

$$\sum M := \sum_{i\le a}\sum_{j\le b} m_{ij}.$$

Next definition introduces the objects we need in this section. The $\le$ symbol denote the partial order on $\mathbb{N}^2$ given by componentwise comparison.

Definition 5. A *Ferrers matrix* of size (a, b) is a matrix

$$M = (m_{ij}) \in \mathcal{M}(\{0, 1\})^{a\times b}$$

such that

$$\text{if } m_{ij} = 1, \text{ then } m_{i'j'} = 1 \text{ for any } (i', j') \le (i, j).$$

We set by $\mathcal{F}^{a\times b}$ the family of all the Ferrers matrices of size (a, b).

In the following definition we introduce expansions of a matrix.

Definition 6. Let $M \in \mathcal{M}(U)^{a\times b}$ be a matrix of size (a, b) and let $\mathbf{v} := (v_1, \dots, v_a) \in \mathbb{N}^a$ and $\mathbf{w} := (w_1, \dots, w_b) \in \mathbb{N}^b$ be vectors of non negative integers. We denote by $M^{\langle \mathbf{v},\bullet\rangle}$ an element in $\mathcal{M}(U)^{\sum \mathbf{v}\times b}$ constructed by

$$\text{repeating the } i\text{-th row of } M\ v_i \text{ times, for } i = 1, \dots, a.$$

We denote by $M^{\langle\bullet,\mathbf{w}\rangle}$ an element in $\mathcal{M}(U)^{a\times\sum \mathbf{w}}$ constructed by

$$\text{repeating the } j\text{-th column of } M\ w_j \text{ times, for } j = 1, \dots, b.$$

Remark 4.2. The expansions of a Ferrers matrix are also Ferres matrices. Take, for instance,

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{F}^{(4,3)}.$$

Set $\mathbf{v} := (2, 1, 0, 3)$ and $\mathbf{w} := (3, 1, 3)$. Then

$$M^{\langle \mathbf{v},\bullet\rangle} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{F}^{(6,3)}, \quad M^{\langle\bullet,\mathbf{w}\rangle} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{F}^{(4,7)}.$$

Given $M, N \in \mathcal{F}^{a\times b}$ we say that $M \leq N$ if and only if $m_{ij} \leq n_{ij}$ for any i, j.

We are ready to introduce the fractal functions.

Definition 7. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. We say that H is a *fractal function* if $H(0,0) = 1$ and, for any $(i,j) \in \mathbb{N}^2$, there exists a matrix $M_{ij} \in \mathcal{F}^{\binom{i-1+n}{n-1}\times\binom{j-1+m}{m-1}}$ with $\sum M_{ij} = H(i,j)$ and such that all the matrices satisfy the condition

$$\begin{cases} M_{ij} \leq M_{i-1\ j}^{\langle [n]^{i-1}, \bullet\rangle} & \text{if } i > 0; \\ \\ M_{ij} \leq M_{i\ j-1}^{\langle \bullet, [m]^{j-1}\rangle} & \text{if } j > 0. \end{cases}$$

Remark 4.3. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be the numerical function

$$H(i,j) = \binom{i-1+n}{n-1}\binom{j-1+m}{m-1}.$$

For any $i, j \in \mathbb{N}$, there is only one element in $M_{ij} \in \mathcal{F}^{\binom{i-1+n}{n-1}\times\binom{j-1+m}{m-1}}$ satisfying the condition in Definition 7 that is the matrix with all "1" entries. Therefore H is a fractal function.

Remark 4.4. If $n = m = 2$, the definition of fractal functions agrees with Definition 3.3 in [5]. Indeed it is enough to write each partition $\alpha_{ij} = (a_1, a_2, \ldots)$ as a matrix $M_{ij} = (m_{hk}) \in \mathcal{F}^{(i+1)\times(j+1)}$ where $m_{hk} = 1$ if and only if $k \leq a_k$ otherwise $m_{hk} = 0$. In this case the expansions are given by the elements in $\Phi(2) := ((2), (1,2), (1,1,2), (1,1,1,2), \ldots)$.

