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WEIGHTED MOORE–PENROSE INVERSES OF ADJOINTABLE OPERATORS ON INDEFINITE INNER-PRODUCT SPACES

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Abstract. Necessary and sufficient conditions are provided under which the weighted Moore–Penrose inverse $A^\dagger_{MN}$ exists, where A is an adjointable operator between Hilbert C^*-modules, and the weights M and N are only self-adjoint and invertible. Relationship between weighted Moore–Penrose inverses $A^\dagger_{MN}$ is clarified when A is fixed, whereas M and N are variable. Perturbation analysis for the weighted Moore–Penrose inverse is also provided.

1. Introduction and preliminaries

During the past decades, the weighted Moore–Penrose inverse (In brief, weighted M-P inverse) and its various applications have been intensely studied. When the weights M and N are both positive definite, the study of the weighted M-P inverse $A^\dagger_{MN}$ can be found in [3,9–11,15] for matrices, in [12] for Hilbert space operators, in [1,8] for elements in a C^*-algebra or in a Banach algebra, and in [13] for Hilbert C^*-module operators, respectively.

Some new phenomena may happen if the weights M and N are not positive definite (positive and invertible), since in this case the weighted spaces induced by M and N are usually indefinite. Along this direction, the weighted M-P inverse $A^\dagger_{MN}$ was generalized in [2] for Hilbert space operators to the case when M and N are only positive semi-definite, and in [4] for matrices when M and N are only Hermitian and nonsingular.

Before stating our results, let us recall some basic facts about Hilbert C^*-modules and introduce our notation; more details can be found e.g. in [6,7].

An inner product module over a C^*-algebra $\mathfrak{A}$ is a (right) $\mathfrak{A}$-module H equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle$, which is $\mathbb{C}$-linear and $\mathfrak{A}$-linear

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in the second variable and has the properties $\langle x,y\rangle^* = \langle y,x\rangle$ as well as $\langle x,x\rangle \geq 0$ with equality if and only if $x=0$. H is called a (right) Hilbert $\mathfrak{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$.

Suppose that H and K are two Hilbert $\mathfrak{A}$-modules, let $\mathcal{L}(H,K)$ be the set of operators $T: H\to K$ for which there is an operator $T^*: K\to H$ such that

$$\langle Tx,y\rangle = \langle x,T^*y\rangle \quad \text{for every } x\in H \text{ and } y\in K.$$

It is known that (see, e.g., [6, p. 8]) every element $T\in\mathcal{L}(H,K)$ must be a bounded linear operator, which is also $\mathfrak{A}$-linear. We call $\mathcal{L}(H,K)$ the set of adjointable operators from H to K. Note that some bounded linear operators between Hilbert C^*-modules cannot be adjointable (see [6, p. 8]).

For each $T\in\mathcal{L}(H,K)$, the range and the null space of T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. In case $H=K$, $\mathcal{L}(H,H)$ which is abbreviated to $\mathcal{L}(H)$, is a C^*-algebra. Let $\mathcal{L}(H)_+$ be the set of positive elements in $\mathcal{L}(H)$. The notation $T\geq 0$ is also used to indicate that T is an element of $\mathcal{L}(H)_+$.

Let H be a Hilbert $\mathfrak{A}$-module. If H_1 and H_2 are submodules of H such that $H_1\cap H_2=\{0\}$, then their direct sum is defined by

$$H_1 \dotplus H_2 = \{h_1+h_2 : h_i\in H_i,\ i=1,2\}\subseteq H.$$

Given a subset M of H, let $M^\perp = \big\{x\in H : \langle x,y\rangle = 0 \text{ for every } y\in M\big\}$. In the special case that M is a closed submodule of H and $H = M\dotplus M^\perp$, M is said to be orthogonally complemented in H.

Although Hilbert C^*-modules generalize Hilbert spaces by allowing inner products to take values in a certain C^*-algebra instead of the C^*-algebra of complex numbers, some fundamental properties of Hilbert spaces are no longer valid in Hilbert C^*-modules in their full generality. Therefore, when we are studying Hilbert C^*-modules, it is always of interest under which conditions the results analogous to those for Hilbert spaces can be reobtained, as well as which more general situations might appear.

In this paper, inspired by [4], we focus on the case that the weights are only self-adjoint and invertible, and study the weighted M-P inverse in the general setting of adjointable operators on Hilbert C^*-modules.

