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ON WEAKLY EINSTEIN ALMOST CONTACT MANIFOLDS

Xiaomin Chen

Abstract. In this article we study almost contact manifolds admitting weakly Einstein metrics. We first prove that if a $(2n+1)$-dimensional Sasakian manifold admits a weakly Einstein metric, then its scalar curvature s satisfies $-6 \leqslant s \leqslant 6$ for $n=1$ and $-2n(2n+1)\frac{4n^2-4n+3}{4n^2-4n-1} \leqslant s \leqslant 2n(2n+1)$ for $n \geqslant 2$. Secondly, for a $(2n+1)$-dimensional weakly Einstein contact metric (κ,μ)-manifold with $\kappa < 1$, we prove that it is flat or is locally isomorphic to the Lie group $SU(2)$, $SL(2)$, or $E(1,1)$ for $n=1$ and that for $n \geqslant 2$ there are no weakly Einstein metrics on contact metric (κ,μ)-manifolds with $0<\kappa<1$. For $\kappa<0$, we get a classification of weakly Einstein contact metric (κ,μ)-manifolds. Finally, it is proved that a weakly Einstein almost cosymplectic (κ,μ)-manifold with $\kappa<0$ is locally isomorphic to a solvable non-nilpotent Lie group.

1. Introduction

An n-dimensional Riemannian manifold (M,g) is said to be *weakly Einstein* if its Riemannian tensor R satisfies

$$\breve{R} = \frac{|R|^2}{n} g. \tag{1}$$

Here $\breve{R}$ is a $(0,2)$-type tensor defined as

$$\breve{R}(X,Y) = \sum_{i,j,k=1}^{n} R(X,e_i,e_j,e_k)R(Y,e_i,e_j,e_k)$$

for an orthonormal frame $\{e_i\}$, $i=1,2,\ldots,n$. The concept was introduced by Euh, Park and Sekigawa in [11]. We also notice that if a weakly Einstein metric is critical to the functional

$$g \mapsto \int_M |s_g|^2 dv_g,$$

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where s_g is the scalar curvature of M (see [2]), then it becomes an Einstein metric. Moreover, it is easy to verify that for a 4-dimensional manifold, Einstein metrics are in fact weakly Einstein metrics. However, when $\dim M > 4$ a generic Einstein metric is not necessary a weakly Einstein metric. Based on the fact, Hwang-Yun considered whether an n-dimensional weakly Einstein metric that is a nontrivial solution to the critical point equation is Einstein (cf. [12]). More recently, Baltazar-Silva-Oliveira [1] classified a four dimensional weakly Einstein manifold with Miao-Tam critical metric under some assumptions on scalar curvature.

In the present paper, we study odd-dimensional manifolds with weakly Einstein metrics. First we consider a Sasakian manifold admitting a weakly Einstein metric and obtain the following result.

Theorem 1.1. *Let M^{2n+1} be a weakly Einstein Sasakian manifold. Then the scalar curvature s satisfies*

$$\begin{cases} -6 \leqslant s \leqslant 6, & \textit{for} \quad n=1; \\ -2n(2n+1)\frac{4n^2-4n+3}{4n^2-4n-1} \leqslant s \leqslant 2n(2n+1), & \textit{for} \quad n \geqslant 2, \end{cases}$$

and the right equality holds if and only if M is a conformal flat Einstein manifold.

On the other hand, we observe that a remarkable class of contact metric manifolds is a (κ, μ)-space, originally introduced by D. E. Blair, T. Koufogiorgos and V. J. Papantoniou in [4], whose curvature tensor satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2}$$

for any vector fields X, Y, where κ and μ are constants and $h := \frac{1}{2}\mathcal{L}_\xi \phi$ is a self-dual operator. Moreover, Blair et al. proved the following classification theorem.

Theorem 1.2 ([4, Theorem 3])**.** *Let M be a 3-dimensional (κ, μ)-manifold. Then $\kappa \leqslant 1$. If $\kappa = 1$, then $h = 0$ and M is a Sasakian manifold. If $\kappa < 1$, then M is locally isometric to one of the unimodular Lie groups $SU(2)$, $SL(2, \mathbb{R}), E(2), E(1,1)$ with a left-invariant metric.*

Moreover, this structure can occur on $SU(2)$ or $SO(3)$ when $1-\lambda-\mu/2 > 0$ and $1+\lambda-\mu/2 > 0$, on $SL(2,\mathbb{R})$ or $O(1,2)$ when $1-\lambda-\mu/2 < 0$ and $1+\lambda-\mu/2 > 0$ or $1-\lambda-\mu/2 < 0$ and $1+\lambda-\mu/2 < 0$, on $E(2)$ when $1-\lambda-\mu/2 = 0$ and $\mu < 2$, including a flat structure when $\mu = 0$, and on $E(1,1)$ when $1+\lambda-\mu/2 = 0$ and $\mu > 2$.

