# REGULARITY AND MULTIPLICITY OF SOLUTIONS FOR A NONLOCAL PROBLEM WITH CRITICAL SOBOLEV-HARDY NONLINEARITIES 

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Abstract. In this work we investigate the nonlocal elliptic equation with critical Hardy-Sobolev exponents as follows,

$$
\text { (P) } \begin{cases}\left(-\Delta_{p}\right)^{s} u=\lambda|u|^{q-2} u+\frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}} & \text { in } \Omega, \\ u=0 & \text { in } \mathbb{R}^{N} \backslash \Omega,\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$ is an open bounded domain with Lipschitz boundary, $0<s<1, \lambda>0$ is a parameter, $0<t<s p<N, 1<q<p<p_{s}^{*}$ where $p_{s}^{*}=\frac{N p}{N-s p}, p_{s}^{*}(t)=\frac{p(N-t)}{N-s p}$, are the fractional critical Sobolev and Hardy-Sobolev exponents respectively. The fractional $p$-laplacian $\left(-\Delta_{p}\right)^{s} u$ with $s \in(0,1)$ is the nonlinear nonlocal operator defined on smooth functions by
$\left(-\Delta_{p}\right)^{s} u(x)=2 \lim _{\epsilon \searrow 0} \int_{\mathbb{R}^{N} \backslash B_{\epsilon}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+p s}} \mathrm{~d} y, \quad x \in \mathbb{R}^{N}$.
The main goal of this work is to show how the usual variational methods and some analysis techniques can be extended to deal with nonlocal problems involving Sobolev and Hardy nonlinearities. We also prove that for some $\alpha \in(0,1)$, the weak solution to the problem $(\mathrm{P})$ is in $C^{1, \alpha}(\bar{\Omega})$.

## 1. Introduction

The purpose of this paper is mainly to study the nonlocal elliptic equation with critical Hardy-Sobolev exponents as follows,

$$
\text { (P) } \begin{cases}\left(-\Delta_{p}\right)^{s} u=\lambda|u|^{q-2} u+\frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}} & \text { in } \Omega \\ u=0 & \text { in } \mathbb{R}^{N} \backslash \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$ is an open bounded domain with Lipschitz boundary, $0<s<1$, $\lambda>0$ is a parameter, $0<t<s p<N, 1<q<p<p_{s}^{*}$. As usual $p_{s}^{*}=\frac{N p}{N-s p}$ and $p_{s}^{*}(t)=\frac{p(N-t)}{N-s p}$ are the fractional critical Sobolev and Hardy-Sobolev exponents

[^0]respectively. The fractional $p$-laplacian $\left(-\Delta_{p}\right)^{s} u$ with $s \in(0,1)$ is the nonlinear nonlocal operator defined on smooth functions by
$$
\left(-\Delta_{p}\right)^{s} u(x)=2 \lim _{\epsilon \searrow 0} \int_{\mathbb{R}^{N} \backslash B_{\epsilon}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+p s}} \mathrm{~d} y, \quad x \in \mathbb{R}^{N} .
$$

The problems of this type are important in many fields of sciences, notably the fields of physics, probability, finance, electromagnetism, astronomy, and fluid dynamics, it also they can be used to accurately describe the jump Lévy processes in probability theory and fluid potentials for more details see $[1,10]$ and references therein.

Before giving the important result that we will investigate in this work, let us briefly recall the literature concerning related problems with Sobolev and Hardy nonlinearity. In [9] Chen, Mosconi and Squassina using Nehari manifold approach and fibering maps established existence and multiplicity of solutions for a class of nonlinear nonlocal problems with Sobolev and Hardy nonlinearity at subcritical and critical growth

$$
\begin{cases}\left(-\Delta_{p}\right)^{s} u=\lambda|u|^{r-2} u+\mu \frac{|u|^{q-2} u}{|x|^{\alpha}} & \text { in } \Omega,  \tag{1}\\ u=0 & \text { in } \mathbb{R}^{n} \backslash \Omega,\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with Lipschitz boundary, $0<s<1$, $\lambda, \mu>0$ are two parameters, $0 \leq \alpha \leq s p<N, 1<p \leq r \leq p_{s}^{*}, p \leq q \leq$ $p_{s}^{*}(\alpha)$ where $p_{s}^{*}=\frac{N p}{N-s p}, p_{s}^{*}(\alpha)=\frac{p(N-\alpha)}{N-s p}$ are the fractional critical Sobolev and Hardy-Sobolev exponents respectively. Yan [36] using abstract critical point theorems proved the existence, multiplicity, and bifurcation results for the Brezis-Nirenberg problem for the fractional $p$-Laplacian operator involving critical Hardy-Sobolev exponents

$$
\begin{cases}\left(-\Delta_{p}\right)^{s} u=\lambda|u|^{p-2} u+\frac{|u|^{p_{s}^{*}(\alpha)-2} u}{|x|^{\alpha}} & \text { in } \Omega,  \tag{2}\\ u=0 & \text { in } \mathbb{R}^{N} \backslash \Omega,\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with Lipschitz boundary, $0<s<1$, $\lambda>0,0<\alpha<s p<N$.

If $t=0$, then the problem ( P ) is reduced to

$$
\begin{cases}\left(-\Delta_{p}\right)^{s} u=\lambda|u|^{q-2} u+|u|^{p_{s}^{*}-2} u & \text { in } \Omega  \tag{3}\\ u=0 & \text { in } \mathbb{R}^{N} \backslash \Omega\end{cases}
$$

where $\lambda>0$ is a parameter and $p_{s}^{*}=\frac{N p}{N-s p}$. We point out that many researchers are paying a lot of attention on this nonlocal problem (3). Iannizzotto-Liu-Perera-Squassina [20] established an existence result via Morse theory in the subcritical case. The critical case is treated in Perera-Squassina-Yang [26] with additional new abstract based on a pseudo-index related to the $\mathbb{Z}_{2^{-}}$ cohomological index. These restrictions are used to prove the existence of a range of the validity of the Palais-Smale condition. Note that, in this work,
the bifurcation and multiplicity results are obtained for some restrictions on the parameter $\lambda$. The Brezis-Nirenberg type existence result is studied in [24]. In [15] Ghanmi-Saoudi using the method of Nehari manifold and fibering maps proved the existence and multiplicity of nonnegative solutions to the nonlocal problem for subcritical concave-convex nonlinearities. Also, the Dirichlet boundary value problem in the case of fractional Laplacian with concave-convex type nonlinearity using variational methods has been studied in $[2,14,29]$ and references therein.

In the local setting $(s=1)$, the problem ( P ) becomes

$$
\begin{cases}-\Delta_{p} u=\lambda|u|^{q-2} u+\frac{|u|^{p^{*}-2} u}{|x|^{t}} & \text { in } \Omega,  \tag{4}\\ u=0 & \text { in } \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with smooth boundary, $1<p<N$, $\lambda>0$ is a parameter, $0<t<p, 1<p \leq q \leq p^{*}(s)$ where $p^{*}=\frac{N p}{N-p}$, is the critical Sobolev exponent and $\Delta_{p} u=\nabla \cdot\left(|\nabla u|^{p-2} \nabla u\right)$ denotes the $p$ Laplace operator. The results obtained by Ghoussoub-Yuan [17] is the starting point on quasilinear problems with Hardy-Sobolev exponent. The authors obtained the existence of infinitely many non-trivial solutions for the problem (4). From this pioneering work, a lot of contributions have been done related to existence, multiplicity, stability and regularity results on problems involving Hardy-Sobolev exponent see [19, 23, 27, 30, 35] and references therein. In [23], the authors studied an elliptic equation involving the critical Sobolev-Hardy exponents and singular potential. They obtained the existence of infinitely many small solutions using concentration-compactness principle and a new version of the symmetric mountain-pass lemma due to Kajikiya. In [19, 35], the concave convex problems with multiple Hardy type terms is studied where multiplicity results are obtained using Nehari manifold approach and fibering maps. Perera-Zou [27], using critical point theorems based on a cohomological index and a related pseudo-index proved the existence, multiplicity, and bifurcation results for the case $\lambda \geq \lambda_{1}$ and extend results in the literature for $0<\lambda<\lambda_{1}$, where $\lambda_{1}>0$ is the first eigenvalue of the eigenvalue problem

$$
\begin{cases}-\Delta_{p} u=\lambda|u|^{p-2} u & \text { in } \Omega  \tag{5}\\ u=0 & \text { in } \Omega\end{cases}
$$

In the local setting case ( $s=1$ and $p=2$ ) the problem $(\mathrm{P})$ is reduced to the semilinear problem with Sobolev-Hardy exponents

$$
\begin{cases}-\Delta u=\lambda u+\frac{|u|^{2^{*}-2} u}{|x|^{2}} & \text { in } \Omega  \tag{6}\\ u=0 & \text { in } \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$ is an open bounded domain with smooth boundary, $\lambda>0$ is a parameter and $2^{*}=\frac{2 N}{N-2}$ is the critical Hardy-Sobolev exponent. This problem has been paid more attention by many authors. We refer the readers
to $[8,22,34]$ and references therein. Cao and Han [8] proved the existence of sign-changing solutions of problem (6) and obtained some estimates for all weak solutions in $H^{1}\left(\mathbb{R}^{N}\right)$. In [34], the authors studied problem (6) with two critical Hardy-Sobolev exponents and boundary singularities. Using Ekeland's variational principle and strong maximum principle, they proved the existence and multiplicity of positive solutions. Jiang and Tang [22], using variational methods obtained the existence of positive solutions of problem (6) when the Hardy-Sobolev-Mazáya potential is concerned.

Now, we state the theorems that we will proved in this paper.
Theorem 1.1. There exists $\Lambda \in(0, \infty)$ such that,
(i) $\forall \lambda \in(0, \Lambda)$, the problem (P) has a minimal solution.
(ii) For $\lambda=\Lambda$ the problem (P) has at least one solution.
(iii) $\forall \lambda \in(\Lambda, \infty)$ the problem ( P ) has no solution.

Theorem 1.2. For every $\lambda \in(0, \Lambda)$, the problem (P) has multiple solutions.

## 2. A functional framework for the nonlocal problems

In this section, we start by recalling some notations which will be frequently used throughout the rest of this work. We start by defining the following function space. Define the fractional Sobolev space

$$
W_{0}^{s, p}\left(\mathbb{R}^{N}\right):=\left\{u \in L^{p}\left(\mathbb{R}^{N}\right): u \text { measurable, }|u|_{s, p}<+\infty, u \equiv 0 \text { a.e. on } \Omega^{c}\right\},
$$

and the homogeneous fractional Sobolev space

$$
W^{s, p}\left(\mathbb{R}^{N}\right):=\left\{u \in L^{p}\left(\mathbb{R}^{N}\right): u \text { measurable, }|u|_{s, p}<+\infty\right\},
$$

the usual fractional Sobolev space with the Gagliardo norm

$$
\|u\|_{s, p}:=\left(\|u\|_{p}^{p}+|u|_{s, p}^{p}\right)^{\frac{1}{p}} .
$$

For a detailed account on the properties of $W^{s, p}\left(\mathbb{R}^{N}\right)$ we refer the reader to [25].

Let $\Omega \subset \mathbb{R}^{N}$ and define $Q=\mathbb{R}^{2 N} \backslash\left(\left(\mathbb{R}^{N} \backslash \Omega\right) \times\left(\mathbb{R}^{N} \backslash \Omega\right)\right)$, then the space $\left(X,\|\cdot\|_{X}\right)$ is defined by

$$
X=\left\{u: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R} \text { is measurable, }\left.u\right|_{\Omega} \in L^{p}(\Omega) \text { and } \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+p s}{p}}} \in L^{p}(Q)\right\}
$$

equipped with the Gagliardo norm

$$
\|u\|_{X}=\|u\|_{p}+\left(\int_{Q} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+p s}} d x d y\right)^{\frac{1}{p}}
$$

Here $\|u\|_{p}$ refers to the $L^{p}$-norm of $u$. We further define the space

$$
X_{0}=\left\{u \in X: u=0 \text { a.e. in } \mathbb{R}^{N} \backslash \Omega\right\}
$$

equipped with the norm

$$
\|u\|=\left(\int_{Q} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+p s}} d x d y\right)^{\frac{1}{p}}
$$

The best Sobolev constant is defined as

$$
\begin{equation*}
S=\inf _{u \in X_{0} \backslash\{0\}} \frac{\int_{Q} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+p s}} d x d y}{\left(\int_{\Omega}\left|p_{s}^{*}\right| d x\right)^{\frac{p}{p_{s}^{*}}}} \tag{7}
\end{equation*}
$$

We now state the following definitions to the problem (P). At first, associated to the problem (P) we have the functional energy $E_{\lambda}: X_{0} \rightarrow \mathbb{R}$ defined by

$$
E_{\lambda}(u)=\frac{1}{p} \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y-\frac{\lambda}{q} \int_{\Omega}|u|^{q} \mathrm{~d} x-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{|u|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x
$$

Remark 2.1. Obviously, every critical point of $E_{\lambda}$ is a weak solution of the problem (P).

Now, we define a weak solution to the problem (P) as follows.
Definition 1. We say that $u \in X_{0}$ is a weak solution of the problem (P) if for all $\phi \in X_{0}$, one has

$$
\begin{aligned}
& \int_{Q} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
= & \lambda \int_{\Omega}|u|^{q-2} u \phi \mathrm{~d} x+\int_{\Omega} \frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}} \phi \mathrm{~d} x .
\end{aligned}
$$

Definition 2. One calls a solution $u_{\lambda}$ of problem (P) is minimal if $u_{\lambda} \leq v$ almost every where in $\Omega$ for any further solution $v$ of problem ( P ).

Then, we state the definitions of a sub and a super-solution to the problem (P).