The paper is organized as follows. In Section 2 the existence and the uniqueness of the weighted M-P inverse are investigated. In the case when A is an adjointable operator and the weights M,N are only self-adjoint and invertible, some necessary and sufficient conditions are provided in Theorem 2.4 under which the weighted M-P inverse $A^\dagger_{MN}$ exists. Consequently, a generalization of both [4, Theorem 1] and [13, Theorem 1.3] is obtained. Two examples are also provided in Section 2 to illustrate certain new phenomena. In the case when A is fixed, the relationship between weighted M-P inverses $A^\dagger_{MN}$ for variable weights M and N is clarified in Section 3. Results obtained in [13, Section 2] and [15, Section 2] are then generalized, since all the weights considered in [13] and [15] are positive definite, whereas in Section 3 of this paper they are only needed to be self-adjoint and invertible. Finally, we study the perturbation analysis for the weighted Moore–Penrose inverse.

Throughout the rest of this paper, $\mathbb{C}^{m\times n}$ is the set of $m\times n$ complex matrices, I_n is the identity matrix in $\mathbb{C}^{n\times n}$, $\mathfrak{A}$ is a C^*-algebra, H and K are Hilbert $\mathfrak{A}$-modules, I_H (or simply I) is the identity operator on H.

2. Conditions of the existence of the weighted M-P inverse

The purpose of this section is, in the general setting of self-adjoint and invertible weights, to figure out necessary and sufficient conditions under which the weighted M-P inverse exists.

Lemma 2.1 (cf. [6, Theorem 3.2] and [14, Remark 1.1])**.** *Let $A\in\mathcal{L}(H,K)$. Then the closedness of any one of the following sets implies the closedness of the remaining three sets*:

$$\mathcal{R}(A),\quad \mathcal{R}(A^*),\quad \mathcal{R}(AA^*)\quad \textit{and}\quad \mathcal{R}(A^*A).$$

Furthermore, if $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A)=\mathcal{R}(AA^*)$, $\mathcal{R}(A^*)=\mathcal{R}(A^*A)$ and the following orthogonal decompositions are valid:

$$H=\mathcal{N}(A)\dotplus\mathcal{R}(A^*)\quad \textit{and}\quad K=\mathcal{R}(A)\dotplus\mathcal{N}(A^*).$$

Definition. An element M of $\mathcal{L}(K)$ is said to be a weight if $M=M^*$ and M is invertible in $\mathcal{L}(K)$. If furthermore M is positive, then M is said to be positive definite.

Definition. Let $M\in\mathcal{L}(K)$ be a weight. The indefinite inner-product on K induced by M is given by

$$\langle x,y\rangle_M=\langle x,My\rangle\quad \text{for every } x,y\in K.$$

Lemma 2.2 (cf. [13, Remark 1.1])**.** *Let $M\in\mathcal{L}(K)$ and $N\in\mathcal{L}(H)$ be two weights. Then for each $T\in\mathcal{L}(H,K)$, it has*

$$\langle Tx,y\rangle_M=\langle x,T^{\#}y\rangle_N\quad \textit{for every } x\in H \textit{ and } y\in K,$$

where $T^{\#}$ is called the weighted adjoint operator of T and is given by

$$T^{\#}=N^{-1}T^*M\in\mathcal{L}(K,H). \tag{1}$$

Definition. Let $M\in\mathcal{L}(K)$ and $N\in\mathcal{L}(H)$ be weights and $A\in\mathcal{L}(H,K)$. The weighted M-P inverse $A^{\dagger}_{MN}$ (if it exists) is the element X of $\mathcal{L}(K,H)$ which satisfies

$$AXA=A,\quad XAX=X,\quad (MAX)^*=MAX\quad \text{and}\quad (NXA)^*=NXA. \tag{2}$$

If $M=I_K$ and $N=I_H$, then $A^{\dagger}_{MN}$ is denoted simply by $A^{\dagger}$ and is called the M-P inverse of A.

The following lemma indicates that the weighted M-P inverse is unique whenever it exists.