For a non-Sasakian (κ, μ)-manifold M, Boeckx [5] introduced an invariant

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$$

and proved the following conclusion:

Theorem 1.3 ([5, Corollary 5])**.** *Let M be a non-Sasakian (κ,μ)-space. Then it is locally isometric, up to a D-homothetic transformation, to the unit tangent sphere bundle of some space of constant curvature* (*different from* 1) *if and only if $I_M > -1$.*

In view of Theorem 1.2 and Theorem 1.3, we obtain:

Theorem 1.4. *A 3-dimensional weakly Einstein contact metric (κ,μ)-manifold for $\kappa < 1$ is flat, or is locally isomorphic to the Lie group $SU(2)$, $SL(2,\mathbb{R})$, $E(1,1)$ endowed with a left-invariant metric.*

For the dimensions $\geqslant 5$ there are no weakly Einstein metrics on contact metric (κ,μ)-manifolds with $0 < \kappa < 1$. If $M^{2n+1}(n > 1)$ is a weakly Einstein contact metric (κ,μ)-manifold with $\kappa < 0$, then $\mu > \frac{n-2+\sqrt{9n^2-16n+8}}{2n-1}$ or $\mu < \frac{n-2-\sqrt{9n^2-16n+8}}{2n-1}$. In particular, when $\mu < \frac{n-2-\sqrt{9n^2-16n+8}}{2n-1}$, M is locally isometric, up to a D-homothetic transformation, to the unit tangent sphere bundle of some space of constant curvature.

Finally, we notice that Endo considered another class of odd-dimensional manifolds, which are said to be *almost cosymplectic (κ,μ)-manifolds*, and it is proved that $\kappa \leqslant 0$ and the equality holds if and only if the almost cosymplectic (κ,μ)-manifolds are cosymplectic (cf. [10]). Since Blair [5] proved that a cosymplectic manifold is locally the product of a Kähler manifold and an interval or unit circle S^1, we are only require to consider the case where $\kappa < 0$. For an almost cosymplectic (κ,μ)-manifold with $\kappa < 0$, if it is equipped with a weakly Einstein metric, we obtain the following conclusion.

Theorem 1.5. *A weakly Einstein almost cosymplectic (κ,μ)-manifold for $\kappa < 0$ is locally isomorphic to a solvable non-nilpotent Lie group G_λ endowed with an almost cosymplectic structure, where $\lambda = \sqrt{-\kappa}$.*

In order to prove these conclusions, in Section 2 we recall some basic concepts and formulas. The proofs of theorems will be given in Section 3, Section 4 and Section 5, respectively.

2. Preliminaries

2.1. Weakly Einstein metrics

In a local coordinate system the components of the $(0,4)$-Riemannian curvature tensor are given by $R_{ijkl} = g(R(e_i,e_j)e_k,e_l)$. Throughout the paper the Einstein convention of summing over the repeated indices will be adopted. The Ricci tensor Ric is obtained by the contraction $(Ric)_{jk} = R_{jk} = g^{il}R_{ijkl}$. $s = g^{ik}R_{ik}$ will denote the scalar curvature and $(\mathring{Ric})_{ik} = R_{ik} - \frac{s}{n}g_{ik}$ the traceless Ricci tensor.

We say that a Riemannian manifold (M^n,g) is *weakly Einstein* if the Riemannian tensor R satisfies (1), i.e.,

$$\breve{R}_{ij} = \frac{|R|^2}{n}g_{ij}$$

for an orthonormal frame $\{e_i\}, i = 1, 2, \ldots, n$, where the 2-tensor $\breve{R}_{ij}$ is defined as $\breve{R}_{ij} = R_{ipqr}R_{jpqr}$ and $|R|^2 = R_{ijkl}R_{ijkl}$.

On an n-dimensional Riemannian manifold (M^n, g) for $n \geqslant 3$, the Weyl tensor is defined by

$$W_{ijkl} = R_{ijkl} + \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) - \frac{s}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{il}g_{jk}). \tag{3}$$

Here, we remark that the curvature tensor of Blair [3] is different from ours by a sign. It is well-known that the Weyl tensor W identically vanishes for $n = 3$. From (3), we conclude (see [12, Eq. (6)])

$$|R|^2 = \frac{2s^2}{n(n-1)} + \frac{4}{n-2}|\mathring{Ric}|^2 + |W|^2, \tag{4}$$

where s denotes the scalar curvature of M and $\mathring{Ric} = Ric - \frac{s}{n}g$ is the traceless Ricci tensor.