Definition 3. A function $\underline{u}_{\lambda} \in X_{0}$ is called a weak subsolution to the problem (P), if
(i) $\underline{u}_{\lambda}>0$, and
(ii) $\int_{Q} \frac{\left|\underline{u}_{\lambda}(x)-\underline{u}_{\lambda}(y)\right|^{p-2}\left(\underline{u}_{\lambda}(x)-\underline{u}_{\lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+p s}} d x d y$

$$
-\lambda \int_{\Omega}\left|\underline{u}_{\lambda}\right|^{q-2} \underline{u}_{\lambda} \phi \mathrm{d} x-\int_{\Omega} \frac{\left|\underline{u}_{\lambda}\right|^{p_{s}^{*}(t)-2} \underline{u}_{\lambda}}{|x|^{t}} \phi \mathrm{~d} x \leq 0
$$

for every positive $\phi \in X_{0}$.
Definition 4. A function $\bar{u}_{\lambda} \in X_{0}$ is called a weak supersolution to the problem (P), if
(i) $\bar{u}_{\lambda}>0$, and
(ii) $\int_{Q} \frac{\left|\bar{u}_{\lambda}(x)-\bar{u}_{\lambda}(y)\right|^{p-2}\left(\bar{u}_{\lambda}(x)-\bar{u}_{\lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+p s}} d x d y$

$$
-\lambda \int_{\Omega}\left|\bar{u}_{\lambda}\right|^{q-2} \bar{u}_{\lambda} \phi \mathrm{d} x-\int_{\Omega} \frac{\left|\bar{u}_{\lambda}\right|^{p_{s}^{*}(t)-2} \bar{u}_{\lambda}}{|x|^{t}} \phi \mathrm{~d} x \geq 0
$$

for all positive $\phi \in X_{0}$.
We now list out the embedding results (See $[31,32]$ for more details).
Lemma 2.2. The following embedding results holds for the space $X_{0}$.
(1) If $\Omega$ has a Lipshitz boundary, then the embedding $X_{0} \hookrightarrow L^{q}(\Omega)$ for $q \in\left[1, p_{s}^{*}\right)$, where $p_{s}^{*}=\frac{N p}{N-p s}$.
(2) The embedding $X_{0} \hookrightarrow L^{p_{s}^{*}}(\Omega)$ is continuous.

For $u \in L^{p_{s}^{*}(t)}\left(\mathbb{R}^{N}\right)$, we denote by

$$
|u|_{p_{s}^{*}(t)}=\left(\int_{\mathbb{R}^{N}} \frac{|u|^{p_{s}^{*}(t)}}{|x|^{t}} d x\right)^{1 / p_{s}^{*}(t)}
$$

We now must recall fractional Hardy-Sobolev inequality.
Lemma 2.3 (Hardy Sobolev inequality [9]). Assume that $0 \leq t \leq p s<N$. Then there exists a positive constant $C$ such that

$$
\begin{equation*}
\left(\int_{\Omega} \frac{|u|^{p_{s}^{*}(t)}}{|x|^{t}} d x\right)^{1 / p_{s}^{*}(t)} \leq C\left(\int_{\mathbb{R}^{2 N}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+s p}} d x d y\right)^{1 / p} \tag{8}
\end{equation*}
$$

for every $u \in W_{0}^{s, p}(\Omega)$.
The following embedding results has been proved in [9].
Lemma 2.4. (1) The embedding $W_{0}^{s, p}(\Omega) \rightarrow L^{q}\left(\Omega, \frac{d x}{|x|^{t}}\right)$ is continuous for $q \in\left[1, p_{s}^{*}(t)\right]$ and compact for $q \in\left[1, p_{s}^{*}(t)\right)$.
(2) For $p>1, W_{0}^{s, p}(\Omega)$ and $D^{s, p}\left(\mathbb{R}^{N}\right)$ are separable reflexive Banach space with respect to the norm $[\cdot]_{s, p}$.
We will also define for any $\alpha \in[0, p s]$ the positive numbers

$$
S_{t}=\inf \left\{\int_{\mathbb{R}^{2 N}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+s p}} d x d y: u \in W_{0}^{s, p}(\Omega) \text { with } \int_{\Omega} \frac{|u|^{p_{s}^{*}(t)}}{|x|^{t}} d x=1\right\} .
$$

## 3. Existence of weak solutions

This section is devoted to show the existence of a solution to the problem ( P ). Our first result is to show that the functional $E_{\lambda}$ possesses a local minimum in a small neighborhood of the origin in $X_{0}$. We start by proving the coerciveness of the functional $E_{\lambda}$. Precisely, we have the following result.

Lemma 3.1. There exist $\lambda_{0}>0, R_{0}>0$ and $\delta_{0}>0$ such that $E_{\lambda}(u) \geq \delta_{0}>0$ for all $\|u\|=R_{0}$.

Proof. Using Hölder inequality and the fractional Sobolev-Hardy inequality, we have

$$
\begin{aligned}
E_{\lambda}(u) & =\frac{1}{p} \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y-\frac{\lambda}{q} \int_{\Omega}|u|^{q} \mathrm{~d} x-\frac{1}{p^{*}(t)} \int_{\Omega} \frac{|u|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \\
& \geq \frac{1}{p}\|u\|_{X_{0}}^{p}-\frac{\lambda}{q} C_{0}\|u\|_{q}^{q}-\frac{C_{1}}{p_{s}^{*}(t)}\|u\|^{p_{s}^{*}(t)} \\
& =\|u\|^{q}\left(\frac{1}{p}\|u\|^{p-q}-\frac{\lambda}{q} C_{0}-\frac{C_{1}}{p_{s}^{*}(t)}\|u\|^{p_{s}^{*}(t)-q}\right)
\end{aligned}
$$

where $C_{0}, C_{1}$ are two positive constants. Put $f(x)=\frac{1}{p} x^{p-q}-\frac{C_{1}}{p_{s}^{*}(t)} x^{p_{s}^{*}(t)-q}-$ $\frac{\lambda}{q} C_{0}$. We find that there is a constant $R=\left(\frac{p_{s}^{*}(t)(p-q)}{p C_{1}\left(p_{s}^{*}(t)-q\right)}\right)^{1 / p_{s}^{*}(t)-p}>0$ such that $f(R)=\max _{k>0} f(k)>0$. Choosing $\lambda_{0}=\frac{q f(R)}{C_{0}}$, we deduce the existence of a constant $\delta_{0}>0$ satisfying

$$
E_{\lambda}(u) \geq \delta_{0}>0
$$

for all $\lambda \in\left(0, \lambda_{0}\right)$. The proof of Lemma 3.1 is now completed.
Lemma 3.2. $E_{\lambda}$ possesses a local minimum close to the origin in $X_{0}$ for all $\lambda \in\left(0, \lambda_{0}\right)$.

Proof. Let $\lambda_{0}, R_{0}$ and $\delta_{0}$ as in Lemma 3.1. Noting that for all $\varphi \in X_{0}, \varphi \geq 0$, $\varphi \neq 0$ and $r>1$, we have

$$
E_{\lambda}(r \varphi)=\frac{|r|^{p}}{p} \|\left.\varphi\right|^{p}-\frac{\lambda}{q}|r|^{q} \int_{\Omega}|\varphi|^{q} d x-\frac{r^{p_{s}^{*}(t)}}{p_{s}^{*}(t)} \int_{\Omega} \frac{|\varphi|^{p_{s}^{*}(t)}}{|x|^{t}} d x
$$

Hence, $E_{\lambda}(r \varphi) \rightarrow-\infty$ as $r \rightarrow+\infty$ since $q<p<p_{s}^{*}(t)$. So, $E_{\lambda}(r u)<0$ as $r \rightarrow \infty$ for all $u>0$. That is, for $\|u\|<R_{0}$ sufficiently small, we have $c=\inf _{u \in B_{R_{0}}} E_{\lambda}(u)<0$.

Now, by the definition of the infimum, we consider a minimizing sequence $\left\{u_{n}\right\}$ for $c$. Then, using the reflexivity of $X_{0}$, there exists a subsequence, still denoted by $u_{n}$, there exists $u$ such that

$$
\begin{array}{cc}
u_{n} \rightarrow u & \text { weakly in } X_{0} \\
u_{n} \rightarrow u & \text { strongly in } L^{k}\left(\Omega, \frac{d x}{|x|^{t}}\right) \text { for } 1 \leq k<p_{s}^{*}(t) \\
u_{n} \rightarrow u & \text { pointwise a.e. in } \Omega .
\end{array}
$$

Thus, by Brèzis-lieb Lemma [5] we get,

$$
\begin{align*}
\left|u_{n}\right|_{p_{s}^{*}(t)}^{p_{s}^{*}(t)} & =|u|_{p_{s}^{*}(t)}^{p_{s}^{*}(t)}+\left|u_{n}-u\right|_{p_{s}^{*}(t)}^{p_{s}^{*}(t)}+o(1),  \tag{9}\\
\left|u_{n}\right|_{q}^{q} & =|u|_{q}^{q}+\left|u_{n}-u\right|_{q}^{q}+o(1),  \tag{10}\\
\left\|u_{n}\right\|^{p} & =\|u\|^{p}+\| u_{n}-u| |^{p}+o(1) . \tag{11}
\end{align*}
$$

Therefore, using Equations (9), (10) and (11), we conclude that

$$
E_{\lambda}\left(u_{n}\right)=E_{\lambda}(u)+\frac{1}{p}| | u_{n}-u \|^{p}-\frac{1}{q}\left|u_{n}-u\right|^{q}-\frac{1}{p_{s}^{*}(t)}\left|u_{n}-u\right|_{p_{s}^{*}(t)}^{p_{s}^{*}(t)}+o(1) .
$$

Moreover, we observe that from Equations (9), (10) and (11) $u, u_{n}-u \in B_{r}$ for $u$ sufficiently large and

$$
\frac{1}{p}\left|\left|u_{n}-u\right|^{p}-\frac{1}{q}\right| u_{n}-\left.u\right|_{q} ^{q}-\frac{1}{p_{s}^{*}(t)}\left|u_{n}-u\right|_{p_{s}^{*}(t)}^{p_{s}^{*}(t)} \geq o(1) .
$$

Hence, as $n \rightarrow \infty$, we deduce that

$$
E_{\lambda}\left(u_{n}\right) \geq E_{\lambda}\left(u_{\lambda}\right)+o(1) .
$$

Since $c=\inf _{\| u x_{0} \leq r} E_{\lambda}(u)$ we have $u \neq 0$, which is a minimizer of $E_{\lambda}$ over $X_{0}$ for all $\lambda \in\left(0, \lambda_{0}\right)$.

Let us define

$$
\Lambda:=\inf \{\lambda>0:(\mathrm{P}) \text { has no weak solution }\}
$$

We show the following result regarding $\Lambda$.
Lemma 3.3. Assume $1<p<q<p_{s}^{*}$. Then $0<\Lambda<\infty$.
Proof. From Lemma $3.2 u \neq 0$ is a local minimizer of $E_{\lambda}$ over $X_{0}$. So, since $E_{\lambda}(r u)<0$ for $r$ small, we have $c<0$. Hence, there exists $u_{\lambda} \in \mathcal{B}_{r}$ satisfying $E_{\lambda}\left(u_{\lambda}\right)=c$ and

$$
\begin{cases}\left(-\Delta_{p}\right)^{s} u_{\lambda}=\lambda\left|u_{\lambda}\right|^{q-2} u_{\lambda}+\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2}}{|x|^{t}} & \text { in } \Omega,  \tag{12}\\ u_{\lambda}=0 & \text { in } \mathbb{R}^{n} \backslash \Omega .\end{cases}
$$

From the strong maximum principle of the fractional $p$-Laplacian we deduce that $u_{\lambda}>0$ in $\Omega$. Consequently, $u_{\lambda}$ is a weak solution to the problem ( P ). Hence $\Lambda>0$.

Now suppose $\Lambda=\infty$. Then, we know from above that the problem (P) has a solution for all $\lambda$. Choose $\lambda_{*}>0$ such that

$$
\lambda|r|^{q-2} r+\frac{|r|^{p_{s}^{*}(t)-2} r}{|x|^{t}} \geq\left(\lambda_{1}+\epsilon\right) r^{p-1} \text { for all } r>0, \epsilon \in(0,1) \text { and } \lambda>\lambda_{*} .
$$

Clearly $\bar{u}=u_{\lambda}$ is a supersolution of the eigenvalue problem

$$
\begin{equation*}
u \in X_{0} \quad \text { and } \quad(-\Delta)_{p}^{s} u=\left(\lambda_{1}+\epsilon\right)|u|^{p-2} u \text { in } \Omega \tag{13}
\end{equation*}
$$

for all $\epsilon \in(0,1)$. Moreover, we can choose $k$ small enough such that $\underline{u}:=k \phi_{1}$ is a subsolution of the problem (13) ( $\phi_{1}$ is the eigenfunction associated to the eigenvalue $\lambda_{1}$ ). Therefore, using the boundedness of $u_{\lambda}$ (see Theorem 6.3) combine with the boundedness of $\phi_{1}$, we can choose $k$ small enough such that $\underline{u} \leq \bar{u}$.

Now, we define the following monotone iterative scheme:

$$
\left\{\begin{array}{l}
u_{0}=k \phi_{1}, \\
u_{n} \in X_{0} \text { and }(-\Delta)_{p}^{s} u_{n}=\left(\lambda_{1}+\epsilon\right)\left|u_{n}\right|^{p-2} u_{n} \text { in } \Omega .
\end{array}\right.
$$

From the weak maximum principle, it is easy to see that

$$
u_{0} \leq u_{1} \leq \cdots \leq u_{n} \leq u_{n+1} \leq \cdots \leq u_{\lambda} .
$$

Hence, the sequence $\left\{u_{n}\right\}$ is bounded in $X_{0}$ and consequently, has a weakly convergent subsequence $\left\{u_{n}\right\}$ that converges to $u_{0}$. Then, $u_{0}$ is a solution to the problem (13) for any arbitrary $\epsilon \in(0,1)$, contradicting the fact that $\lambda_{1}$ is an isolated and simple point in the spectrum of $(-\Delta)_{p}^{s}$ in $X_{0}$ and by consequence $\Lambda<\infty$. The proof of Lemma 3.3 is now completed.