Lemma 2.3. *Let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be weights, and $A \in \mathcal{L}(H, K)$. If $A^\dagger_{MN}$ exists, then it is unique.*

Proof. Put $P = AA^\dagger_{MN}$ and $Q = A^\dagger_{MN}A$. Then from (2) and Lemma 2.2, both P and Q are idempotent such that

$$\mathcal{R}(P) = \mathcal{R}(A), \quad P^\# = P \quad \text{and} \quad Q^\# = Q.$$

If (2) is satisfied for any $X \in \mathcal{L}(K, H)$, then both $P' = AX$ and $Q' = XA$ are also idempotent such that

$$\mathcal{R}(P') = \mathcal{R}(A), \quad (P')^\# = P' \quad \text{and} \quad (Q')^\# = Q'.$$

It follows that $\mathcal{R}(P) = \mathcal{R}(P')$, hence $PP' = P'$ and $P'P = P$, which yield

$$P = P^\# = (P'P)^\# = P^\#(P')^\# = PP' = P'.$$

In addition, we have

$$\mathcal{R}(Q) = \mathcal{R}(Q^\#) = \mathcal{R}\big(A^\#(A^\dagger_{MN})^\#\big) = \mathcal{R}(A^\#) = \mathcal{R}(A^\# X^\#) = \mathcal{R}(Q'),$$

which leads to $Q = Q'$ as illustrated before. Therefore,

$$A^\dagger_{MN} = A^\dagger_{MN}P = A^\dagger_{MN}P' = QX = Q'X = X. \qquad \square$$

Now we present the main result of this section as follows:

Theorem 2.4. *Let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be weights, and $A \in \mathcal{L}(H, K)$. Then $A^\dagger_{MN}$ exists if and only if the following conditions are all satisfied*:

(i) $\mathcal{R}(A)$ *is closed*;
(ii) $\mathcal{R}(AN^{-1}A^*) = \mathcal{R}(A)$;
(iii) $\mathcal{R}(A^*MA) = \mathcal{R}(A^*)$.

Proof. "$\Longrightarrow$": For simplicity, we put $X = A^\dagger_{MN}$. Then $\mathcal{R}(A) = \mathcal{R}(AX)$ is closed. Also, since XA is idempotent, H can be decomposed directly as $H = \mathcal{R}(XA) \dotplus \mathcal{N}(XA)$, which leads furthermore to the direct decomposition of H as

$$H = \mathcal{R}(N^{-1}A^*) \dotplus \mathcal{N}(A). \tag{3}$$

Indeed, the equalities of $\mathcal{N}(XA) = \mathcal{N}(A)$ and $\mathcal{R}(A^*X^*) = \mathcal{R}(A^*)$ can be derived clearly from the first equation in (2). We can then use the last equation in (2) together with the invertibility of N to conclude that

$$\mathcal{R}(XA) = \mathcal{R}(N^{-1}A^*X^*N) = \mathcal{R}\big(N^{-1}(A^*X^*)\big) = \mathcal{R}(N^{-1}A^*).$$

It follows directly from (3) that $\mathcal{R}(A) \subseteq \mathcal{R}(AN^{-1}A^*)$, which can happen only if $\mathcal{R}(A) = \mathcal{R}(AN^{-1}A^*)$. The proof of $\mathcal{R}(A^*MA) = \mathcal{R}(A^*)$ is similar.

"$\Longleftarrow$": Suppose that conditions (i)–(iii) are all satisfied. In what follows, we construct an operator $X \in \mathcal{L}(K, H)$ which satisfies (2).

Firstly, we provide the direct decompositions of K and H, respectively. Given every $x \in K$, by item (iii) we have $A^*Mx = A^*MAu$ for some $u \in H$, then $x - Au \in \mathcal{N}(A^*M)$, therefore $x = Au + (x - Au) \in \mathcal{R}(A) + \mathcal{N}(A^*M)$. By the arbitrariness of x in K, we have

$$K = \mathcal{R}(A) + \mathcal{N}(A^*M). \tag{4}$$

Furthermore, given every $w \in \mathcal{R}(A) \cap \mathcal{N}(A^*M)$, we have $A^*Mw = 0$ and $w = Av$ for some $v \in H$, hence $v \in \mathcal{N}(A^*MA)$. By item (i) and Lemma 2.1, we know that $\mathcal{R}(A^*)$ is also closed such that $\mathcal{R}(A^*)^\perp = \mathcal{N}(A)$. So $\mathcal{R}(A^*MA)$ is also closed by item (iii), hence by Lemma 2.1 once again we have

$$v \in \mathcal{N}(A^*MA) = \mathcal{R}(A^*MA)^\perp = \mathcal{R}(A^*)^\perp = \mathcal{N}(A),$$