2.2. Almost contact manifolds

In the following we suppose that M is a $(2n+1)$-dimensional smooth manifold. An *almost contact structure* on M is a triple (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ a unit vector field, called Reeb vector field, η a one-form dual to ξ satisfying $\phi^2 = -I + \eta \otimes \xi$, $\eta \circ \phi = 0$, $\phi \circ \xi = 0$. A smooth manifold with such a structure is called an *almost contact manifold.*

A Riemannian metric g on M is called compatible with the almost contact structure if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X)$$

for any $X, Y \in \mathfrak{X}(M)$. An almost contact structure together with a compatible metric is called an *almost contact metric structure* and (M, ϕ, ξ, η, g) is called an almost contact metric manifold. Such an almost contact metric manifold is called a *contact metric manifold* if $d\eta = \omega$, where ω denotes the fundamental 2-form on M defined by $\omega(X, Y) := g(\phi X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. An almost contact structure (ϕ, ξ, η) is said to be *normal* if the corresponding complex structure J on $M \times \mathbb{R}$ is integrable. If a contact metric manifold M is normal, it is said be a *Sasakian manifold.* For a Sasakian manifold, the following equations hold ([3]):

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{5}$$

$$Q\xi = 2n\xi. \tag{6}$$

Here Q is the Ricci operator defined by $Ric(X, Y) = g(QX, Y)$ for any vectors X, Y.

Now we recall that there is an operator $h = \frac{1}{2}\mathcal{L}_\xi \phi$ which is a self-dual operator. For a contact metric manifold, it is proved the following relations [4]:

$$\mathrm{trace}(h) = 0, \quad h\xi = 0, \quad \phi h = -h\phi, \quad g(hX, Y) = g(X, hY), \quad \forall X, Y \in \mathfrak{X}(M).$$

We notice that the above formulas also hold in almost cosymplectic manifolds (see [9]).

As the generalization of the condition $R(X, Y)\xi = 0$, Blair et al. defined a so-called (κ, μ)-*nullity distribution* on a contact metric manifold:

$$\begin{aligned} N_x(\kappa, \mu) := \{Z \in T_xM \mid R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y) \\ + \mu(g(Y, Z)hX - g(X, Z)hY)\} \end{aligned}$$

for two real numbers $\kappa, \mu \in \mathbb{R}$ (see [4]). A contact metric manifold is called a *contact metric* (κ, μ)*-manifold* if the Reeb vector field ξ belongs to the (κ, μ)-nullity distribution, namely the condition (2) is satisfied.

Let (M, ϕ, ξ, η) be an almost contact metric manifold. If the fundamental 2-form ω and 1-form η are closed, then M is called an *almost cosymplectic manifold*. Moreover, if M is normal, it is said to be cosymplectic. An *almost cosymplectic* (κ, μ)*-manifold* is an almost cosymplectic manifold satisfying (2). This class of almost contact manifold was firstly considered in [10]. In particular, any cosymplectic manifold is an almost cosymplectic (κ, μ)-manifold with $\kappa = 0$ and any μ. Endo proved that if $\kappa \neq 0$ any almost cosymplectic (κ, μ)-manifolds are not cosymplectic ([10]). When $\kappa < 0$ and $\mu = 0$, Dacko in [8] proved that M is necessarily an almost cosymplectic manifold with Kählerian leaves, moreover gave a full description of the local structure of this class.

We briefly recall the structure, referring to [8] for more details. Let λ be a real positive number and $\mathfrak{g}_\lambda$ be the solvable non-nilpotent Lie algebra with basis $\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ and non-zero Lie brackets

$$[\xi, X_i] = -\lambda X_i, \quad [\xi, Y_i] = \lambda Y_i$$

for each $i \in \{1, \ldots, n\}$. Let G_λ be a Lie group whose Lie algebra is $\mathfrak{g}_\lambda$ and let (ϕ, ξ, η, g) be the left-invariant almost cosymplectic structure defined by

$$\begin{gathered} g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}, \quad g(X_i, Y_j) = 0, \quad g(\xi, X_i) = g(\xi, Y_i) = 0, \\ \phi\xi = 0, \quad \phi X_i = Y_i, \quad \phi Y_i = -X_i, \quad \eta = g(\cdot, \xi). \end{gathered}$$

Theorem 2.1 ([8, Theorem 4])**.** *An almost cosymplectic* $(\kappa, 0)$*-manifold for some* $\kappa < 0$ *is locally isomorphic to the above Lie group* G_λ *endowed with the above almost cosymplectic structure, where* $\lambda = \sqrt{-\kappa}$.