Combining the result Lemma 3.3 with the previous notations provides the following existence result.

Lemma 3.4. Suppose that $\underline{u}_{\lambda}$ is a weak sub-solution while $\bar{u}_{\lambda}$ is a weak supersolution to problem $(\mathrm{P})$ such that $\underline{u}_{\lambda} \leq \bar{u}_{\lambda}$. Then, there exists a weak solution $u_{\lambda}$ to (P) such that $\underline{u}_{\lambda} \leq u_{\lambda} \leq \bar{u}_{\lambda}$ a.e. in $\Omega$.

Proof. Our proof is inspired from Ghanmi-Saoudi [14]. Consider

$$
M=\left\{u_{\lambda} \in X_{0}: \underline{u}_{\lambda} \leq u_{\lambda} \leq \bar{u}_{\lambda}\right\} .
$$

So, it is simple to see that $M$ is a closed convex set. Moreover, it is clear that $E_{\lambda}$ is weakly lower semicontinuous on $M$. Applying Lemma 5.4, we obtain the existence of $u_{\lambda} \in M$ satisfying

$$
E_{\lambda}\left(u_{\lambda}\right)=\inf _{u_{0} \in M} E_{\lambda}\left(u_{0}\right) .
$$

Now, we proceed to prove that $u_{\lambda}$ is a weak solution to the problem (P). For this, we introduce $\psi \in M$ define by $\psi_{\epsilon}=u_{\lambda}+\epsilon \phi-\phi^{\epsilon}+\phi_{\epsilon} \in M$ where $\phi^{\epsilon}=\left(u_{\lambda}+\epsilon \phi-\bar{u}_{\lambda}\right)^{+} \geq 0$ and $\phi_{\epsilon}=\left(u_{\lambda}+\epsilon \phi-\underline{u}_{\lambda}\right)^{-} \geq 0$ for any $\phi \in X_{0}$ and $\epsilon>0$. Then, again from Lemma 3.2, since $u_{\lambda}$ is a local minimizer of $E_{\lambda}$ on $M$, one has

$$
\left\langle E_{\lambda}^{\prime}\left(u_{\lambda}\right), \psi_{\epsilon}-u_{\lambda}\right\rangle \geq 0
$$

So,

$$
\left\langle E_{\lambda}^{\prime}\left(u_{\lambda}\right), \phi\right\rangle \geq \frac{1}{\epsilon}\left(\left\langle E_{\lambda}^{\prime}\left(u_{\lambda}\right), \phi^{\epsilon}\right\rangle-\left\langle E_{\lambda}^{\prime}\left(u_{\lambda}\right), \phi_{\epsilon}\right\rangle\right)
$$

Which gives,

$$
\begin{align*}
& \int_{\mathcal{Q}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
& -\int_{\Omega}\left(\lambda\left|u_{\lambda}\right|^{q-2} u_{\lambda}+\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}\right) \phi d x \geq \frac{1}{\epsilon}\left(H^{\epsilon}-H_{\epsilon}\right) \tag{14}
\end{align*}
$$

where

$$
\begin{aligned}
H^{\epsilon}= & \int_{\mathcal{Q}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)\left(\phi^{\epsilon}(x)-\phi^{\epsilon}(y)\right)}{|x-y|^{N+s p}} d x d y \\
& -\int_{\Omega}\left(\lambda\left|u_{\lambda}\right|^{q-2} u_{\lambda}+\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}\right) \phi^{\epsilon} d x
\end{aligned}
$$

and

$$
H_{\epsilon}=\int_{\mathcal{Q}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)\left(\phi_{\epsilon}(x)-\phi_{\epsilon}(y)\right)}{|x-y|^{N+s p}} d x d y
$$

$$
-\int_{\Omega}\left(\lambda\left|u_{\lambda}\right|^{q-2} u_{\lambda}+\frac{\left|u_{\lambda}\right|^{p_{s}^{*}}(t)-2}{|x|^{t}}\right) \phi_{\epsilon} d x
$$

Now, let us recall the following inequality

$$
\begin{equation*}
|a-b|^{p} \leq 2^{p-2}\left(|a|^{p-2} a-|b|^{p-2} b\right)(a-b) \quad \text { for } p \geq 2, \quad a, b \in \mathbb{R} . \tag{15}
\end{equation*}
$$

Putting $\Omega^{\epsilon}=\left\{u_{\lambda}+\epsilon \phi \geq \bar{u}_{\lambda}>u_{\lambda}\right\}$ and $\Omega_{\epsilon}=\left\{u_{\lambda}+\epsilon \phi<\underline{u}_{\lambda}\right\}$. So, as $\bar{u}_{\lambda}$ is a super-solution, we obtain

$$
\begin{aligned}
& \left\langle E_{\lambda}^{\prime}\left(u_{\lambda}\right), \phi^{\epsilon}\right\rangle \\
& \geq\left\langle E_{\lambda}^{\prime}\left(u_{\lambda}\right)-E_{\lambda}^{\prime}\left(\bar{u}_{\lambda}\right), \phi^{\epsilon}\right\rangle \\
& =\int_{\mathcal{Q}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)\left(\phi^{\epsilon}(x)-\phi^{\epsilon}(y)\right)}{|x-y|^{N+s p}} d x d y \\
& -\lambda \int_{\Omega}\left(\left|u_{\lambda}\right|^{q-2} u_{\lambda}-\left|\bar{u}_{\lambda}\right|^{q-2} \bar{u}_{\lambda}\right) \phi^{\epsilon} d x \\
& -\int_{\Omega}\left(\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}-\frac{\left|\bar{u}_{\lambda}\right|^{p_{s}^{*}(t)-2} \bar{u}_{\lambda}}{|x|^{t}}\right) \phi^{\epsilon} d x \\
& =\left(\int_{\Omega^{\epsilon} \times \Omega^{\epsilon}}+\int_{\Omega^{\epsilon} \times \Omega_{\epsilon}}+\int_{\Omega_{\epsilon} \times \Omega^{\epsilon}}\right) \\
& \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)\left(\phi^{\epsilon}(x)-\phi^{\epsilon}(y)\right)}{|x-y|^{N+s p}} d x d y \\
& -\lambda \int_{\Omega^{\epsilon}}\left(\left|u_{\lambda}\right|^{q-2} u_{\lambda}-\left|\bar{u}_{\lambda}\right|^{q-2} u_{\lambda}\right) \phi^{\epsilon} d x \\
& -\int_{\Omega^{\epsilon}}\left(\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}-\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}\right) \phi^{\epsilon} d x \\
& =\int_{\Omega^{\epsilon} \times \Omega^{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)\left(\left(u_{\lambda}-\bar{u}_{\lambda}\right)(x)-\left(u_{\lambda}-\bar{u}_{\lambda}\right)(y)\right)}{|x-y|^{N+s p}} d x d y \\
& +\epsilon \int_{\Omega^{\epsilon} \times \Omega^{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
& +\int_{\Omega^{\epsilon} \times \Omega_{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)\left(u_{\lambda}-\bar{u}_{\lambda}\right)(x)}{|x-y|^{N+s p}} d x d y \\
& +\epsilon \int_{\Omega^{\epsilon} \times \Omega_{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)}{|x-y|^{N+s p}} \phi(x) d x d y \\
& -\int_{\Omega_{\epsilon} \times \Omega^{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)\left(u_{\lambda}-\bar{u}_{\lambda}\right)(y)}{|x-y|^{N+s p}} d x d y \\
& -\epsilon \int_{\Omega^{\epsilon} \times \Omega_{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)}{|x-y|^{N+s p}} \phi(y) d x d y \\
& -\lambda \int_{\Omega^{\epsilon}}\left(\left|u_{\lambda}\right|^{q-2} u_{\lambda}-\left|\bar{u}_{\lambda}\right|^{q-2} \bar{u}_{\lambda}\right)\left(u_{\lambda}-\bar{u}_{\lambda}\right) d x
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\Omega^{\epsilon}}\left(\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}-\frac{\left|\bar{u}_{\lambda}\right|^{p_{s}^{*}}(t)-2 \bar{u}_{\lambda}}{|x|^{t}}\right)\left(u_{\lambda}-\bar{u}_{\lambda}\right) d x \\
&-\epsilon \lambda \int_{\Omega^{\epsilon}}\left(\left|u_{\lambda}\right|^{q-2} u_{\lambda}-\left|\bar{u}_{\lambda}\right|^{q-2} \bar{u}_{\lambda}\right) \phi d x \\
&-\epsilon \int_{\Omega^{\epsilon}}\left(\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}-\frac{\left|\bar{u}_{\lambda}\right|^{p_{s}^{*}(t)-2} \bar{u}_{\lambda}}{|x|^{t}}\right) \phi d x \\
& \geq \frac{3}{2^{p-2}} \int_{\Omega^{\epsilon} \times \Omega^{\epsilon}} \frac{\left|\left(u_{\lambda}-\bar{u}_{\lambda}\right)(x)-\left(u_{\lambda}-\bar{u}_{\lambda}\right)(y)\right|^{p}}{|x-y|^{N+s p}} d x d y \\
&+\epsilon \int_{\Omega^{\epsilon} \times \Omega^{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
&-\lambda \int_{\Omega^{\epsilon}}\left(\left|u_{\lambda}\right|^{q}-\left|\bar{u}_{\lambda}\right|^{q}\right) d x-\int_{\Omega^{\epsilon}}\left(\frac{\left|u_{\lambda}\right|^{p}(t)}{|x|^{t}}-\frac{\left|\bar{u}_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}}\right) d x \\
&-\epsilon \lambda \int_{\Omega^{\epsilon}}\left(\left|u_{\lambda}\right|^{q-2} u-\left|\bar{u}_{\lambda}\right|^{q-2} \bar{u}_{\lambda}\right) \phi \\
&-\epsilon \int_{\Omega^{\epsilon}}\left(\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}-\frac{\left|\bar{u}_{\lambda}\right|^{p_{s}^{*}(t)-2} \bar{u}_{\lambda}}{|x|^{t}}\right) \phi d x \\
& \geq \epsilon \int_{\Omega^{\epsilon} \times \Omega^{\epsilon}} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
&-\epsilon \lambda \int_{\Omega^{\epsilon}}\left(\left|u_{\lambda}\right|^{q-2} u-\left|\bar{u}_{\lambda}\right|^{q-2} \bar{u}_{\lambda}\right) \phi d x \\
&-\epsilon \int_{\Omega^{\epsilon}}\left(\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{\left.|x|^{t}-\frac{\left|\bar{u}_{\lambda}\right|^{p_{s}^{*}(t)-2} \bar{u}_{\lambda}}{|x|^{t}}\right) \phi d x}\right. \\
&
\end{aligned}
$$

follow from the inequality (15). Now, since the measure of the domain of integration $\Omega^{\epsilon}$ go to zero as $\epsilon \rightarrow 0$. So $\frac{1}{\epsilon} H^{\epsilon} \geq o(1)$.

In the same way, we prove that $\frac{1}{\epsilon} H^{\epsilon} \leq o(1)$. Therefore, using (14) and letting $\epsilon \rightarrow 0$, we obtain

$$
\begin{aligned}
& \int_{Q} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p-2}\left(u_{\lambda}(x)-u_{\lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
& -\int_{\Omega}\left(\lambda\left|u_{\lambda}\right|^{q-2} u_{\lambda}+\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda}}{|x|^{t}}\right) \phi d x \geq o(1)
\end{aligned}
$$

since $\phi$ is an arbitrary test function, we obtain the equality if we change $\phi$ by $-\phi$. Hence, $u_{\lambda}$ is a weak solution to the problem (P).

As a consequence of Lemma 3.4 we obtain the following crucial result.
Lemma 3.5. For all $\lambda \in(0, \Lambda]$, the problem $(\mathrm{P})$ has a weak solution $u_{\lambda} \in X_{0}$. Moreover, $u_{\lambda}$ is a local minimum for $\left.E_{\lambda}\right|_{C^{1}(\bar{\Omega})}$.

Proof. From the definition of $\Lambda$ there exists $\mu \in(0, \Lambda)$ such that the problem (P) has a solution by Lemma 3.3, say $w_{\mu}$. So, $\bar{u}_{\lambda}=w_{\mu}$ becomes a supersolution to the problem ( P ) with $\lambda<\mu$. Consider the eigenvalue problem as follows:

$$
\begin{cases}\left(-\Delta_{p}\right)^{s} \phi_{1}=\lambda_{1}\left|\phi_{1}\right|^{p-2} \phi_{1} & \text { in } \Omega, \\ \phi_{1}>0 & \text { in } \Omega, \\ \phi_{1}=0 & \text { in } \mathbb{R}^{N} \backslash \Omega,\end{cases}
$$

where $\lambda_{1}$ is the smallest eigenvalue and $\phi_{1}$ is the corresponding eigenfunction.
Now, let $\epsilon>0$, satisfying $\epsilon \phi_{1} \leq \bar{u}$ and $\epsilon^{p-q} \phi_{1}^{p-q} \leq \frac{\lambda}{\lambda_{1}}$.
Putting, $\underline{u}_{\lambda}=\epsilon \phi_{1}$ ones has

$$
\begin{aligned}
\left(-\Delta_{p}\right)^{s} \underline{u}_{\lambda} & =\lambda_{1} \epsilon^{p-1} \phi_{1}^{p-1} \\
& \leq \lambda \epsilon^{q-1}\left|\phi_{1}\right|^{q-2} \phi_{1}+\frac{\epsilon^{p_{s}^{*}(t)-1}\left|\phi_{1}\right|^{p_{s}^{*}(t)-2} \phi_{1}}{|x|^{t}} \\
& =\lambda\left|\underline{u}_{\lambda}\right|^{q-2} \underline{u}_{\lambda}+\frac{\left|\underline{u}_{\lambda}\right|^{p_{s}^{*}(t)-2} \underline{u}_{\lambda}}{|x|^{t}} .
\end{aligned}
$$

So, $\underline{u}_{\lambda}$ is a sub-solution to the problem (P). Hence, applying the weak maximum principle, we get $\underline{u}_{\lambda} \leq \bar{u}_{\lambda}$. Consequently, from Lemma 3.4, we obtain that problem (P) has a solution $u_{\lambda}$ for all $\lambda \in(0, \Lambda)$ such that $\underline{u}_{\lambda} \leq u_{\lambda} \leq \bar{u}_{\lambda}$. Hence, by the strong maximum principle it follows that $\underline{u}_{\lambda}<u_{\lambda}<\bar{u}_{\lambda}$ and by the regularity results $u_{\lambda} \in C^{1, \beta}(\bar{\Omega})$ for some $\beta \in(0,1)$. Therefore, we can find $\delta>0$ small enough such that if $u \in C=\left\{u \in C^{1}(\bar{\Omega}) \backslash\left\|u-u_{\lambda}\right\|_{C^{1}(\bar{\Omega})}<1\right\}$. Thus, $\underline{u}_{\lambda}<u<\bar{u}_{\lambda}$ in $\Omega$. Further, $u_{\lambda}$ is a local minimum of $E_{\lambda}$ this completes the proof of Lemma 3.5.