In general the matrices M_{ij} are not uniquely determined by the conditions of Definition 7, even in "small" cases.

Example 4.5. Set $n = m = 2$ and let $H : \mathbb{N}^2 \to \mathbb{N}$ be the numerical function:

$$H := \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & \ldots \\ \hline 0 & 1 & 2 & 3 & 0 & \ldots \\ 1 & 2 & 4 & 3 & 0 & \ldots \\ 2 & 3 & 3 & 3 & 0 & \ldots \\ 3 & 0 & 0 & 0 & 0 & \ldots \end{array}$$

The only possibility for M_{11} is

$$M_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{F}^{(2,2)}.$$

Thus, we have no restriction on M_{21} and M_{12} but the number of non zero entries:

$$M_{12} \in \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\} \subseteq \mathcal{F}^{(2,3)},$$

$$M_{21} \in \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \subseteq \mathcal{F}^{(3,2)}.$$

Now we check if all these choices are allowed. We have to look at the conditions on M_{22}. We have $M_{22} \leq M_{12}^{\langle(1,2),\bullet\rangle}$ and $M_{22} \leq M_{21}^{\langle\bullet,(1,2)\rangle}$. In the next table we collect all the possibilities for M_{22} depending on M_{12} and M_{21}.

$M_{22} \leq$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Since $H(2,2) = 3$, we note that

- only one of the above four matrices in the table must be excluded;
- in the other three cases M_{22} is uniquely determined by M_{12} and M_{21}.

Therefore, H is a fractal function and three different set of matrices satisfy the conditions in Definition 7.

In the following we denote by $X := \{x_1, \dots, x_n\}$ and $Y := \{y_1, \dots, y_m\}$ the set of the variables of degree $(1,0)$ and $(0,1)$ respectively.

Next lemma is useful for our purpose. It is an immediate consequence of Lemma 2.11.

Lemma 4.6. $x_1 \cdot X_a^{(d)} = X_{a^{\langle d\rangle}}^{(d+1)}$.

To shorten the notation we set $\alpha_i := \binom{n+i-1}{n-1}$ and $\beta_j := \binom{m+j-1}{m-1}$. In order to relate fractal functions and Hilbert functions of bigraded algebras we need to introduce a correspondence between Ferrers matrices and monomials.

Let $M = (m_{ab}) \in \mathcal{F}^{\alpha_i \times \beta_j}$. We denote by $\lambda(M)_{(ij)}$ the set of the monomials

$$\lambda(M) := \{X_a^{(i)} \cdot Y_b^{(j)} \mid m_{ab} = 0\}.$$

Let $L \subseteq R_{(i,j)}$ be a bilex set of monomials of bidegree (i,j). We denote by $\mu(L) \in \mathcal{M}(\{0,1\})^{\alpha_i \times \beta_j}$ the matrix (m_{ab}) such that $m_{ab} = 1$ if and only if $X_a^{(i)} \cdot Y_b^{(j)} \notin L$ otherwise $m_{ab} = 0$.

Proposition 4.7. *There is a one to one correspondence between bilex sets of monomials of degree (i,j) and elements in $\mathcal{F}^{\alpha_i \times \beta_j}$.*

Proof. Let $M = (m_{ab}) \in \mathcal{F}^{\alpha_i \times \beta_j}$, we claim that

$$\lambda(M) \ \text{ is a bilex set of monomials of bidegree } (i,j).$$

To prove the claim we use Lemma 2.9 and Remark 3.1. Let $X_a^{(i)} \cdot Y_b^{(j)}$ be an element in $\lambda(M)$ and $X_u^{(i)} > X_a^{(i)}$. Since $(u,b) > (a,b)$ and $m_{ab} = 0$ we get

$m_{ub} = 0$, i.e., $X_u^{(i)} \cdot Y_b^{(j)} \in \lambda(M)$. In a similar way it follows that $X_a^{(i)} \cdot Y_v^{(j)} \in \lambda(M)$ for $v > b$.