which gives $w = Av = 0$. This shows that $\mathcal{R}(A) \cap \mathcal{N}(A^*M) = \{0\}$. Hence from (4) and item (ii), K can be decomposed directly as

$$K = \mathcal{R}(AN^{-1}A^*) \dotplus \mathcal{N}(A^*M). \tag{5}$$

Similarly, H can be decomposed directly as

$$H = \mathcal{R}(N^{-1}A^*MA) \dotplus \mathcal{N}(A). \tag{6}$$

Secondly, we construct two operators X and Y based on the obtained direct decompositions. Let $X : K \to H$ be given by

$$X(AN^{-1}A^*u_1 + u_2) = N^{-1}A^*u_1 \quad \text{for every } u_1 \in K, u_2 \in \mathcal{N}(A^*M). \tag{7}$$

In view of (5) and

$$\mathcal{N}(AN^{-1}A^*) = \mathcal{R}(AN^{-1}A^*)^\perp = \mathcal{R}(A)^\perp = \mathcal{N}(A^*) = \mathcal{N}(N^{-1}A^*),$$

we know that X is well-defined. It follows from (7) and (1) that

$$\mathcal{R}(X) = \mathcal{R}(N^{-1}A^*) = \mathcal{R}(A^\#) \quad \text{and} \quad \mathcal{N}(X) = \mathcal{N}(A^*M) = \mathcal{N}(A^\#). \tag{8}$$

Similarly, the operator $Y : H \to K$ defined by

$$Y(N^{-1}A^*MAv_1 + v_2) = Av_1 \quad \text{for every } v_1 \in H, v_2 \in \mathcal{N}(A) \tag{9}$$

is also well-defined such that

$$\mathcal{R}(Y) = \mathcal{R}(A) \quad \text{and} \quad \mathcal{N}(Y) = \mathcal{N}(A).$$

Thirdly, we show that the constructed operator X is adjointable. To this end, we show that

$$\langle Xu, v\rangle_N = \langle u, Yv\rangle_M \quad \text{for every } u \in K \text{ and } v \in H.$$

In fact, for $u \in K$ and $v \in H$, according to (5) and (6) we know that

$$u = AN^{-1}A^*u_1 + u_2 \quad \text{and} \quad v = N^{-1}A^*MAv_1 + v_2 \tag{10}$$

for some $u_1 \in K$, $u_2 \in \mathcal{N}(A^*M)$, $v_1 \in H$ and $v_2 \in \mathcal{N}(A)$. Therefore, from (7) and (9) we have

$$\begin{aligned}\langle Xu, v\rangle_N &= \langle N^{-1}A^*u_1, N^{-1}A^*MAv_1 + v_2\rangle_N \\ &= \langle N^{-1}A^*u_1, A^*MAv_1 + Nv_2\rangle \\ &= \langle N^{-1}A^*u_1, A^*MAv_1\rangle + \langle u_1, Av_2\rangle\end{aligned}$$

$$\begin{aligned}
&= \langle MAN^{-1}A^*u_1, Av_1\rangle \\
&= \langle MAN^{-1}A^*u_1, Av_1\rangle + \langle A^*Mu_2, v_1\rangle \\
&= \langle MAN^{-1}A^*u_1 + Mu_2, Av_1\rangle \\
&= \langle AN^{-1}A^*u_1 + u_2, Av_1\rangle_M \\
&= \langle u, Yv\rangle_M.
\end{aligned}$$

Now, we put

$$X^* := MYN^{-1}. \tag{11}$$

Note that $\mathcal{R}(N) = H$ and for every $u \in K$ and $v \in H$, by (11), we have

$$\langle Xu, Nv\rangle = \langle Xu, v\rangle_N = \langle u, Yv\rangle_M = \langle u, MYN^{-1}Nv\rangle = \langle u, X^*Nv\rangle,$$

which shows that X^* is the adjoint operator of X.

Finally, we prove that X is exactly the weighted M-P inverse $A^\dagger_{MN}$. Note that every $v \in H$ can be decomposed as (10) such that $v_1 \in H$ and $v_2 \in \mathcal{N}(A)$, so by (7), (11) and (9) we have

$$AXAv = AX(AN^{-1}A^*MAv_1) = A(N^{-1}A^*MAv_1) = Av$$

and

$$(NXA)^*v = A^*MYv = A^*MAv_1 = NN^{-1}A^*MAv_1 = NXAv.$$

This completes the proof that $AXA = A$ and $(NXA)^* = NXA$.