Now, we show the following result.
Lemma 3.6. Problem (P) has at least one solution if $\lambda=\Lambda$.
Proof. Consider an increasing sequence $\left\{\lambda_{k}\right\}_{k \in \mathbb{N}}$ such that $\lambda_{k}$ converge to $\Lambda$ as $k \rightarrow \infty$. So, from Lemma $3.5 u_{k}=u_{\lambda_{k}}$ be a weak solution to the problem (P) for $\lambda=\lambda_{k}$. Therefore,

$$
\begin{aligned}
& \int_{Q} \frac{\left|u_{k}(x)-u_{k}(y)\right|^{p-2}\left(u_{k}(x)-u_{k}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
= & \lambda_{k} \int_{\Omega}\left|u_{k}\right|^{q-2} u_{k} \phi d x+\int_{\Omega} \frac{\left|u_{k}\right|^{p_{s}^{*}(t)-2} u_{k}}{|x|^{t}} \phi d x .
\end{aligned}
$$

Then, take $\phi=u_{k}$ in (16), we obtain

$$
\begin{equation*}
\int_{Q} \frac{\left|u_{k}(x)-u_{k}(y)\right|^{p}}{|x-y|^{N+s p}} d x d y-\lambda_{k} \int_{\Omega}\left|u_{k}\right|^{q} d x-\int_{\Omega} \frac{\left|u_{k}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x=0 . \tag{17}
\end{equation*}
$$

Again, from Lemma 3.5

$$
\begin{align*}
E_{\lambda_{k}}\left(u_{k}\right)= & \frac{1}{p} \int_{Q} \frac{\left|u_{k}(x)-u_{k}(y)\right|^{p}}{|x-y|^{N+s p}} d x d y  \tag{18}\\
& -\frac{\lambda_{k}}{q} \int_{\Omega}\left|u_{k}\right|^{q} d x-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|u_{k}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x \leq A,
\end{align*}
$$

where $A$ is a positive constant. Now, injecting (17) in (18), we get

$$
\begin{align*}
& \frac{1}{p}\left(\lambda_{k} \int_{\Omega}\left|u_{k}\right|^{q} d x+\int_{\Omega} \frac{\left|u_{k}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x\right)  \tag{19}\\
& -\frac{\lambda_{k}}{q} \int_{\Omega}\left|u_{k}\right|^{q} d x-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|u_{k}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x \leq A
\end{align*}
$$

Which implies that,

$$
\begin{equation*}
\left(\frac{1}{p}-\frac{1}{p_{s}^{*}(t)}\right) \int_{\Omega} \frac{\left|u_{k}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x \leq A+\lambda_{k}\left(\frac{1}{q}-\frac{1}{p}\right) \int_{\Omega}\left|u_{k}\right|^{q} d x \tag{20}
\end{equation*}
$$

Again, injecting (20) in (17), it follows that,

$$
\begin{equation*}
\left\|u_{k}\right\|_{X_{0}}^{p-q} \leq A_{1}+\frac{A_{2}}{\left\|u_{k}\right\|_{X_{0}}^{q}} \tag{21}
\end{equation*}
$$

where $A_{1}$ and $A_{2}$ are two positive constants.
Therefore, from Equation (21), we can easily obtain that $\sup _{k \in \mathbb{N}}\left\|u_{\lambda_{k}}\right\|_{X_{0}}<\infty$. So, since the space $X_{0}$ is reflexive, we obtain the existence of a sub sequence which still denoted by $\left\{u_{k}\right\}$, satisfying $u_{k} \rightharpoonup u_{\Lambda}$ in $X_{0}$ as $k \rightarrow \infty$. Taking the limit in (16) as $k \rightarrow \infty$, we obtain that:

$$
\begin{aligned}
& \int_{Q} \frac{\left|u_{\Lambda}(x)-u_{\Lambda}(y)\right|^{p-2}\left(u_{\Lambda}(x)-u_{\Lambda}(y)\right)(\phi(x)-\phi(y))}{|x-y|^{N+s p}} d x d y \\
= & \Lambda \int_{\Omega}\left|u_{\Lambda}\right|^{q-2} u_{\Lambda} \phi d x+\int_{\Omega} \frac{\left|u_{\Lambda}\right|^{p_{s}^{*}(t)-2} u_{\Lambda}}{|x|^{t}} \phi d x .
\end{aligned}
$$

Therefore, $u_{\Lambda}$ is a weak solution to the problem (P). This completes the proof of Lemma 3.6.

We next prove that problem $(\mathrm{P})$ possesses a minimal solution.
Corollary 3.7. For all $\lambda \in(0, \Lambda]$ problem $(\mathrm{P})$ has a minimal solution $u_{\lambda}$ in $X_{0}$.

Proof. From Lemma 3.5 we know that problem (P) has a solution for all $\lambda \in$ $(0, \Lambda)$. Now, we define a sequence $\left\{u_{n}\right\}$ by the following monotone iterative
scheme:

$$
\begin{cases}u_{0}=\underline{u}_{\lambda}, & \\ (-\Delta)_{p}^{s} u_{n+1}=\lambda\left|u_{n}\right|^{\mid-2} u_{n}+\frac{\left|u_{n}\right|^{p_{s}^{*}(t)-2} u_{n}}{|x|^{t}} & \text { in } \Omega, \\ u_{n}>0 & \text { in } \Omega, \\ u_{n}=0 & \text { in } \mathbb{R}^{N} \backslash \Omega,\end{cases}
$$

for each $n \in \mathbb{N}$ and $u_{0}$ is a weak sub-solution to the problem (P). Now, by the choice of $u_{0}$ we have $u_{0} \leq u_{\lambda}$ where $u_{\lambda}$ is any solution to the problem (P), whose existence follow from Lemma 3.5 for all $\lambda \in(0, \Lambda)$ and from Lemma 3.6 for $\lambda=\Lambda$. From the weak maximum principle, it is easy to see that

$$
u_{0} \leq u_{1} \leq \cdots \leq u_{n} \leq u_{n+1} \leq \cdots \leq u_{\lambda}
$$

Moreover, from Theorem 6.3, we know that $u_{\lambda}$ is in $L^{\infty}(\Omega)$, which gives that $\left\{u_{n}\right\}$ is uniformly bounded in $X_{0}$. Again, as in Lemma 3.5 it is simple to show that $\left\{u_{n}\right\}$ converges $\hat{u}_{\lambda}$ to be a weak solution to the problem (P). Now, to prove that $\hat{u}_{\lambda}$ is the minimal solution, we let that $w_{\lambda}$ to be a weak solution such that $u_{0}=\underline{u}_{\lambda} \leq w_{\lambda}$. Using again the weak maximum principle we have $u_{n} \leq w_{\lambda}$ for each $n \in \mathbb{N}$. Passing to limit, we get $\hat{u}_{\lambda} \leq w_{\lambda}$ and that $\hat{u}_{\lambda}$ is the smallest solution. The proof of Corollary 3.7 is now completed.

## 4. $C^{1}$ versus $W^{s, p}$ local minimizers of the functional energy

This section is devoted to prove a crucial lemma in showing multiplicity of solutions. It has been shown in the case $p=2$ in [6] for the case of critical growth functionals $E_{\lambda}: H_{0}^{1}(\Omega) \rightarrow \mathbb{R}, \Omega \subset \mathbb{R}^{N}, N \geq 3$, and later for critical growth functionals $E_{\lambda}: W_{0}^{1, p}(\Omega) \rightarrow \mathbb{R}, 1<p<N, \Omega \subset \mathbb{R}^{N}, N \geq 3$ in [13]. A key feature of these latter works is the uniform $C^{1, \alpha}$ estimate they obtain for equations like $\left(\mathrm{P}_{\epsilon}\right)$ but involving two $p$-Laplace operators. Using constraints based on $L^{p}$-norms rather than Sobolev norms as in [13], the equations for which uniform estimates are required can be simplified to a standard type involving only one $p$-Laplace operator. This approach was followed in [7] in the subcritical case, in [11] in the critical case adopted in [18, 28] and also adopted in this work to deal with the nonlocal elliptic equation with critical Hardy-Sobolev exponents. More precisely, we have the following result:

Theorem 4.1. Let $u \in X_{0}$ be a local minimizer of $E_{\lambda}$ in $C^{1}$-topology; that is,

$$
\exists r_{1}>0 \text { such that } u \in C^{1}(\Omega),\left\|u-u_{0}\right\|_{C^{1}(\Omega)}<r_{1} \Rightarrow E_{\lambda}\left(u_{0}\right) \leq E_{\lambda}(u) .
$$

Then, also $u_{0}$ is a local minimum of $E_{\lambda}$ in $X_{0}$ topology; that is,

$$
\exists r_{2}>0 \text { such that } u \in X_{0},\left\|u-u_{0}\right\|_{X_{0}}<r_{2} \Rightarrow E_{\lambda}\left(u_{0}\right) \leq E_{\lambda}(u) .
$$

Proof. Firstly, using the smoothness of $\Omega$, we have from Theorem 6.4, that $u \in C^{1, \beta}(\bar{\Omega})$ for some $\beta \in(0,1)$. Define

$$
\begin{equation*}
\chi(w)=\frac{1}{p_{s}^{*}} \int_{\Omega}\left|w-u_{0}\right|^{p_{s}^{*}} d x, w \in X_{0} \tag{22}
\end{equation*}
$$

and

$$
C_{\epsilon}=\left\{u \in X_{0}: \chi(u) \leq \epsilon\right\} .
$$

Now, we proceed by contradiction, i.e., suppose that the conclusion of the Theorem 4.1 does not holds. $u$ is not a local minimizer. Then, as in [6], we first make a truncation argument to get the weak lower semi-continuity property of the energy functional. Further, consider the truncated functional

$$
E_{\lambda, j}(u)=\frac{1}{p}\|u\|_{X}^{p}-\frac{1}{q} \int_{\Omega}|u|^{q} d x-\int_{\Omega} \frac{T_{j}(u)^{p_{s}^{*}(t)-2} T_{j}(u)}{p_{s}^{*}(t)|x|^{t}} d x, \forall u \in X_{0},
$$

where

$$
T_{j}(w)= \begin{cases}-j & w \leq-j \\ w & -j \leq w \leq j \\ j & w \geq j\end{cases}
$$

By the 'Lebesgue theorem' we have, for any $u \in X_{0}, E_{\lambda, j}(u) \rightarrow E_{\lambda}(u)$ as $j \rightarrow \infty$. It follows, from the truncation and this convergence that for each $\epsilon>0$, there is some $j_{\epsilon}$ (with $j_{\epsilon} \rightarrow \infty$ as $\epsilon \rightarrow 0^{+}$) such that $E_{\lambda, j_{\epsilon}}\left(u_{\epsilon}\right) \leq E_{\lambda}\left(u_{\epsilon}\right) \leq E_{\lambda}\left(u_{0}\right)$.

On the other hand, since $C_{\epsilon}$ is closed, convex and since $E_{\lambda, j_{\epsilon}}$ is weakly lower semicontinuous we deduce that $E_{\lambda, j_{\epsilon}}$ achieves its infimum at some $u_{\epsilon} \in C_{\epsilon}$. Therefore, for $\epsilon>0$ small enough, we have

$$
E_{\lambda, j_{\epsilon}}\left(u_{\epsilon}\right) \leq E_{\lambda}\left(u_{\epsilon}\right)<E_{\lambda}\left(u_{0}\right)
$$

By Lagrange multiplier, there exists $\mu_{\epsilon}$ such that $E_{\lambda, j_{\epsilon}}^{\prime}\left(u_{\epsilon}\right)=\mu_{\epsilon} \chi^{\prime}\left(u_{\epsilon}\right)$. We will first show that $\mu_{\epsilon} \leq 0$. Suppose $\mu_{\epsilon}>0$, then $\exists \phi \in X_{0}$ such that

$$
\left\langle E_{\lambda, j}^{\prime}\left(u_{\epsilon}\right), \phi\right\rangle<0 \text { and }\left\langle\chi^{\prime}\left(u_{\epsilon}\right), \phi\right\rangle<0 .
$$

Then for small $\delta>0$ we have

$$
\begin{aligned}
E_{\lambda, j}\left(u_{\epsilon}+\delta \phi\right) & <E_{\lambda, j}\left(u_{\epsilon}\right), \\
\chi\left(u_{\epsilon}+\delta \phi\right) & <\chi\left(u_{\epsilon}\right)=\epsilon
\end{aligned}
$$

which is a contradiction to $u_{\epsilon}$ being a minimizer of $E_{\lambda, j}$ in $C_{\epsilon}$ it follows that $\mu_{\epsilon} \leq 0$.