Throughout this paper we write the indices $i, j, k, l \in \{0, 1, 2, \ldots, 2n\}$, $a, b, c, d \in \{1, 2, \ldots, 2n\}$, $\alpha, \beta, \gamma, \delta \in \{1, 2, \ldots, n\}$ and $A, B, C, D \in \{n+1, n+2, \ldots, 2n\}$. Write

$$h_{ij} = g(he_i, e_j), \quad R_{ij0k} = g(R(e_i, e_j)\xi, e_k), \quad Ric(\xi, \xi) = R_{00} = R_{i00i}.$$

3. Proof of Theorem 1.1

In this section we assume that M is a $2n+1$-dimensional Sasakian manifold and $\{e_i\}_{i=0}^{2n}$ is a local orthonormal frame of M such that $e_0=\xi, e_{n+i}=\phi e_i$ for $i=1,2,\dots,n$.

Using (5) and (6), it follows from (3) that

$$\begin{aligned} W_{ij0l} &= R_{ij0l}+\frac{1}{2n-1}(g_{i0}R_{jl}-g_{il}R_{j0}-g_{j0}R_{il}+g_{jl}R_{i0}) \\ &\quad -\frac{s}{2n(2n-1)}(g_{jl}g_{i0}-g_{il}g_{j0}) \\ &= \Big[1-\frac{1}{2n-1}\Big(2n-\frac{s}{2n}\Big)\Big](g_{j0}g_{il}-g_{i0}g_{jl}) \\ &\quad +\frac{1}{2n-1}(g_{i0}R_{jl}-g_{j0}R_{il}). \end{aligned} \tag{7}$$

Since W is totally trace-free, we have

$$\begin{aligned} W_{ija0}W_{ija0} &= \Big[1-\frac{1}{2n-1}\Big(2n-\frac{s}{2n}\Big)\Big](g_{j0}g_{ia}-g_{i0}g_{ja})W_{ij0a} \\ &\quad +\frac{1}{2n-1}(g_{i0}R_{ja}-g_{j0}R_{ia})W_{ij0a} \\ &= \Big[1-\frac{1}{2n-1}\Big(2n-\frac{s}{2n}\Big)\Big](W_{b00a}g_{ba}-W_{0b0a}g_{ba}) \\ &\quad +\frac{1}{2n-1}(W_{0j0a}R_{ja}-W_{i00a}R_{ia}) \\ &= 2\Big[1-\frac{1}{2n-1}\Big(2n-\frac{s}{2n}\Big)\Big]W_{a00a}+\frac{2}{2n-1}W_{0b0a}R_{ba} \\ &= -\frac{2}{2n-1}\Big[1-\frac{1}{2n-1}\Big(2n-\frac{s}{2n}\Big)\Big]R_{aa}+\frac{2}{(2n-1)^2}R_{ba}^2 \\ &= -\Big[1-\frac{1}{2n-1}\Big(2n-\frac{s}{2n}\Big)\Big]\frac{2(s-2n)}{2n-1} \\ &\quad +\frac{2}{(2n-1)^2}\Big(-4n^2+|\mathring{Ric}|^2+\frac{s^2}{2n+1}\Big) \\ &= \frac{2}{(2n-1)^2}\Big[|\mathring{Ric}|^2-\frac{s^2}{2n(2n+1)}+2s-2n(2n+1)\Big]. \end{aligned} \tag{8}$$

Here we have used $R_{aa}=s-2n$ and $\mathring{Ric}=Ric-\frac{s}{2n+1}$.

On the other hand, using (5) we directly compute

$$\begin{aligned} \check{R}(\xi,\xi) &= R_{0ijk}R_{0ijk}=(g_{0j}g_{ik}-g_{0i}g_{jk})R_{ij0k} \\ &= R_{i00i}-R_{0j0j}=2R_{00}=4n. \end{aligned}$$

Therefore for a $2n+1$-dimensional contact metric manifold, (4) should become

$$4n=\frac{1}{2n+1}\Big(\frac{2s^2}{2n(2n+1)}+\frac{4}{2n-1}|\mathring{Ric}|^2+|W|^2\Big). \tag{9}$$

It is well know that when $n=1$, $W=0$, then

$$4|\mathring{Ric}|^2 = 12 - \frac{s^2}{3}.$$

Thus we have

$$-6 \leqslant s \leqslant 6.$$

Next we assume $n \geqslant 2$ and the following lemma is clear.