Let $u \in K$ be any given by (10) such that $u_1 \in K$ and $u_2 \in \mathcal{N}(A^*M)$. Then since $u_2 \in \mathcal{N}(A^*M)$, we have

$$A^*Mu = A^*MAN^{-1}A^*u_1. \tag{12}$$

Similarly, from (7), (11), (12) and (9) we can obtain

$$Xu = N^{-1}A^*u_1 = XA(N^{-1}A^*u_1) = XAXu$$

and

$$(MAX)^*u = MYN^{-1}A^*MAN^{-1}A^*u_1 = MAN^{-1}A^*u_1 = MAXu.$$

This completes the proof that $XAX = X$ and $(MAX)^* = MAX$. □

Remark 2.5. Let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be weights, and $A \in \mathcal{L}(H, K)$ be given such that $A^\dagger_{MN}$ exists. We can see from (8) that

$$\mathcal{R}(A^\dagger_{MN}) = \mathcal{R}(N^{-1}A^*) = \mathcal{R}(A^\#)$$

and

$$\mathcal{N}(A^\dagger_{MN}) = \mathcal{N}(A^*M) = \mathcal{N}(A^\#). \tag{13}$$

Moreover, items (ii) and (iii) in Theorem 2.4 can be rephrased as

$$\mathcal{R}(AA^\#) = \mathcal{R}(A) \quad \text{and} \quad \mathcal{R}(A^\#A) = \mathcal{R}(A^\#). \tag{14}$$

Indeed, $\mathcal{R}(AN^{-1}A^*) = \mathcal{R}(AN^{-1}A^*M) = \mathcal{R}(AA^{\#})$, and

$$\begin{aligned}\mathcal{R}(A^*MA) = \mathcal{R}(A^*) &\Longleftrightarrow \mathcal{R}(N^{-1}A^*MA) = \mathcal{R}(N^{-1}A^*M)\\ &\Longleftrightarrow \mathcal{R}(A^{\#}A) = \mathcal{R}(A^{\#}).\end{aligned}$$

If both H and K are finite-dimensional spaces, then $\mathcal{R}(A)$ is always closed and $\mathrm{rank}(A^{\#}) = \mathrm{rank}(A)$, so (14) can be reduced to

$$\mathrm{rank}(AA^{\#}) = \mathrm{rank}(A^{\#}A) = \mathrm{rank}(A). \tag{15}$$

In view of the observation above, we have the following corollary.

Corollary 2.6 ([4, Theorem 1])**.** *Let $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be weights, and $A \in \mathbb{C}^{m\times n}$. Then $A^{\dagger}_{MN}$ exists if and only if* (15) *is satisfied.*

Theorem 2.4 can also be simplified in the infinite-dimensional case if the weights M and N are both positive definite.

Corollary 2.7 ([13, Theorem 1.3])**.** *Let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be both positive definite. Then for every $A \in \mathcal{L}(H,K)$, $A^{\dagger}_{MN}$ exists if and only if $\mathcal{R}(A)$ is closed.*

Proof. It needs only to prove the sufficiency. Let $T = AN^{-\frac{1}{2}}$. It is clear that $AN^{-1}A^* = TT^*$ and $\mathcal{R}(T) = \mathcal{R}(A)$, which means by Lemma 2.1 that $\mathcal{R}(AN^{-1}A^*)$ is closed whenever $\mathcal{R}(A)$ is closed. Similarly,

$$\mathcal{R}(A^*MA) \text{ is closed} \Longleftrightarrow \mathcal{R}(A^*) \text{ is closed} \Longleftrightarrow \mathcal{R}(A) \text{ is closed.} \qquad \square$$

Another special case of Theorem 2.4 is as follows.