By the construction we have $u_{\epsilon} \rightarrow u_{0}$ in $L^{p_{s}^{*}}(\Omega)$ as $\epsilon>0$ and we deduce the boundedness of $u_{\epsilon}$ in $X_{0}$.
Claim: $\left\{u_{\epsilon}\right\}$ is bounded in $L^{\infty}(\Omega)$ as $\epsilon \rightarrow 0$.
Case $i: \inf _{0<\epsilon<1}\left\{\mu_{\epsilon}\right\}>-\infty\left(\mu_{\epsilon} \in(-l, 0)\right.$ where $\left.l>-\infty\right)$.
Look at

$$
\left(P_{\epsilon}\right):\left(-\Delta_{p}\right)^{s} u=|u|^{q-2} u+\frac{\left|T_{j}(u)\right|^{p_{s}^{*}(t)-2} T_{j}(u)}{|x|^{t}}+\mu_{\epsilon}\left|u-u_{0}\right|^{p_{s}^{*}-2}\left(u-u_{0}\right),
$$

which is satisfied weakly by $u_{\epsilon}$. Further, since $-l \leq \mu_{\epsilon} \leq 0$, there exist $M, c$ such that

$$
\left(-\Delta_{p}\right)^{s}\left(u_{\epsilon}-1\right)^{+} \leq M+c\left|\left(u_{\epsilon}-1\right)^{+}\right|^{p_{s}^{*}-2}\left(u_{\epsilon}-1\right)^{+} .
$$

By the Moser iteration method we get $\left\{u_{\epsilon}\right\}$ is bounded in $L^{\infty}(\Omega)$. By the compact embedding $\left.C^{1, \beta}(\bar{\Omega}) \hookrightarrow C^{1, \kappa} \bar{\Omega}\right)$ for any $\kappa<\beta$, we have $u_{\epsilon} \rightarrow u_{0}$ which contradicts the assumption made.
Case ii: $\inf _{0<\epsilon<1}\left\{\mu_{\epsilon}\right\}=-\infty$.
Let us assume $\mu_{\epsilon} \leq-1$. In this case, there exist $M>0$, independent of $\epsilon$, and $\epsilon_{0}>0$ such that for $0<\epsilon<\epsilon_{0}$
$|w|^{q-2} w+\frac{\left|T_{j}(w)\right|^{p_{s}^{*}(t)-2} T_{j}(w)}{|x|^{t}}+\mu_{\epsilon}\left|w-u_{0}(x)\right|^{p_{s}^{*}-2}\left(w-u_{0}(x)\right)<0$, if $w>M$.
Then from the weak comparison principle on $\left(-\Delta_{p}\right)^{s}$, we get $u_{\epsilon} \leq M$ for $\epsilon>0$ sufficiently small. Since $u_{0}$ is a local $C^{1}$-minimizer, $u_{0}$ is a weak solution to (P) and hence

$$
\begin{equation*}
\left\langle\left(-\Delta_{p}\right)^{s} u_{0}, \phi\right\rangle=\int_{\Omega}\left|u_{0}\right|^{q-2} u_{0} \phi d x+\int_{\Omega} \frac{\left|T_{j}\left(u_{0}\right)\right|^{p_{s}^{*}(t)-2} T_{j}\left(u_{0}\right)}{|x|^{t}} \phi d x \tag{23}
\end{equation*}
$$

$\forall \phi \in C_{c}^{\infty}(\Omega)$. In fact, we have for every function $w \in X_{0}, u_{0}$ satisfies

$$
\begin{equation*}
\left\langle\left(-\Delta_{p}\right)^{s} u_{0}, w\right\rangle=\int_{\Omega}\left|u_{0}\right|^{q-2} u_{0} w d x+\int_{\Omega} \frac{\left|T_{j}\left(u_{0}\right)\right|^{p_{s}^{*}(t)-2} T_{j}\left(u_{0}\right)}{|x|^{t}} w d x . \tag{24}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\left\langle\left(-\Delta_{p}\right)^{s} u_{\epsilon}, w\right\rangle=\int_{\Omega}\left|u_{\epsilon}\right|^{q-2} u_{\epsilon} w d x+\int_{\Omega} \frac{\left|T_{j}\left(u_{\epsilon}\right)\right|^{p_{s}^{*}(t)-2} T_{j}\left(u_{\epsilon}\right)}{|x|^{t}} w d x \tag{25}
\end{equation*}
$$

On subtracting Eq. (24) from Eq. (25) and testing with $w=\left|u_{\epsilon}-u_{0}\right|^{\beta-1}\left(u_{\epsilon}-\right.$ $u_{0}$ ), where $\beta \geq 1$, we obtain

$$
\begin{align*}
0 \leq & \left.\beta\left\langle\left(-\Delta_{p}\right)^{s} u_{\epsilon}-\left(-\Delta_{p}\right)^{s} u_{0},\right| u_{\epsilon}-\left.u_{0}\right|^{\beta-1}\left(u_{\epsilon}-u_{0}\right)\right\rangle \\
& -\int_{\Omega}\left(\left|u_{\epsilon}\right|^{q-2} u_{\epsilon}-\left|u_{0}\right|^{q-2} u_{0}\right)\left|u_{\epsilon}-u_{0}\right|^{\beta-1}\left(u_{\epsilon}-u_{0}\right) d x \\
= & \int_{\Omega}\left(\frac{\left|T_{j}\left(u_{\epsilon}\right)\right|^{p_{s}^{*}(t)-2} T_{j}\left(u_{\epsilon}\right)}{|x|^{t}}-\frac{\left|T_{j}\left(u_{0}\right)\right|^{p_{s}^{*}(t)-2} T_{j}\left(u_{0}\right)}{|x|^{t}}\right) \\
& \left|u_{\epsilon}-u_{0}\right|^{\beta-1}\left(u_{\epsilon}-u_{0}\right) d x+\mu_{\epsilon} \int_{\Omega}\left|u_{\epsilon}-u_{0}\right|^{p_{s}^{*}+\beta-1} d x . \tag{26}
\end{align*}
$$

By the Hölder's inequality and the bounds of $u_{\epsilon}, u_{0}$ we obtain

$$
\begin{equation*}
-\mu_{\epsilon}\left\|u_{\epsilon}-u_{0}\right\|_{p_{s}^{*}+\beta-1}^{p_{s}^{*}-1} \leq C|\Omega|^{\frac{p_{s}^{*}-1}{p_{s}^{*}+\beta-1}} . \tag{27}
\end{equation*}
$$

Here $C$ is independent of $\epsilon$ and $\beta$. On passing the limit $\beta \rightarrow \infty$ we get $-\mu_{\epsilon}\left\|u_{\epsilon}-u_{0}\right\|_{\infty}^{p_{s}^{*}-1} \leq C$. Working on similar lines we end up getting $u_{\epsilon}$ is bounded in $C^{1, \beta}(\bar{\Omega})$ independent of $\epsilon$ and the conclusion follows. This marks an end to the prove of the claim of Case ii. The proof of Theorem 4.1 is now completed.

## 5. Multiplicity of weak solutions

This section is devoted to show the existence of a second critical point $v_{\lambda}$ different from the critical point $u_{\lambda}$ of the functional $E_{\lambda}$ energy obtained in Section 3. The critical point $v_{\lambda}$ of $E_{\lambda}$ is also a point where the Gâteaux derivative of the functional $E_{\lambda}$ vanishes. Therefore, $v_{\lambda}$ will solve the problem (P). We will prove $v_{\lambda} \neq u_{\lambda}$. First, we now introduce a generalized notion of Palais Smale sequence for $E_{\lambda}$.
Definition 5. Let $F \subset \Omega$, be closed and $c \in \mathbb{R}$. Then a sequence $\left\{v_{n}\right\} \subset X_{0}$ is said be a Palais Smale sequence [in short $(P S)_{F, c}$ ] for the functional energy $E_{\lambda}$ around $F$ at the level $c$, if

$$
\lim _{n \rightarrow \infty} \operatorname{dist}\left(x_{n}, F\right)=0, \lim _{n \rightarrow \infty} E_{\lambda}\left(x_{n}\right)=c \& \lim _{n \rightarrow \infty}\left\|E_{\lambda}^{\prime}\left(x_{n}\right)\right\|=0 .
$$

Now, we start by proving the compactness property for the functional energy $E_{\lambda}$.

Lemma 5.1. Let $F \subset \Omega$ be a closed and $c \in \mathbb{R}$. Let $\left\{v_{n}\right\} \subset X_{0}$ be a $(P S)_{F, c}$ sequence for the functional energy $E_{\lambda}$. Then, $E_{\lambda}$ satisfies the $(P S)_{c}$ for all

$$
c<\left(\frac{1}{p}-\frac{1}{p_{s}^{*}(t)}\right) S_{t}^{\frac{N-t}{p s-t}} .
$$

Proof. Since $\left\{v_{n}\right\}$ is a $(P S)_{F, c}$ sequence for the functional energy $E_{\lambda}$, so from Definition 5, we have

$$
\begin{equation*}
E_{\lambda}\left(v_{n}\right)=c+o_{n}(1), \quad\left\langle E_{\lambda}^{\prime}\left(v_{n}\right), v_{n}\right\rangle \leq c\left\|v_{n}\right\|_{X_{0}} \tag{28}
\end{equation*}
$$

That is,

$$
\begin{align*}
& \frac{1}{p}\left\|v_{n}\right\|^{p}-\int_{\Omega}\left(\frac{\lambda\left|v_{n}\right|^{q}}{q}+\frac{\left|v_{n}\right|^{p_{s}^{*}(t)}}{p_{s}^{*}(t)|x|^{t}}\right) \mathrm{d} x=E_{\lambda}\left(v_{n}\right)=c+o_{n}(1)  \tag{29}\\
& \left\|v_{n}\right\|^{p}-\int_{\Omega}\left(\lambda\left|v_{n}\right|^{q}+\frac{\left|v_{n}\right|^{p_{s}^{*}(t)}}{|x|^{t}}\right) \mathrm{d} x=\left\langle E_{\lambda}^{\prime}\left(v_{n}\right), v_{n}\right\rangle=o_{n}(1)\left\|v_{n}\right\| \tag{30}
\end{align*}
$$

as $n \rightarrow \infty$. Therefore, it follows that

$$
\begin{align*}
c+o_{n}(1)\left\|v_{n}\right\| \geq & p E_{\lambda}\left(v_{n}\right)-\left\langle E_{\lambda}^{\prime}\left(v_{n}\right), v_{n}\right\rangle=\lambda\left(1-\frac{p}{q}\right) \int_{\Omega}\left|v_{n}\right|^{q} \mathrm{~d} x \\
& +\left(1-\frac{p}{p_{s}^{*}(t)}\right) \int_{\Omega}\left|v_{n}\right|^{p_{s}^{*}(t)} \mathrm{d} x \tag{31}
\end{align*}
$$

Now, since $p_{s}^{*}(t)$ is greater than $p$, then (31) implies

$$
\begin{equation*}
\int_{\Omega}\left|v_{n}\right|^{p_{s}^{*}(t)} \mathrm{d} x \leq C\left(1+\left\|v_{n}\right\|\right) \tag{32}
\end{equation*}
$$

for some constant $C>0$. Moreover, using the Hölder inequality and Eq. (31), we obtain

$$
\int_{\Omega}\left|v_{n}\right|^{q} \mathrm{~d} x=\int_{\Omega} \frac{\left|v_{n}\right|^{q}}{|x|^{\frac{q t}{p_{S}^{\xi}(t)}}}|x|^{\frac{q t}{p_{s}^{*}(t)}} \mathrm{d} x
$$

$$
\begin{align*}
& \leq C_{1}\left(\int_{\Omega} \frac{\left|v_{n}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x\right)^{\frac{q}{p_{s}^{(t)}}} \\
& \leq C_{1}\left(1+\left\|v_{n}\right\|^{\frac{q}{p_{s}^{(t)}}}\right) \tag{33}
\end{align*}
$$

for some constant $C_{1}>0$. So Eqs. (29)-(30) combine with Eqs. (32)-(33) gives

$$
\begin{equation*}
\left\|v_{n}\right\|^{p} \leq C_{2}\left(1+\left\|v_{n}\right\|\right) \tag{34}
\end{equation*}
$$

for some constant $C_{2}>0$. Which implies the boundedness of $\left\{v_{n}\right\}_{n}$ in $X_{0}$. Now, since the space $X_{0}$ is a reflexive space, there exists $v_{\lambda} \in X_{0}$ such that $v_{n} \rightharpoonup v_{\lambda}$ in $X_{0}$, strongly in $L^{k}(\Omega)$ for all $k \in\left[1, p_{s}^{*}\right]$, and a.e. in $\Omega$ (see [25], Corollary 7.2). Let $p^{\prime}$ the Hölder conjugate of $p$ given by $p^{\prime}=\frac{p}{p-1}$, then $\left|v_{n}(x)-v_{n}(y)\right|^{p-2}\left(v_{n}(x)-v_{n}(y)\right) /|x-y|^{(N+s p) / p^{\prime}}$ is bounded in $L^{p^{\prime}}\left(\mathbb{R}^{2 N}\right)$ and converges to $\left|v_{\lambda}(x)-v_{\lambda}(y)\right|^{p-2}\left(v_{\lambda}(x)-v_{\lambda}(y)\right) /|x-y|^{(N+s p) / p^{\prime}}$ in $\mathbb{R}^{2 N}$ and $\left(v_{\lambda}(x)-v_{\lambda}(y)\right) /|x-y|^{(N+s p) / p} \in L^{p}\left(\mathbb{R}^{2 N}\right)$, so

$$
\int_{Q} \frac{\left|v_{n}(x)-v_{n}(y)\right|^{p-2}\left(v_{n}(x)-v_{n}(y)\right)(\Phi(x)-\Phi(y))}{|x-y|^{N+s p}} d x d y
$$