Lemma 3.1.

$$|W|^2 = 2W_{ija0}W_{ija0} + W_{dcab}W_{dcab}.$$

Proof. For the indices $i, j, k, l \in \{0, 1, 2, \ldots, 2n\}$ and $a, b, c, d \in \{1, 2, \ldots, 2n\}$, we directly compute

$$\begin{aligned}
|W|^2 &= W_{ijkl}W_{ijkl} = W_{ij0l}W_{ij0l} + W_{ijal}W_{ijal} \\
&= W_{ij0l}W_{ij0l} + W_{ija0}W_{ija0} + W_{ijab}W_{ijab} \\
&= 2W_{ija0}W_{ija0} + W_{ijab}W_{ijab} \\
&= 2W_{ija0}W_{ija0} + W_{d0ab}W_{d0ab} + W_{0cab}W_{0cab} + W_{dcab}W_{dcab} \\
&= 2W_{ija0}W_{ija0} + 2W_{d0ab}W_{d0ab} + W_{dcab}W_{dcab} \\
&= 2W_{ija0}W_{ija0} + W_{dcab}W_{dcab}
\end{aligned}$$

since $W_{d0ab} = 0$ for any $a, b, d \in \{1, 2, \ldots, 2n\}$ by (7). □

By Lemma 3.1, substituting (8) into (9) gives

$$\begin{aligned}
4n(2n+1) = \frac{2s^2}{2n(2n+1)} + \frac{4}{2n-1}|\mathring{Ric}|^2 + \frac{4}{(2n-1)^2}\Big[|\mathring{Ric}|^2 \\
- \frac{s^2}{2n(2n+1)} + 2s - 2n(2n+1)\Big] + W_{dcab}W_{dcab}.
\end{aligned}$$

Since $W_{dcab}W_{dcab} \geqslant 0$, we conclude

$$0 \geqslant \frac{4n^2-4n-1}{n(2n+1)}s^2 + 8n|\mathring{Ric}|^2 + 8s - 4n(2n+1)(4n^2-4n+3),$$

that is,

$$4n|\mathring{Ric}|^2 \leqslant -\frac{4n^2-4n-1}{2n(2n+1)}s^2 - 4s + 2n(2n+1)(4n^2-4n+3).$$

Hence

$$-\frac{4n^2-4n-1}{2n(2n+1)}s^2 - 4s + 2n(2n+1)(4n^2-4n+3) \geqslant 0.$$

Because $n \geqslant 2$, we obtain

$$-2n(2n+1)\frac{4n^2-4n+3}{4n^2-4n-1} \leqslant s \leqslant 2n(2n+1).$$

Moreover, when the right equality holds, from (9) we find $W=0$, i.e., M is conformal flat. We complete the proof of Theorem 1.1.

4. Proof of Theorem 1.4

In this section we suppose that $(M^{2n+1}, \phi, \eta, \xi, g)$ is a contact metric (κ, μ)-manifold. It is proved that $\kappa \leqslant 1$ and if $\kappa = 1$, then $h = 0$, i.e., M is a Sasakian manifold by Theorem 1.2. Thus we only need to consider the case where $\kappa < 1$. For the contact metric (κ, μ)-manifold the following lemma was given.

Lemma 4.1 ([13]). *Let $(M^{2n+1}, \phi, \eta, \xi, g)$ be a contact metric (κ, μ)-manifold with $\kappa < 1$. For every $p \in M$, there exist an open neighborhood W of p and orthonormal local vector fields $X_i, \phi X_i$, and ξ for $i = 1, \ldots, n$, defined on W, such that*

$$hX_i = \lambda X_i, \quad h\phi X_i = -\lambda \phi X_i, \quad h\xi = 0$$

for $i = 1, \ldots, n$, where $\lambda = \sqrt{1-\kappa}$.

By Lemma 4.1, we can take a local orthonormal frame $\{e_0 = \xi, e_1, \ldots, e_{2n}\}$ of M such that $e_{n+i} = \phi e_i$ and $he_i = \lambda e_i$ and $he_{n+i} = -\lambda e_{n+i}$ for $i = 1, 2, \ldots, n$.

If M admits a weakly Einstein metric, by (1) we have

$$\breve{R}(\xi, \xi) = \frac{1}{2n+1}|R|^2. \tag{10}$$

We first compute $\breve{R}(\xi, \xi)$. Since $h^2 = (\kappa - 1)\phi^2$ (see [14, Eq. (3.3)]), by (2) we obtain

$$\begin{aligned}\breve{R}(\xi, \xi) &= R_{ij0k}R_{ij0k} = [\kappa(g_{0j}g_{ik} - g_{0i}g_{jk}) + \mu(g_{0j}h_{ik} - g_{0i}h_{jk})]R_{ij0k}\\ &= 2\kappa R_{i00i} + 2\mu(R_{i00k}h_{ik})\\ &= 2\kappa R_{00} + 2\mu[\kappa(g_{ik} - g_{0i}g_{0k}) + \mu(h_{ik})]h_{ik}\\ &= 4n(\kappa^2 - \mu^2(\kappa - 1)).\end{aligned} \tag{11}$$

We can prove the following lemma.