Let $L \subseteq R_{(i,j)}$ be a bilex set of monomials of bidegree (i, j). We claim that

$$\mu(L) \in \mathcal{F}^{\alpha_i \times \beta_j}.$$

The claim follows by using Lemma 2.9 and Remark 3.1. Indeed, say $m_{ab} = 0$ for an entry of $\mu(L)$. This implies $X_a^{(i)} \cdot Y_b^{(j)} \in L$. Thus, for $u > a$, we have $X_u^{(i)} \cdot Y_b^{(j)} \in L$, i.e., $m_{ub} = 0$. Analogously, we see that $m_{av} = 0$ for $v > b$. $\square$

We are ready to prove the main result of this paper.

Theorem 4.8. *Let $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a numerical function. Then the following are equivalent*:

(1) *H is a fractal function*;
(2) *$H = H_{R/I}$ for some bilex ideal $I \subseteq R = k[x_1, \ldots, x_n, y_1, \ldots, y_m]$.*

Proof. (1) $\Rightarrow$ (2) Let H be a fractal function. For each $(i, j) \in \mathbb{N}^2$, let $I_{(i,j)}$ be the k-vector space spanned by the elements in $\lambda(M_{ij})$. We claim that $I := \oplus_{(i,j)\in\mathbb{N}^2} I_{(i,j)}$ is an ideal of R. To prove the claim, it is enough to show that if $X_a^{(i)} \cdot Y_b^{(j)} \in I_{(i,j)}$, then $x_u \cdot X_a^{(i)} \cdot Y_b^{(j)} \in I_{(i+1,j)}$ for any $x_u \in X$ and $y_v \cdot X_a^{(i)} \cdot Y_b^{(j)} \in I_{(i,j+1)}$ for any $y_v \in Y$. We have, see Lemma 4.6,

$$x_u \cdot X_a^{(i)} \cdot Y_b^{(j)} \geq x_1 \cdot X_a^{(i)} \cdot Y_b^{(j)} = X_{a^{\langle i \rangle}}^{(i+1)} \cdot Y_b^{(j)}.$$

Then, by Definition 7 and Theorem 2.12, the entry $(a^{\langle i \rangle}, b)$ in the matrix M_{i+1j} is 0. Thus $X_{a^{\langle i \rangle}}^{(i+1)} \cdot Y_b^{(j)} \in I_{(i+1,j)}$ and furthermore $x_u \cdot X_a^{(i)} \cdot Y_b^{(j)} \in I_{(i+1,j)}$. In a similar way it follows that $y_v \cdot X_a^{(i)} \cdot Y_b^{(j)} \in I_{(i,j+1)}$.

(2) $\Rightarrow$ (1) Let $I \subseteq R$ be a bilex ideal such that $H_{R/I} = H$. Set $M_{ij} := \mu(I_{ij})$, we claim that the M_{ij}s satisfy the condition in Definition 7. By Theorem 2.12 it is enough to show that if $M_{ij}(a, b) = 0$ (the entry (a, b) in M_{ij} is 0), then also $M_{i+1j}(a^{\langle i \rangle}, b) = 0$ (the entry $(a^{\langle i \rangle}, b)$ in M_{i+1j} is 0). Set $J := (X_u^{(h)} \mid M_{hj}(u, b) = 0)$, then the claim is an immediate consequence of the fact that J is a lex ideal of $k[X]$. $\square$

Remark 4.9. Note that since the M_{ij} are not uniquely determined, as shown in Example 4.5, there could be several bilex ideals having the same Hilbert function.

The following question is motivated by the arguments in Section 3.

Question 4.10. Can the bigraded Betti numbers of a bilex ideal $I = \oplus I_{(i,j)}$ be computed from the matrices $\mu(I_{(i,j)})$?

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