$\rightarrow \int_{Q} \frac{\left|v_{\lambda}(x)-v_{\lambda}(y)\right|^{p-2}\left(v_{\lambda}(x)-v_{\lambda}(y)\right)(\Phi(x)-\Phi(y))}{|x-y|^{N+s p}} d x d y$ for any $\Phi \in X_{0}$.
Moreover,

$$
\int_{\Omega}\left|v_{n}\right|^{q-2} v_{n} \Phi \mathrm{~d} x \rightarrow \int_{\Omega}\left|v_{\lambda}\right|^{q-2} v_{\lambda} \Phi \mathrm{d} x
$$

and

$$
\int_{\Omega} \frac{\left|v_{n}\right|^{p_{s}^{*}(t)-2}}{|x|^{t}} v_{n} \Phi \mathrm{~d} x \rightarrow \int_{\Omega} \frac{\left|v_{\lambda}\right|^{p_{s}^{*}(t)-2}}{|x|^{t}} v_{\lambda} \Phi \mathrm{d} x
$$

since $\frac{\left|v_{n}\right|^{p_{s}^{*}(t)-2} v_{n}}{|x|^{p_{s}^{*}(t)}}$ is bounded in $\left(L^{p_{s}^{*}(t)}\right)^{\prime}$ and converges to $\frac{\left|v_{\lambda}\right|^{p_{s}^{*}(t)-2} v_{\lambda}}{|x|^{p_{s}^{*}(t)^{\prime}}}$ a.e. in $\Omega$, and $\frac{v_{n}}{|x|^{p_{s}^{t}(t)}} \in L^{p_{s}^{*}(t)}$. Hence, taking the limit when $n \rightarrow \infty$ in (30) and applying the embedding result in Lemma 2.4 we conclude $E_{\lambda}^{\prime}\left(v_{\lambda}\right)=0$. Consequently, since $\left\langle E_{\lambda}^{\prime}\left(v_{\lambda}\right), v_{\lambda}\right\rangle=0$, it follows that

$$
\begin{equation*}
E_{\lambda}\left(v_{\lambda}\right)=\lambda\left(\frac{1}{p}-\frac{1}{q}\right) \int_{\Omega}\left|v_{\lambda}\right|^{q} \mathrm{~d} x+\left(\frac{1}{p}-\frac{1}{p_{s}^{*}(t)}\right) \int_{\Omega} \frac{\left|v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \geq 0 \tag{35}
\end{equation*}
$$

Next, we prove that $v_{n} \rightarrow v_{\lambda}$ strongly in $X_{0}$. Indeed, combine Lemma in [9] with the boundedness of $\left\{v_{n}\right\}_{n}$, we get

$$
\begin{equation*}
E_{\lambda}\left(v_{n}\right)=E_{\lambda}\left(v_{\lambda}\right)+\frac{1}{p} \| v_{n}-\left.v_{\lambda}\right|^{p}-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|v_{n}-v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x+o_{n}(1), \tag{36}
\end{equation*}
$$

and

$$
\begin{align*}
o_{n}(1) & =\left\langle E_{\lambda}^{\prime}\left(v_{n}\right), v_{n}\right\rangle-\left\langle E_{\lambda}^{\prime}\left(v_{\lambda}\right), v_{\lambda}\right\rangle \\
& =\left\|v_{n}-v_{\lambda}\right\|^{p}-\int_{\Omega} \frac{\mid v_{n}-v_{\lambda} p_{s}^{*}(t)}{|x|^{t}} \mathrm{~d} x+o_{n}(1) . \tag{37}
\end{align*}
$$

Therefore, Equation (37) gives

$$
\begin{align*}
& \frac{1}{p}\left\|v_{n}-v_{\lambda}\right\|^{p}-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|v_{n}-v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x  \tag{38}\\
= & \left(\frac{1}{p}-\frac{1}{p_{s}^{*}(t)}\right)\left\|v_{n}-v_{\lambda}\right\|^{p}+o_{n}(1) .
\end{align*}
$$

Now, using Eq. (35) and Eq. (36), we obtain

$$
\frac{1}{p}\left\|v_{n}-v_{\lambda}\right\|^{p}-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|v_{n}-v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \leq E_{\lambda}\left(v_{n}\right)+o_{n}(1)=c+o_{n}(1)
$$

Then, for $c<\left(\frac{1}{p}-\frac{1}{p_{s}^{*}(t)}\right) S_{t}^{\frac{N-t}{p s-t}}$, it follows that

$$
\begin{equation*}
\limsup _{n}\left(\frac{1}{p}-\frac{1}{p_{s}^{*}(t)}\right)\left\|v_{n}-v_{\lambda}\right\|^{p}<\left(\frac{1}{p}-\frac{1}{p_{s}^{*}(t)}\right) S_{t}^{\frac{N-t}{p s-t}} \tag{39}
\end{equation*}
$$

So Equation (39) combine with the fractional Sobolev-Hardy inequality, gives

$$
\begin{aligned}
o_{n}(1) & =\left\|v_{n}-v_{\lambda}\right\|^{p}-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|v_{n}-v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \\
& \geq\left\|v_{n}-v_{\lambda}\right\|^{p}-S_{t}^{\frac{-p_{s}^{*}(t)}{p}}\left\|v_{n}-v_{\lambda}\right\|^{p_{s}^{*}(t)} \\
& =\left\|v_{n}-v_{\lambda}\right\|^{p}\left(1-S_{t}^{\frac{-p_{s}^{*}(t)}{p}}\left\|v_{n}-v_{\lambda}\right\|^{p_{s}^{*}(t)-p}\right) \\
& =C_{3}\left\|v_{n}-v_{\lambda}\right\|^{p}
\end{aligned}
$$

for some constant $C_{3}>0$. Therefore, the proof of the claim follow and completes the proof of Lemma 5.1.

Now using Lemma 3.2 with the fact that $E_{\lambda}(t u) \rightarrow-\infty$ as $t \rightarrow \infty$ for all $u \in X_{0}, u>0$, we conclude that the functional $E_{\lambda}$ has the mountain pass geometry close to $u_{\lambda}$. Consequently, we may fix $e \in X_{0}, e>0$ satisfying $E_{\lambda}(e)<0$. Define the complete metric space

$$
\Gamma=\left\{\eta \in C\left([0,1], X_{0}\right) \mid \eta \text { is continuous, } \eta(0)=0, \eta(1)=e\right\}
$$

and the mini max value for mountain pass level

$$
\gamma_{0}=\inf _{\eta \in \Gamma} \max _{t \in[0,1]} E_{\lambda}(\eta(t)) .
$$

Let $R=\left\|e-u_{\lambda}\right\|$ and $r_{0}>0$ be small enough such that $u_{\lambda}$ is a minimizer of $E_{\lambda}$ on $\overline{B\left(u_{\lambda}, r_{0}\right)}$. We distinguish between the following two cases:
(P1) "Zero altitude case" $\inf \left\{E_{\lambda}(u) \mid u \in X_{0},\|u\|=l\right\} \leq 0$ for all $l<R_{0}$.
(P2) $\exists l_{1}<R_{0}$ such that $\inf \left\{E_{\lambda}(u) \mid u \in X_{0},\|u\|=l_{1}\right\}>0$.
Note that (P1) (resp. (P2)) implies that $\gamma_{0}=0$ (resp. $\gamma_{0}>0$ ). In case where "Zero altitude case" occurs, we can construct a $\left(P S_{\mathcal{F}, \gamma_{0}}\right)$ sequence with $\mathcal{F}=\{\|u\|=l\}$ where $l \leq l_{0}$ and obtain at least a second weak solution to (P). Precisely, we obtain the following result.

Lemma 5.2. Suppose Case 1 holds, then for $1<p<\infty, 1<q-1<p-1<$ $p_{s}^{*}-1$, and $\lambda \in(0, \Lambda)$, there exists a weak solution $v_{\lambda}$ of $(\mathrm{P})$ satisfying $v_{\lambda} \neq u_{\lambda}$.
Proof. From Theorem 1 in [16], we can guarantee the existence of a $(P S)_{F, \delta_{0}}$ sequence $\left\{v_{n}\right\}$ for every $r \leq R_{0}$. From Lemma 5.1, we can conclude that the sequence $\left\{v_{n}\right\}$ is bounded in $X_{0}$ and it converges, upto a subsequence, to a weak solution $v_{\lambda}$ of the problem ( P ). To prove $v_{\lambda} \neq u_{\lambda}$, it is sufficient to prove the strong convergence of $\left\{v_{n}\right\}$ to $v_{\lambda}$ in $X_{0}$. So, since $v_{n} \rightharpoonup v_{\lambda}$ weakly as $n \rightarrow \infty$, from the embedding result $v_{n} \rightarrow v_{\lambda}$ in $L^{k}(\Omega)$ for $1 \leq k<p_{s}^{*}$, and pointwise almost everywhere in $\Omega$. Now, we recall the following result from [9].

$$
\begin{align*}
\left\|v_{n}\right\| & =\left\|v_{n}-v_{\lambda}\right\|+\left\|v_{\lambda}\right\|+o_{n}(1) \text { and } \\
\left\|v_{n}\right\|_{L^{q}(\Omega)} & =\left\|v_{n}-v_{\lambda}\right\|_{L^{q}(\Omega)}+\left\|v_{\lambda}\right\|_{L^{q}(\Omega)}+o_{n}(1) . \tag{40}
\end{align*}
$$

and for all $p<q_{k} \leq p_{s}^{*}(t)$ such that $q_{k} \rightarrow p_{s}^{*}(t)$ as $k \rightarrow \infty$

$$
\begin{equation*}
\int_{\Omega} \frac{\left|v_{n}\right|^{q_{k}}}{|x|^{t}} d x=\int_{\Omega} \frac{\left|v_{n}-v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x+\int_{\Omega} \frac{\left|v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x+o_{n}(1) . \tag{41}
\end{equation*}
$$

So, since $v_{\lambda}$ is a weak solution to the problem (P), we get

$$
\begin{equation*}
\left\|v_{\lambda}\right\|^{p}-\lambda\left|v_{\lambda}\right|_{L^{q}(\Omega)}^{q}-\int_{\Omega} \frac{\left|v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x=0 . \tag{42}
\end{equation*}
$$

Hence, letting $n \rightarrow \infty$ we get

$$
\begin{aligned}
& \int_{Q} \frac{\left|v_{n}(x)-v_{n}(y)\right|^{p-2}\left(v_{n}(x)-v_{n}(y)\right)\left(\left(v_{n}-v_{\lambda}\right)(x)-\left(v_{n}-v_{\lambda}\right)(y)\right)}{|x-y|^{N+p s}} d x d y \\
(43)= & \lambda \int_{\Omega}\left|v_{n}\right|^{q-2} v_{n}\left(v_{n}-v_{\lambda}\right) d x+\int_{\Omega} \frac{\left|v_{n}\right|^{p_{s}^{*}(t)-2} v_{n}}{|x|^{t}}\left(v_{n}-v_{\lambda}\right) d x+o_{n}(1) .
\end{aligned}
$$

Therefore, from Eqs. (40), (43) and Eq. (42) the following holds as $n \rightarrow \infty$

$$
\begin{equation*}
\left\|v_{n}-v_{\lambda}\right\|^{p}=\lambda \int_{\Omega}\left|v_{n}-v_{\lambda}\right|^{q} d x+\int_{\Omega} \frac{\left|v_{n}-v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x+o_{n}(1) \tag{44}
\end{equation*}
$$

We now consider the following two cases:

1. $E_{\lambda}\left(u_{\lambda}\right) \neq E_{\lambda}\left(v_{\lambda}\right)$,
2. $E_{\lambda}\left(u_{\lambda}\right)=E_{\lambda}\left(v_{\lambda}\right)$.

In Case 1 holds, then we are through. In Case 2 holds, from Eq. (40) and (41) we get,

$$
\begin{equation*}
E_{\lambda}\left(v_{n}-v_{\lambda}\right)=E_{\lambda}\left(v_{n}\right)-E_{\lambda}\left(v_{\lambda}\right)+o_{n}(1) \text { as } n \rightarrow \infty . \tag{45}
\end{equation*}
$$

Hence, from Eq. (42) it follows that
(46) $\frac{1}{p}\left\|v_{n}-v_{\lambda}\right\|^{p}-\frac{\lambda}{q}\left\|v_{n}-v_{\lambda}\right\|_{L^{q}(\Omega)}^{q}-\int_{\Omega} \frac{\left|v_{n}-v_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} d x \leq o_{n}(1)$ as $n \rightarrow \infty$.

Therefore, from Eq. (44) and Eq. (46), we get $\left\|v_{n}-v_{\lambda}\right\| \rightarrow 0$ as $n \rightarrow \infty$. Hence $\left\|u_{\lambda}-v_{\lambda}\right\|=r$ and $v_{\lambda} \neq u_{\lambda}$. The proof of Lemma 5.2 is now completed.

Before we state the multiplicity result for Case 2, let us recall the necessary tools for this (for more details see [9]).