Corollary 2.8. *Let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be weights, and $A \in \mathcal{L}(H,K)$. If both M and N commute with A, then $A^{\dagger}_{MN}$ exists if and only if $\mathcal{R}(A)$ is closed.*

Proof. Since both M and N commute with A, we have

$$\mathcal{R}(AN^{-1}A^*) = \mathcal{R}(AA^*N^{-1}) = \mathcal{R}(AA^*)$$

and

$$\mathcal{R}(A^*MA) = \mathcal{R}(A^*AM) = \mathcal{R}(A^*A).$$

So if $\mathcal{R}(A)$ is closed, then items (ii) and (iii) of Theorem 2.4 will be satisfied by Lemma 2.1. $\square$

There exists a weighted M-P inverse $A^{\dagger}_{MN}$ such that neither M nor N is positive definite, and also neither M nor N commutes with A. Such an example in the infinite-dimensional case is as follows:

Example 2.9. Let S be the unilateral shift on the Hilbert space $\ell^2(\mathbb{N})$, that is, $Se_n = e_{n+1}$ for every $n \in \mathbb{N}$, where $\{e_n : n \in \mathbb{N}\}$ is the orthonormal basis of $\ell^2(\mathbb{N})$. Put $\mathfrak{A} = \mathbb{C}, H = K = \ell^2(\mathbb{N})$ and $A = S$. Then $A^*A = I$ and

$AA^* = I - P_1$ is a diagonal operator, where P_1 is the projection from H onto its linear subspace spanned by e_1. In particular, $\mathcal{R}(A)$ is closed.

Given positive numbers c_1, c_2, d_1 and d_2, and two sequences $\{a_n\}$ and $\{b_n\}$ taken in the real line such that

$$c_1 \le |a_n| \le d_1 \quad \text{and} \quad c_2 \le |b_n| \le d_2 \quad \text{for every } n \in \mathbb{N},$$

let $M, N \in \mathcal{L}(H)$ be diagonal operators determined by

$$Me_n = a_n e_n \quad \text{and} \quad Ne_n = b_n e_n \quad \text{for every } n \in \mathbb{N}.$$

Then both M and N are self-adjoint and invertible, whereas M fails to be positive if there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} < 0$. The same is true for N.

Since $A^*A = I$, it is obvious that

$$AA^*A = A,\ A^*AA^* = A^* \quad \text{and} \quad (NA^*A)^* = NA^*A.$$

Also, we have $(MAA^*)^* = MAA^*$, since both M and AA^* are diagonal and self-adjoint. Therefore, by (2) we know that $A^\dagger_{MN} = A^*$.

Remark 2.10. Let $A \in \mathcal{L}(H, K)$ be given such that $\mathcal{R}(A)$ is closed. Put

$$W(A) = \Big\{ M \in \mathcal{L}(K) : M \text{ is a weight such that } \mathcal{R}(A^*MA) = \mathcal{R}(A^*) \Big\}.$$

Clearly, a weight M in $\mathcal{L}(K)$ is a member of $W(A)$ if and only if $\mathcal{R}(A^*MA)$ is closed and $A^*MA(A^*MA)^\dagger = A^*(A^*)^\dagger$. Assume that $\{M_n\}$ is a sequence taken in $W(A)$ such that $M_n \to M$ as $n \to +\infty$. Then obviously, M is self-adjoint and $A^*M_nA \to A^*MA$. If M is also invertible and $\mathcal{R}(A^*MA)$ is closed, then since

$$A^*M_nA(A^*M_nA)^\dagger = A^*(A^*)^\dagger \quad \text{for every } n \in \mathbb{N},$$

we know from [5, Theorem 1.6] that

$$\begin{aligned} M \in W(A) &\Longleftrightarrow A^*MA(A^*MA)^\dagger = A^*(A^*)^\dagger \\ &\Longleftrightarrow \lim_{n\to\infty} A^*M_nA(A^*M_nA)^\dagger = A^*MA(A^*MA)^\dagger \\ &\Longleftrightarrow \sup\big\{ \big\|(A^*M_nA)^\dagger\big\| : n \in \mathbb{N} \big\} < +\infty. \end{aligned} \tag{16}$$

An example can be constructed as follows, in which (16) is not satisfied.

Example 2.11. Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$, $N = I_2$ and put

$$M_n = \begin{pmatrix} \frac{1}{n} & 2 \\ 2 & \frac{1}{n} \end{pmatrix} \quad \text{and} \quad N_n = I_2 \quad \text{for every } n \in \mathbb{N}.$$

Then $\mathcal{R}(AN_n^{-1}A^*) = \mathcal{R}(A)$ and $\mathcal{R}(A^*M_nA) = \mathcal{R}(A^*) = \mathbb{C} \oplus \{0\}$, therefore $A^\dagger_{M_nN_n}$ exists for each $n \in \mathbb{N}$, whereas $A^\dagger_{MN}$ does not exist, since the supremum in (16) turns out to be $+\infty$.