Lemma 4.2.

$$\begin{aligned}|R|^2 = {}& 2\breve{R}(\xi, \xi) + R_{\alpha\beta\delta\gamma}R_{\alpha\beta\delta\gamma} + 4R_{\alpha\beta\delta A}R_{\alpha\beta\delta A} + 2R_{\alpha\beta AB}R_{\alpha\beta AB}\\ &+ 4R_{\alpha A\beta B}R_{\alpha A\beta B} + 4R_{AB\alpha C}R_{AB\alpha C} + R_{ABCD}R_{ABCD}.\end{aligned}$$

Proof. First, similar to the proof of Lemma 3.1 we derive

$$|R|^2 = R_{ijkl}R_{ijkl} = 2R_{ija0}R_{ija0} + R_{abcd}R_{abcd} = 2\breve{R}(\xi, \xi) + R_{abcd}R_{abcd}. \tag{12}$$

Moreover, we compute

$$\begin{aligned}R_{cdab}R_{cdab} &= R_{\alpha dab}R_{\alpha dab} + R_{Adab}R_{Adab}\\ &= R_{\alpha\beta ab}R_{\alpha\beta ab} + R_{\alpha Aab}R_{\alpha Aab} + R_{A\alpha ab}R_{A\alpha ab} + R_{ABab}R_{ABab}\\ &= R_{\alpha\beta\delta b}R_{\alpha\beta\delta b} + R_{\alpha\beta Ab}R_{\alpha\beta Ab} + 2(R_{\alpha A\beta b}R_{\alpha A\beta b} + R_{\alpha ABb}R_{\alpha ABb})\\ &\quad + R_{AB\alpha b}R_{AB\alpha b} + R_{ABCb}R_{ABCb}\\ &= R_{\alpha\beta\delta\gamma}R_{\alpha\beta\delta\gamma} + R_{\alpha\beta\delta A}R_{\alpha\beta\delta A} + R_{\alpha\beta A\delta}R_{\alpha\beta A\delta} + R_{\alpha\beta AB}R_{\alpha\beta AB}\\ &\quad + 2(R_{\alpha A\beta\delta}R_{\alpha A\beta\delta} + R_{\alpha A\beta B}R_{\alpha A\beta B} + R_{\alpha AB\beta}R_{\alpha AB\beta}\\ &\quad + R_{\alpha ABC}R_{\alpha ABC}) + R_{AB\alpha\beta}R_{AB\alpha\beta} + R_{AB\alpha C}R_{AB\alpha C}\end{aligned}$$

$$
\begin{aligned}
&+ R_{ABC\alpha}R_{ABC\alpha} + R_{ABCD}R_{ABCD}\\
=\ & R_{\alpha\beta\delta\gamma}R_{\alpha\beta\delta\gamma} + 4R_{\alpha\beta\delta A}R_{\alpha\beta\delta A} + 2R_{\alpha\beta AB}R_{\alpha\beta AB}\\
&+ 4R_{\alpha A\beta B}R_{\alpha A\beta B} + 4R_{AB\alpha C}R_{AB\alpha C} + R_{ABCD}R_{ABCD}.
\end{aligned}
$$

We complete the proof the lemma by substituting the above formula into (12). □

Proposition 4.3 ([4, Theorem 1])**.** *Let $M^{2n+1}(\phi,\eta,\xi,g)$ be a contact metric manifold with belonging to the (κ,μ)-nullity distribution. If $\kappa < 1$, M^{2n+1} admits three mutually orthogonal and integrable distributions $\mathcal{D}(0), \mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ determined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$. Moreover,*

$$
\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= (\kappa-\mu)[g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda],\\
R(X_{-\lambda}, Y_{-\lambda})Z_{\lambda} &= (\kappa-\mu)[g(\phi Y_{-\lambda}, Z_{\lambda})\phi X_{-\lambda} - g(\phi X_{-\lambda}, Z_{\lambda})\phi Y_{-\lambda}],\\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= \kappa g(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} + \mu g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda},\\
R(X_\lambda, Y_{-\lambda})Z_{\lambda} &= -\kappa g(\phi Y_{-\lambda}, Z_{\lambda})\phi X_{\lambda} - \mu g(\phi Y_{-\lambda}, X_{\lambda})\phi Z_{\lambda},\\
R(X_\lambda, Y_\lambda)Z_\lambda &= [2(1+\lambda)-\mu][g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= [2(1-\lambda)-\mu][g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}],
\end{aligned}
$$

where $X_\lambda, Y_\lambda, Z_\lambda \in \mathcal{D}(\lambda)$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in \mathcal{D}(-\lambda)$.