Let $U(x)=\left(1+|x|^{p^{\prime}}\right)^{-\frac{N-s p}{p}}$ and $U_{t, \epsilon}(x)=\epsilon^{-\frac{N-s p}{p}} U_{t}\left(\frac{|x|}{\epsilon}\right)$, where $\epsilon>0, x \in$ $\mathbb{R}^{N}$ and $p^{\prime}=\frac{p}{p-1}$. Note that $U_{t, \epsilon}(x)$ is a minimizer for $S_{t}$ satisfying

$$
\begin{equation*}
\left(-\Delta_{p}\right)^{s} U_{t}=\frac{U_{t}^{p_{s}^{*}(t)-1}}{|x|^{t}} \quad \text { weakly in } \mathbb{R}^{N} \tag{47}
\end{equation*}
$$

Let us set,

$$
m_{\epsilon, \delta}=\frac{U_{t, \epsilon}(\delta)}{U_{t, \epsilon}(\delta)-U_{t, \epsilon}(\theta \delta)},
$$

where $\epsilon, \delta>0$, and $\theta>1$. For a fixed $\epsilon, \delta>0$, set

$$
g_{\epsilon, \delta}(k)= \begin{cases}0 & \text { if } 0 \leq k \leq U_{t, \epsilon}(\theta \delta), \\ m_{\epsilon, \delta}^{p}\left(k-U_{t, \epsilon}(\theta \delta)\right) & \text { if } U_{t, \epsilon}(\theta \delta) \leq k \leq U_{t, \epsilon}(\delta), \\ k+U_{t, \epsilon}(\delta)\left(m_{\epsilon, \delta}^{p-1}-1\right) & \text { if } k \geq U_{\epsilon}(\delta)\end{cases}
$$

and let

$$
G_{\epsilon, \delta}(k)=\int_{0}^{k} g_{\epsilon, \delta}^{\prime}(\tau) \mathrm{d} \tau= \begin{cases}0 & \text { if } 0 \leq k \leq U_{t, \epsilon}(\theta \delta) \\ m_{\epsilon, \delta}\left(k-U_{t, \epsilon}(\theta \delta)\right) & \text { if } U_{t, \epsilon}(\theta \delta) \leq k \leq U_{t, \epsilon}(\delta), \\ k & \text { if } k \geq U_{t, \epsilon}(\delta)\end{cases}
$$

The functions $g_{\epsilon, \delta}$ and $G_{\epsilon, \delta}$ are nondecreasing and absolutely continuous. Consider the radially symmetric nonincreasing function

$$
u_{t, \epsilon, \delta}(r)=G_{\epsilon, \delta}\left(U_{t, \epsilon}(r)\right)
$$

which satisfies

$$
u_{t, \epsilon, \delta}(r)= \begin{cases}U_{t, \epsilon}(r) & \text { if } r \leq \delta \\ 0 & \text { if } r \geq \theta \delta\end{cases}
$$

for all $r \geq 1$. We follow here the arguments of [9, Lemma 2.10]. For each sufficiently small $\epsilon, \delta>0$, we have the following estimates for $u_{t, \epsilon, \delta}$.

Lemma 5.3. There exists a constant $C=C(N, p, s)>0$ such that for any $0<2 \epsilon \leq \delta<\theta^{-1} \operatorname{dist}(0, \partial \Omega)$, there holds

$$
\begin{equation*}
\left\|u_{t, \epsilon, \delta}\right\|^{p} \leq S_{t}^{\frac{N-t}{p s-t}}+C\left(\frac{\epsilon}{\delta}\right)^{(N-s p) /(p-1)} \tag{48}
\end{equation*}
$$

$$
\int_{\mathbb{R}^{N}} \frac{u_{t, \epsilon, \delta}^{p_{s}^{*}(t)}}{|x|^{t}}(x) \mathrm{d} x \geq S_{t}^{\frac{N-t}{p s-t}}-C\left(\frac{\epsilon}{\delta}\right)^{(N-t) /(p-1)} .
$$

Moreover, for any $\beta>0$, there exists $C_{\beta}$ such that

$$
\int_{\mathbb{R}^{N}} u_{t, \epsilon, \delta}(x)^{\beta} \geq C_{\beta} \begin{cases}\epsilon^{N-\frac{N-p s}{p} \beta}\left|\log \left(\frac{\epsilon}{\delta}\right)\right| & \text { if } \beta=\frac{p_{s}^{*}}{p_{s}^{*}}  \tag{50}\\ \epsilon^{\frac{N-p s}{p(p-p} \beta} \delta^{N-\frac{N}{\rho(p)}} \overline{p(p-1)} \beta & \text { if } \beta<\frac{p_{s}^{*}}{p^{\prime}}, \\ \epsilon^{N-\frac{N-p s}{p} \beta} & \text { if } \beta>\frac{p_{*}^{*}}{p^{\prime}} .\end{cases}
$$

Also, we have the following estimates cf. [9, Lemma 2.11].
Lemma 5.4. For any $\beta>0$, there exists $C_{\beta}$ such that for any $0<2 \epsilon \leq \delta<$ $\theta^{-1} \operatorname{dist}(0, \partial \Omega)$, there holds

$$
\int_{\mathbb{R}^{N}} u_{t, \epsilon, \delta}(x)^{\beta} \leq C_{\beta} \begin{cases}\epsilon^{N-\frac{N-p s}{p} \beta}\left|\log \left(\frac{\epsilon}{\delta}\right)\right| & \text { if } \beta=\frac{p_{s}^{*}}{p^{\prime}}  \tag{51}\\ \epsilon^{\frac{N-p s}{p(p-1)} \beta} \delta^{N-\frac{N-p s}{p(p-1)} \beta} & \text { if } \beta<\frac{p_{s}^{*}}{p^{\prime}} \\ \epsilon^{N-\frac{N-p s}{p} \beta} & \text { if } \beta>\frac{p_{s}^{*}}{p^{\prime}}\end{cases}
$$

We now prove the following Lemma, when Case 2 holds.
Lemma 5.5. Suppose Case 2 holds, then for $1<p<\infty, 1<q-1<p-1<$ $p_{s}^{*}-1$, and $\lambda \in(0, \Lambda)$, there exists a weak solution $v_{\lambda}$ of the problem ( P ) such that $v_{\lambda} \neq u_{\lambda}$.
Proof. From Yang [36], we know that the condition of Palais Smale is satisfied if

$$
\begin{equation*}
\gamma_{0}<E_{\lambda}\left(u_{\lambda}\right)+\frac{s p-t}{p(N-t)} S^{\frac{N-t}{s p-t}} \tag{52}
\end{equation*}
$$

where $S_{t}$ is the best Sobolev constant.
Claim. $\sup _{0 \leq k \leq \frac{1}{2}} E_{\lambda}\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right)<E_{\lambda}\left(u_{\lambda}\right)+\frac{s p-t}{p(N-t)} S^{\frac{N-t}{s p-t}}$.
At first, using the approach as in García Azorero and Peral [12] where the following estimate is proved (see pp. 946 and 949 in [12]):

$$
\begin{align*}
& \int_{\Omega} \frac{\left|u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x  \tag{53}\\
\geq & \int_{\Omega} \frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x+\frac{\left(k R_{0}\right)^{p_{s}^{*}(t)}}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|u_{t, \epsilon, \delta}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \\
& +\left(p_{s}^{*}(t)\right) k R_{0} \int_{\Omega} \frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda} u_{t, \epsilon, \delta}}{|x|^{t}} \mathrm{~d} x \\
& +\left(p_{s}^{*}(t)\right)\left(k R_{0}\right)^{p_{s}^{*}(t)-1} \int_{\Omega} \frac{\left.u_{\lambda}\left|u_{t, \epsilon, \delta}\right|\right|^{p_{s}^{*}(t)-2} u_{t, \epsilon, \delta}}{|x|^{t}} \mathrm{~d} x+O\left(\epsilon^{\alpha}\right)
\end{align*}
$$

with $\alpha>\frac{(N-s p)}{p}$. Moreover, the following estimate is proved (see Proposition 3.2 in [24]):

$$
\begin{align*}
& \int_{Q} \frac{\left|\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right)(x)-\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right)(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y  \tag{54}\\
\leq & \int_{Q} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y+\frac{\left(k R_{0}\right)^{p}}{p} \int_{Q} \frac{\left|u_{t, \epsilon, \delta}(x)-u_{t, \epsilon, \delta}(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y .
\end{align*}
$$

Finally, we follow here the arguments of Tarantello [33]. Let $R_{0} \geq 1$ and consider any $k \in[0,1]$; then for a suitable $\beta \in(0, q)$ it follows that

$$
\left|\int_{\Omega}\right| u_{\lambda}+\left.k R_{0} u_{t, \epsilon, \delta}\right|^{q} \mathrm{~d} x-\int_{\Omega}\left|u_{\lambda}\right|^{q} \mathrm{~d} x-\left(t R_{0}\right)^{q} \int_{\Omega}\left|u_{t, \epsilon, \delta}\right|^{q} \mathrm{~d} x
$$

NONLOCAL PROBLEM WITH CRITICAL SOBOLEV-HARDY NONLINEARITIES 769

$$
\begin{align*}
& -q R_{0} k \int_{\Omega}\left|u_{\lambda}\right|^{q-2} u_{\lambda} U_{t, \epsilon} \mathrm{~d} x-q\left(k R_{0}\right)^{q-1} \int_{\Omega}\left|U_{t, \epsilon}^{q-2} U_{t, \epsilon} u_{\lambda} \mathrm{d} x\right|  \tag{55}\\
= & R_{0}^{\beta} o\left(\epsilon^{\frac{N-s p}{p}}\right) .
\end{align*}
$$

Therefore, using Eqs. (53)-(54) and Eq. (55), we get (56)

$$
\begin{aligned}
& E_{\lambda}\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right) \\
= & \frac{1}{p} \int_{Q} \frac{\left|\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right)(x)-\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right)(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y \\
& -\frac{\lambda}{q} \int_{\Omega}\left|u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right|^{q} \mathrm{~d} x-\frac{1}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|u_{\lambda}+k R_{0} u_{t, \epsilon}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \\
\leq & \int_{Q} \frac{\left|u_{\lambda}(x)-u_{\lambda}(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y+\frac{\left(k R_{0}\right)^{p}}{p} \int_{Q} \frac{\left|u_{t, \epsilon, \delta}(x)-u_{t, \epsilon, \delta}(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y \\
& -\frac{\lambda\left(k R_{0}\right)^{q}}{q} \int_{\Omega}\left|u_{t, \epsilon, \delta}\right|^{q} \mathrm{~d} x-\lambda\left(k R_{0}\right)^{q-1} \int_{\Omega} u_{\lambda}\left|u_{t, \epsilon, \delta}\right|^{q-2} u_{t, \epsilon, \delta} \mathrm{~d} x \\
& +\lambda k R_{0} \int_{\Omega}\left|u_{\lambda}\right|^{q-2} u_{\lambda} u_{t, \epsilon, \delta} \mathrm{~d} x+k R_{0} \int_{\Omega}\left(\lambda\left|u_{\lambda}\right|^{q-2} u_{\lambda}+\frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)}}{|x|^{t}}\right) u_{t, \epsilon, \delta} \mathrm{~d} x \\
& -\frac{\left(k R_{0} p^{p_{s}^{*}(t)}\right.}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|u_{t, \epsilon, \delta} \delta\right|_{s}^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \\
& +p_{s}^{*}(t) k R_{0} \int_{\Omega} \frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda} u_{t, \epsilon, \delta}}{|x|^{t}} \mathrm{~d} x \\
& -\left(k R_{0}\right)^{p_{s}^{*}(t)-1} \int_{\Omega} \frac{u_{\lambda}\left|u_{t, \epsilon, \delta}\right|^{p_{s}^{*}(t)-2} u_{t, \epsilon, \delta}}{|x|^{t}} \mathrm{~d} x+o\left(\epsilon^{\frac{N-s p}{p}}\right) \\
\leq & E_{\lambda}\left(u_{\lambda}\right)+\left(k R_{0}\right)^{p} \int_{Q} \frac{\left|u_{t, \epsilon, \delta}(x)-u_{t, \epsilon, \delta}(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y \\
& -\frac{\left(k R_{0}\right)^{p_{s}^{*}(t)}}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|u_{t, \epsilon, \delta}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x-\left(k R_{0}\right)^{p_{s}^{*}(t)-1} \int_{\Omega} \frac{u_{\lambda}\left|u_{t, \epsilon, \delta}\right|^{p_{s}^{*}(t)-2} u_{t, \epsilon, \delta}}{|x|^{t}} \mathrm{~d} x \\
& +p_{s}^{*}(t) k R_{0} \int_{\Omega} \frac{\left|u_{\lambda}\right|^{p_{s}^{*}(t)-2} u_{\lambda} u_{t, \epsilon, \delta}}{|x|^{t}} \mathrm{~d} x \\
& -\frac{\lambda\left(k R_{0}\right)^{q}}{q} \int_{\Omega}\left|u_{t, \epsilon, \delta}\right|^{q} \mathrm{~d} x-\lambda\left(k R_{0}\right)^{q-1} \int_{\Omega} u_{\lambda}\left|u_{t, \epsilon, \delta}\right|^{\mid-2} u_{t, \epsilon, \delta} \mathrm{~d} x \\
& +\lambda k R_{0} \int_{\Omega}^{\left|u_{\lambda}\right|^{q-2} u_{\lambda} u_{t, \epsilon, \delta} \mathrm{~d} x+o\left(\epsilon e^{\frac{N-s p}{p}}\right)}
\end{aligned}
$$

Hence, from Eq. (53) and Eq. (55), it follows that

$$
\begin{aligned}
& E_{\lambda}\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right) \\
\leq & E_{\lambda}\left(u_{\lambda}\right)+\frac{\left(k R_{0}\right)^{p}}{p} \int_{Q} \frac{\left|u_{\epsilon, \delta}(x)-u_{t, \epsilon, \delta}(y)\right|^{p}}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\left(k R_{0}\right)^{p_{s}^{*}(t)}}{p_{s}^{*}(t)} \int_{\Omega} \frac{\left|u_{t, \epsilon, \delta}\right|^{p_{s}^{*}(t)}}{|x|^{t}} \mathrm{~d} x \mathrm{~d} x \\
& -\left(k R_{0}\right)^{p_{s}^{*}(t)-1} \int_{\Omega} \frac{u_{\lambda}\left|u_{t, \epsilon, \delta}\right|^{p_{s}^{*}(t)-2} u_{t, \epsilon, \delta}}{|x|^{t}} \mathrm{~d} x+o\left(\epsilon^{\frac{N-s p}{p}}\right) .
\end{aligned}
$$

Arguing as in García Azorero and Peral [12] (see p. 947), and using Eqs. (48)(49) and Eq. (50) we get:

$$
\sup _{0 \leq k \leq \frac{1}{2}} E_{\lambda}\left(u_{\lambda}+k R_{0} u_{t, \epsilon, \delta}\right)<E_{\lambda}\left(u_{\lambda}\right)+\frac{s p-t}{p(N-t)} S^{\frac{N-t}{s p-t}}
$$

which completes the proof of the claim. Now, the compactness of $\left\{v_{n}\right\}$ implies that $E_{\lambda}\left(v_{\lambda}\right)=\gamma_{0}>E_{\lambda}\left(u_{\lambda}\right)$. Therefore $v_{\lambda} \neq u_{\lambda}$. Thus, the proof of Lemma 5.5 is now completed.