3. The relationship between weighted M-P inverses

Unless otherwise specified, in this section $A \in \mathcal{L}(H,K)$ is fixed such that $\mathcal{R}(A)$ is closed, $M, M_1, M_2 \in \mathcal{L}(K)$ and $N, N_1, N_2 \in \mathcal{L}(H)$ are weights such that all the weighted M-P inverses exist.

We begin with an auxiliary lemma as follows:

Lemma 3.1. *Let $P, Q \in \mathcal{L}(H)$ be two idempotents. Then $P = Q$ whenever $\mathcal{R}(Q) \subseteq \mathcal{R}(P)$ and $\mathcal{N}(Q) \subseteq \mathcal{N}(P)$.*

Proof. Suppose that $\mathcal{R}(Q) \subseteq \mathcal{R}(P)$ and $\mathcal{N}(Q) \subseteq \mathcal{N}(P)$. Then $PQ = Q$ since $\mathcal{R}(Q) \subseteq \mathcal{R}(P)$, and $P(I-Q) = 0$ since $\mathcal{R}(I-Q) = \mathcal{N}(Q) \subseteq \mathcal{N}(P)$. Therefore, $0 = P(I-Q) = P - PQ = P - Q$, and hence $P = Q$. □

To clarify the relationship between weighted M-P inverses, we need two lemmas:

Lemma 3.2. *The following equations for the weighted M-P inverse are valid*:

$$AA^\dagger_{MN_1} = AA^\dagger_{MN_2} \quad \text{and} \quad A^\dagger_{M_1N}A = A^\dagger_{M_2N}A.$$

Proof. Clearly, $\mathcal{R}(AA^\dagger_{MN_1}) = \mathcal{R}(A) = \mathcal{R}(AA^\dagger_{MN_2})$, and by (13) we have

$$\mathcal{N}(AA^\dagger_{MN_1}) = \mathcal{N}(A^\dagger_{MN_1}) = \mathcal{N}(A^*M) = \mathcal{N}(A^\dagger_{MN_2}) = \mathcal{N}(AA^\dagger_{MN_2}).$$

By Lemma 3.1 we can obtain $AA^\dagger_{MN_1} = AA^\dagger_{MN_2}$, since both of them are idempotents. Similarly, it can be shown that $A^\dagger_{M_1N}A = A^\dagger_{M_2N}A$. □

Lemma 3.3. *The following equations for the weighted M-P inverse are valid*:

$$(I - A^\dagger_{MN_1}A)N_1^{-1}N_2A^\dagger_{MN_2}A = 0, \tag{17}$$

$$AA^\dagger_{M_2N}M_2^{-1}M_1(I - AA^\dagger_{M_1N}) = 0. \tag{18}$$

Proof. Let $\Omega = A^\dagger_{MN_1}AN_1^{-1}N_2A^\dagger_{MN_2}A$. By (2) we have

$$N_2A^\dagger_{MN_2}A = (A^\dagger_{MN_2}A)^*N_2 \quad \text{and} \quad A^\dagger_{MN_1}AN_1^{-1} = N_1^{-1}(A^\dagger_{MN_1}A)^*,$$

which lead to

$$\begin{aligned}\Omega &= N_1^{-1}(A^\dagger_{MN_1}A)^*(A^\dagger_{MN_2}A)^*N_2\\ &= N_1^{-1}\Big(A^\dagger_{MN_2}(AA^\dagger_{MN_1}A)\Big)^*N_2\\ &= N_1^{-1}(A^\dagger_{MN_2}A)^*N_2\\ &= N_1^{-1}N_2A^\dagger_{MN_2}A.\end{aligned}$$

This completes the proof of (17). The proof of (18) is similar. □

The relationship between $A^\dagger_{MN_1}$ and $A^\dagger_{MN_2}$ can be described as follows.