By Proposition 4.3, we can get

$$
\begin{aligned}
R_{\alpha\beta\delta\gamma} &= [2(1+\lambda)-\mu](g_{\beta\delta}g_{\alpha\gamma} - g_{\alpha\delta}g_{\beta\gamma}),\\
R_{\alpha\beta\delta A} &= 0,\\
R_{\alpha\beta AB} &= (\kappa-\mu)(g_{\overline{\beta}A}g_{\overline{\alpha}B} - g_{\overline{\alpha}A}g_{\overline{\beta}B}),\\
R_{\alpha A\beta B} &= -\kappa g_{\overline{A}\beta}g_{\overline{\alpha}B} - \mu g_{\overline{A}\alpha}g_{\overline{\beta}B},\\
R_{AB\alpha C} &= 0,\\
R_{ABCD} &= [2(1-\lambda)-\mu](g_{BC}g_{AD} - g_{AC}g_{BD}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
R^2_{\alpha\beta\delta\gamma} &= 2n(n-1)[2(1+\lambda)-\mu]^2,\\
R^2_{\alpha\beta\delta A} &= 0,\\
R^2_{\alpha\beta AB} &= 2n(n-1)(\kappa-\mu)^2,\\
R^2_{\alpha A\beta B} &= (\kappa^2+\mu^2)n^2 + 2n\kappa\mu,\\
R^2_{AB\alpha C} &= 0,\\
R^2_{ABCD} &= 2n(n-1)[2(1-\lambda)-\mu]^2.
\end{aligned}
$$

Therefore by Lemma 4.2 and (11) we conclude

$$
\begin{aligned}
|R|^2 &= 8n(\kappa^2 - (k-1)\mu^2) + 2n(n-1)[2(1+\lambda)-\mu]^2 + 4n(n-1)(\kappa-\mu)^2\\
&\quad + 4[(\kappa^2+\mu^2)n^2 + 2n\kappa\mu] + 2n(n-1)[2(1-\lambda)-\mu]^2. \qquad (13)
\end{aligned}
$$

Substituting (13) into (10) and using (11), we have

$$-(2n-1)\mu^2\kappa = (n-1)[4(1+\lambda^2)+\mu^2-4\mu]-2(n-2)\kappa\mu. \tag{14}$$

Now we divide into two cases to discuss.

Case I: $n = 1$. Then (14) implies $(\mu+2)\mu\kappa = 0$. If $\kappa = \mu = 0$, M is flat (see [3, Theorem 7.5]).

By Theorem 1.2, when $\kappa = 0, \mu \neq 0$, then $1+\lambda-\frac{\mu}{2} = 2-\frac{\mu}{2}, 1-\lambda-\frac{\mu}{2} = -\frac{\mu}{2}$, and M is locally isometric to the Lie group $SU(2)$, $SL(2,\mathbb{R})$ or $E(1,1)$.

When $0 \neq \kappa < 1$ and $\mu = 0$, we know $1+\lambda-\frac{\mu}{2} = 1+\lambda > 0$. When $0 \neq \kappa < 1$ and $\mu = -2$, then $1+\lambda-\mu/2 = 2+\lambda > 0$. Both cases imply that M is locally isometric to the Lie group $SU(2)$ or $SL(2,\mathbb{R})$ by Theorem 1.2.

Case II: $n > 1$. Since $\lambda^2 = 1-k$, it follows from (14) that

$$[-\mu^2(2n-1)+2\mu(n-2)+4(n-1)]\kappa = (n-1)[4+(\mu-2)^2]. \tag{15}$$

Moreover, when $0 < \kappa < 1$, we find

$$[-\mu^2(2n-1)+2\mu(n-2)+4(n-1)] > (n-1)[4+(\mu-2)^2].$$

That is,

$$(3n-2)\mu^2-2(3n-4)\mu+4(n-1) < 0.$$

Because $n > 1$, it is easy to prove that the above inequality has no solution.

When $\kappa < 0$, Equation (15) implies

$$-\mu^2(2n-1)+2\mu(n-2)+4(n-1) < 0,$$

that is,

$$\mu > \frac{n-2+\sqrt{9n^2-16n+8}}{2n-1} \quad \text{or} \quad \mu < \frac{n-2-\sqrt{9n^2-16n+8}}{2n-1}.$$

In particular, when $\mu < \frac{n-2-\sqrt{9n^2-16n+8}}{2n-1}$, we know $\mu < 0$ since $n > 1$. Hence the invariant I_M (see introduction) must be greater than -1. Therefore, we complete the proof by Theorem 1.3.

5. Proof of Theorem 1.5

In this section let us assume that $(M^{2n+1}, \phi, \eta, \xi, g)$ is an almost cosymplectic (κ, μ)-manifold, namely an almost cosymplectic manifold satisfies (2). First the following relations are provided (see [6, Eq. (3.22), (3.23)]):

$$h^2 = \kappa\phi^2, Q = \mu h + 2n\kappa\eta\otimes\xi. \tag{16}$$

In particular, $Q\xi = 2n\kappa\xi$ because of $h\xi = 0$. From (16), $trace(h^2) = -2n\kappa$. Furthermore, since $\kappa\phi^2 = h^2$, $\kappa \leq 0$ and the equality holds if and only if the almost cosymplectic (κ, μ)-manifolds are cosymplectic. Therefore, we will concentrate on the case $\kappa < 0$.