## 6. Regularity of solutions

This section is devoted to presented some regularity properties to the weak solutions of the problem (P). We will use an adaptation of the classical Moser iteration technique to prove a priori bounds for the bounded weak solutions of the problem (P). Firstly, let us recall some elementary inequality that we use to proof the $L^{\infty}$ estimate. We begin by the following elementary inequality proved in [3].

Lemma 6.1 (Lemma C. 2 in [3]). Let $1<p<\infty$ and $\beta \geq 1$. For every $a, b, M \geq 0$ there holds

$$
|a-b|^{p-2}(a-b)\left(a_{M}^{\beta}-b_{M}^{\beta}\right) \geq \frac{\beta p^{p}}{(\beta+p-1)^{p}}\left|a_{M}^{\frac{\beta+p-1}{p}}-b_{M}^{\frac{\beta+p-1}{p}}\right|
$$

where we set $a_{M}=\min \{a, M\}$ and $b_{M}=\min \{b, M\}$.
Lemma 6.2 (Lemma C. 3 in [4]). Let $1<p<\infty$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. Then we have

$$
|G(a)-G(b)|^{p} \leq|a-b|^{p-2}(a-b)(g(a)-g(b)),
$$

where $G(t)=\int_{0}^{t} g^{\prime}(\tau)^{\frac{1}{p}}$, for $t \in \mathbb{R}$.
Theorem 6.3. Let $u$ be a weak solution of (P). Then $u \in L^{\infty}(\Omega)$.
Proof. The proof is adopted from [4]. Let us define $u_{k}=\min \left\{(u-1)^{+}, k\right\}$ for $k>0, \beta \geq 1$, and consider the non-decreasing function $\Phi=\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}$ for $\rho>0$. Using Lemma 6.1 combine with the triangle inequality and testing the equation in the problem (P) with $\Phi \in X_{0}$, we get

$$
\begin{align*}
& \frac{\beta p^{p}}{(\beta+p-1)^{p}} \int_{\mathbb{R}^{2 N}}\left|u_{k}^{\frac{\beta+p-1}{p}}(x)-u_{k}^{\frac{\beta+p-1}{p}}(y)\right| \mathrm{d} x \mathrm{~d} y  \tag{57}\\
\leq & \int_{\mathbb{R}^{2 N}} \frac{\| u(x)|-|u(y)||^{p-2}(|u(x)|-|u(y)|)\left(\left(u_{k}(x)+\rho\right)^{\beta}-\left(u_{k}(y)+\rho\right)^{\beta}\right)}{|x-y|^{N+s p}} \mathrm{~d} x \mathrm{~d} y
\end{align*}
$$

$$
\leq \int_{\Omega} \lambda|u|^{q-2} u\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x+\int_{\Omega} \frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}}\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x
$$

Now, using the support of $u_{k}$ and using the Hölder inequality, we obtain

$$
\begin{align*}
& \int_{\Omega} \lambda|u|^{q-2} u\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x+\int_{\Omega} \frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}}\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x  \tag{58}\\
= & \int_{u \geq 1} \lambda|u|^{q-2} u\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x+\int_{u \geq 1} \frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}}\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x \\
\leq & C_{1} \int_{u \geq 1}\left(1+\frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}}\right)\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x \\
\leq & 2 C_{1} \int_{u \geq 1} \frac{|u|^{p_{s}^{*}(t)-2} u}{|x|^{t}}\left(\left(u_{k}+\rho\right)^{\beta}-\rho^{\beta}\right) \mathrm{d} x \\
\leq & 2 C_{1}|u|_{p_{s}^{*}}^{p_{s}^{*}(t)-1}\left|\left(u_{k}+\rho\right)^{\beta}\right|_{p^{\prime}},
\end{align*}
$$

where $C_{1}=\max \{\lambda, 1\}$ and with $p^{\prime}=p_{s}^{*} /\left(p_{s}^{*}+1-p_{s}^{*}(t)\right)$.
On the other hand, from Theorem 1 in [25], we have

$$
\begin{align*}
& \int_{\mathbb{R}^{2 N}}\left|\left(u_{k}(x)+\rho\right)^{\frac{\beta+p-1}{p}}-\left(u_{k}(y)+\rho\right)^{\frac{\beta+p-1}{p}}\right|^{p} \mathrm{~d} x \mathrm{~d} y \\
\geq & C_{N, p, s}\left(\int_{\mathbb{R}^{N}}\left(\left(u_{k}(x)+\rho\right)^{\frac{\beta+p-1}{p}}-\rho^{\frac{\beta+p-1}{p}}\right)^{\frac{N p}{N-s p}} \mathrm{~d} x\right)^{\frac{N p}{N-s p}} . \tag{59}
\end{align*}
$$

Now, let us recall the following inequality

$$
\begin{equation*}
\left(u_{k}+\rho\right)^{\beta} \leq\left(u_{k}+\rho\right)^{\beta+p-1} \rho^{1-p} . \tag{60}
\end{equation*}
$$

So, combine the inequality (60) with the triangle inequality, we get

$$
\begin{align*}
& \left(\int_{\mathbb{R}^{N}}\left(\left(u_{k}(x)+\rho\right)^{\frac{\beta+p-1}{p}}-\rho^{\frac{\beta+p-1}{p}}\right)^{\frac{N p}{N-s p}} \mathrm{~d} x\right)^{\frac{N p}{N-s p}}  \tag{61}\\
\geq & \left(\frac{\rho}{2}\right)^{p-1} C_{N, p, s}\left(\int_{\mathbb{R}^{N}}\left(\left(u_{k}(x)+\rho\right)^{\frac{\beta+p-1}{p}}\right)^{\frac{N p}{N-s p}} \mathrm{~d} x\right)^{\frac{N-s p}{N}}-\rho^{\beta+p-1}|\Omega|^{\frac{N p}{N-s p}}
\end{align*}
$$

Therefore, from Equations (58)-(61), we obtain

$$
\begin{aligned}
& \left(\int_{\mathbb{R}^{N}}\left(\left(u_{k}(x)+\rho\right)^{\frac{\beta+p-1}{p}}\right)^{\frac{N p}{N-s p}} \mathrm{~d} x\right)^{\frac{N-s p}{N}} \\
\leq & C_{N, p, s} \frac{|u|_{p s_{s}}^{p_{s}^{*}(t)-1}}{\beta}\left(\frac{\beta+p-1}{p \rho^{\frac{p-1}{p}}}\right)^{p-1}\left|\left(u_{k}+\rho\right)^{\beta}\right|_{p^{\prime}}+\rho^{\beta}|\Omega|^{\frac{N-s p}{N_{p}}}
\end{aligned}
$$

Moreover, observing that for $\beta \geq 1$, we have

$$
\begin{equation*}
\frac{\beta p^{p}}{(\beta+p-1)^{p}} \geq\left(\frac{p}{\beta+p-1}\right)^{p-1} \tag{62}
\end{equation*}
$$

Therefore, from estimate (62), it is easy to check

$$
\rho^{\beta}|\Omega|^{\frac{N-s p}{N p}} \leq \frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^{p}|\Omega|^{1-\frac{1}{p^{\prime}}-\frac{s p}{N}}\left|\left(u_{k}+\rho\right)^{\beta}\right|_{p^{\prime}}
$$

Thus, it follows that

$$
\begin{align*}
& \left(\int_{\mathbb{R}^{N}}\left(\left(u_{k}(x)+\rho\right)^{\frac{\beta+p-1}{p}}\right)^{\frac{N p}{N-s p}} \mathrm{~d} x\right)^{\frac{N-s p}{N}}  \tag{63}\\
\leq & C_{N, p, s}\left(\frac{\beta+p-1}{p \rho^{\frac{p-1}{p}}}\right)^{p-1}\left|\left(u_{k}+\rho\right)^{\beta}\right|_{p^{\prime}} \times\left(\frac{|u|_{p_{s}^{s}}^{p_{s}^{*}(t)-1}}{\rho^{p-1}}+|\Omega|^{1-\frac{1}{p^{\prime}}-\frac{s p}{N}}\right) .
\end{align*}
$$

Then, choose $\rho>0$ by

$$
\rho=\left(|u|_{p_{s}^{*}}^{\frac{p_{s}^{*}(t)-1}{p-1}}+|\Omega|^{\frac{1}{p-1}\left(1-\frac{1}{p^{\prime}}-\frac{s p}{N}\right)}\right) .
$$

Let us now set $v=\beta p^{\prime}$ and $\tau=\frac{N}{N-s p} \frac{1}{p^{\prime}}>1$, then the inequality (63) can be written as

$$
\begin{align*}
& \left(\int_{\Omega}\left(u_{k}(x)+\rho\right)^{v \tau} \mathrm{~d} x\right)^{\frac{1}{v \tau}}  \tag{64}\\
\leq & \left(C_{N, p, s}|\Omega|^{1-\frac{1}{p^{\prime}}-\frac{s p}{N}}\right)^{\frac{p^{\prime}}{v}}\left(\frac{p^{\prime}}{v}\right)^{\frac{p^{\prime}}{v}}\left(\frac{v+p^{\prime} p-p^{\prime}}{q^{\prime} p}\right)^{\frac{p p^{\prime}}{v}}\left|\left(u_{k}+\rho\right)\right|_{v p^{\prime}} .
\end{align*}
$$

We want to iterate the inequality (64), by taking the following sequence $\left\{v_{m}\right\}$ of exponents

$$
v_{0}=1 \quad v_{m+1}=v_{m} \tau=\tau^{m+1} .
$$

Therefore, by starting from $m=0$ at the step $m$, the inequality (64) can be written as
(65) $\left|\left(u_{k}+\rho\right)\right|_{v_{m+1}}$

$$
\leq\left(C_{N, p, s}|\Omega|^{1-\frac{1}{p^{\prime}}-\frac{s p}{N}}\right)^{\sum_{i=0}^{m} \frac{p^{\prime}}{v_{i}}} \Pi_{i=0}^{m}\left(\frac{p^{\prime}}{v_{i}}\right)^{\frac{p^{\prime}}{v_{i}}}\left(\frac{v_{i}+p^{\prime} p-p^{\prime}}{q^{\prime} p}\right)^{\frac{p p^{\prime}}{v_{i}}}\left|\left(u_{k}+\rho\right)\right|_{v}
$$

We now observe that $v_{m}$ diverges at infinity and in addition

$$
\begin{aligned}
& \sum_{m=0}^{\infty} \frac{1}{v_{m}}=\sum_{m=0}^{\infty} \frac{1}{\tau^{m}}=\frac{\tau}{\tau-1} \text { and } \\
& \prod_{m=0}^{\infty}\left(\frac{p^{\prime}}{v_{m}}\right)^{\frac{p^{\prime}}{v_{m}}}\left(\frac{v+p^{\prime} p-p^{\prime}}{q^{\prime} p}\right)^{\frac{p p^{\prime}}{v}}<+\infty .
\end{aligned}
$$

Iterating inequality (65) infinitely many times, we finally obtain

$$
\left|\left(u_{k}+\beta\right)\right|_{L^{\infty}} \leq\left(C_{N, p, s}|\Omega|^{1-\frac{1}{p^{\prime}}-\frac{s p}{N}}\right)^{p^{\prime} \frac{\tau}{\tau-1}}\left|\left(u_{k}+\beta\right)\right|_{p^{\prime}}
$$

Since $u_{k} \leq(u-1)^{+}$, we get

$$
\left|\left(u_{k}+\beta\right)\right|_{L^{\infty}} \leq\left(C_{N, p, s}|\Omega|^{1-\frac{1}{p^{\prime}}-\frac{s p}{N}}\right)^{p^{\prime} \frac{\tau}{\tau-1}}\left(\left|\left(u_{k}-1\right)^{+}\right|_{p^{\prime}}+\rho|\Omega|^{\frac{1}{p^{\prime}}}\right)
$$

Now, letting $k \rightarrow \infty$, we get

$$
\left|(u-1)^{+}\right|_{L^{\infty}} \leq\left(C_{N, p, s}|\Omega|^{1-\frac{1}{p^{\prime}}-\frac{s p}{N}}\right)^{p^{\prime} \frac{\tau}{\tau-1}}\left(\left|\left(u_{k}-1\right)^{+}\right|_{p^{\prime}}+\rho|\Omega|^{\frac{1}{p^{\prime}}}\right)
$$

Then, providing in the end $u$ is uniformly bounded in $L^{\infty}(\Omega)$. Which concludes the proof of Lemma 6.3.

Now by the regularity results of Iannizzotto-Mosconi-Squassina (Theorem 1.1 in [21]), we obtain the following uniform $C^{1, \alpha}$ estimate.

Theorem 6.4. Let $u$ be a weak solution of the problem (P). Then there exists constants $\alpha \in(0, s), M>0$ such that

$$
\|u\|_{C^{1, \alpha}(\bar{\Omega})} \leq M
$$

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