Since $trace(h^2) = -2n\kappa$, as the calculation of (11), making use of (2) we obtain

$$\breve{R}(\xi,\xi) = 4n(\kappa^2-\mu^2\kappa). \tag{17}$$

For an almost cosymplectic (κ,μ)-manifold with $\kappa<0$, we also have a similar lemma to Lemma 4.1.

Lemma 5.1. *Let $(M^{2n+1},\phi,\eta,\xi,g)$ be an almost cosymplectic (κ,μ)-manifold with $\kappa<0$. For every $p\in M$, there exist an open neighborhood W of p and orthonormal local vector fields $X_i,\phi X_i$, and ξ for $i=1,\ldots,n$, defined on W, such that*

$$hX_i=\lambda X_i,\quad h\phi X_i=-\lambda\phi X_i,\quad h\xi=0$$

for $i=1,\ldots,n$, where $\lambda=\sqrt{-\kappa}$.

Thus we can also take a local frame $\{e_i\}$ of M as in Section 4. In this section we will adopt the same index as Section 4. In the following we compute the square $|R|^2$ of curvature tensor R. In order to do that, we notice the following proposition.

Proposition 5.2 ([7, Theorem 3.7])**.** *Let M be an almost cosymplectic (κ,μ)-manifold of dimension greater than or equal to 5 with $\kappa<0$. Then its Riemann curvature tensor can be written as*

$$R=-\kappa R_3-R_{5,2}-\mu R_6,$$

where

$$\begin{aligned}R_3(X,Y)Z&=\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi,\\ R_6(X,Y)Z&=\eta(X)\eta(Z)hY-\eta(Y)\eta(Z)hX+g(hX,Z)\eta(Y)\xi\\ &\quad-g(hY,Z)\eta(X)\xi,\\ R_{5,2}(X,Y)Z&=g(\phi hY,Z)\phi hX-g(\phi hX,Z)\phi hY\end{aligned}$$

for any vector fields X,Y,Z.

In view of Lemma 5.1, $he_a=\lambda_a e_a$ with $\lambda_a=\pm\sqrt{-\kappa}$, thus by Proposition 5.2, we know

$$R_{abcd}=-(h_{b\bar c}h_{a\bar d}-h_{a\bar c}h_{b\bar d})=-\lambda_b\lambda_a[g_{b\bar c}g_{a\bar d}-g_{a\bar c}g_{b\bar d}],\tag{18}$$

where $h_{a\bar d}=g(he_a,\phi e_d)$ and $g_{b\bar c}=g(e_b,\phi e_c)$ for all $a,b,c,d\in\{1,2,\ldots,2n\}$.

Making use of (18), we have

$$\begin{aligned}&R_{\alpha\beta\delta\gamma}=0,\quad R_{\alpha\beta\delta A}=0,\\ &R_{\alpha\beta AB}=-\kappa(g_{\beta\overline{A}}g_{\alpha\overline{B}}-g_{\alpha\overline{A}}g_{\beta\overline{B}}),\\ &R_{\alpha A\beta B}=\kappa g_{A\overline{\beta}}g_{\alpha\overline{B}},\\ &R_{AB\alpha C}=0,\quad R_{ABCD}=0.\end{aligned}$$

Hence

$$\begin{aligned}&R^2_{\alpha\beta\delta\gamma}=0,\quad R^2_{\alpha\beta\delta A}=0,\\ &R^2_{\alpha\beta AB}=2n(n-1)\kappa^2,\\ &R^2_{\alpha A\beta B}=n^2\kappa^2,\end{aligned}$$

$$R^2_{AB\alpha C} = 0, \quad R^2_{ABCD} = 0.$$

Hence we derive from (17) and Lemma 4.2 that

$$|R|^2 = 8n(\kappa^2 - \mu^2\kappa) + 2\kappa^2 2n(n-1) + 4\kappa^2 n^2 = 4n[(2n+1)\kappa^2 - 2\mu^2\kappa].$$

By (10), we have

$$(\kappa^2 - \kappa\mu^2)(2n+1) = (2n+1)\kappa^2 - 2\mu^2\kappa.$$

This shows $\mu = 0$ since $\kappa < 0$.

We complete the proof Theorem 1.5 by Theorem 2.1.

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Xiaomin Chen
College of Science
China University of Petroleum-Beijing
Beijing 102249, P. R. China
Email address: xmchen@cup.edu.cn