Theorem 3.4. *The following equation for the weighted M-P inverse is valid*:

$$A^{\dagger}_{MN_1} = R_{M;N_1,N_2} \cdot A^{\dagger}_{MN_2},$$

where

$$R_{M;N_1,N_2} = A^{\dagger}_{MN_1}A + (I - A^{\dagger}_{MN_1}A)N_1^{-1}N_2. \tag{19}$$

Furthermore, $R_{M;N_1,N_2}$ *is invertible in* $\mathcal{L}(H)$ *if and only if*

$$\mathcal{R}\Big((I - A^{\dagger}_{MN_1}A)^* N_2 (I - A^{\dagger}_{MN_1}A)\Big) = \mathcal{R}\big((I - A^{\dagger}_{MN_1}A)^*\big). \tag{20}$$

Proof. It follows from (19) and (17) that

$$R_{M;N_1,N_2} \cdot A^{\dagger}_{MN_2}A = A^{\dagger}_{MN_1}(AA^{\dagger}_{MN_2}A) = A^{\dagger}_{MN_1}A,$$

which is combined with Lemma 3.2 to conclude that

$$\begin{aligned} R_{M;N_1,N_2} \cdot A^{\dagger}_{MN_2} &= R_{M;N_1,N_2} \cdot A^{\dagger}_{MN_2}AA^{\dagger}_{MN_2} \\ &= R_{M;N_1,N_2} \cdot A^{\dagger}_{MN_2}AA^{\dagger}_{MN_1} \\ &= A^{\dagger}_{MN_1}AA^{\dagger}_{MN_1} = A^{\dagger}_{MN_1}. \end{aligned}$$

Let $P = A^{\dagger}_{MN_1}A$, $H_1 = \mathcal{R}(P)$ and $H_2 = \mathcal{R}(I-P)$. Then P is an idempotent and $H = H_1 \dotplus H_2$. Hence $R_{M;N_1,N_2}$ can be partitioned as

$$R_{M;N_1,N_2} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \begin{pmatrix} I_{H_1} & 0 \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix},$$

where

$$R_{22} = (I - P)N_1^{-1}N_2(I - P)\big|_{H_2} \tag{21}$$

and

$$R_{21} = (I - P)N_1^{-1}N_2P\big|_{H_1}.$$

It follows that

$$R_{M;N_1,N_2} \in \mathcal{L}(H) \text{ is invertible} \Longleftrightarrow R_{22} : H_2 \to H_2 \text{ is a bijection.}$$

In addition, by (2) we have $N_1PN_1^{-1} = P^*$ and thus

$$\mathcal{R}\big(N_1(I - P)\big) = \mathcal{R}\big(N_1(I - P)N_1^{-1}\big) = \mathcal{R}\big((I - P)^*\big),$$

which means by the invertibility of N_1 that the morphism $N_1|_{H_2} : H_2 \to \mathcal{R}\big((I-P)^*\big)$ is a bijection. Therefore, R_{22} is a bijection if and only if $N_1|_{H_2} \cdot R_{22}$ is a bijection. Since $\mathcal{R}(R_{22}) \subseteq H_2$, we know from (21) that

$$N_1|_{H_2} \cdot R_{22} = N_1R_{22} = (I - P)^*N_2(I - P)\big|_{H_2}.$$

It follows that $R_{22} : H_2 \to H_2$ is a bijection if and only if

$$T \overset{def}{=} (I - P)^*N_2(I - P)\big|_{H_2} : H_2 \to \mathcal{R}\big((I - P)^*\big) \text{ is a bijection.} \tag{22}$$

Suppose that (22) is satisfied. Then (20) is valid, since it is obvious that

$$\mathcal{R}\big((I - P)^*N_2(I - P)\big) = \mathcal{R}\Big((I - P)^*N_2(I - P)\big|_{H_2}\Big).$$

Conversely, if (20) holds, then the operator T defined by (22) is surjective. Note that $(I-P)^*$ is an idempotent, so $\mathcal{R}\big((I-P)^*\big)$ is closed, therefore Lemma 2.1 and (20) yield

$$\mathcal{N}(I-P)=\mathcal{R}\big((I-P)^*\big)^{\perp}=\mathcal{R}\big((I-P)^*N_2(I-P)\big)^{\perp}=\mathcal{N}\big((I-P)^*N_2(I-P)\big),$$

which clearly leads to the injectivity of T. □

The relationship between $A^{\dagger}_{M_1N}$ and $A^{\dagger}_{M_2N}$ can be described as follows.

Theorem 3.5. *The following equation for the weighted M-P inverse is valid*:

$$A^{\dagger}_{M_1N}=A^{\dagger}_{M_2N}\cdot L_{M_1,M_2;N},$$

where

$$L_{M_1,M_2;N}=AA^{\dagger}_{M_1N}+M_2^{-1}M_1(I-AA^{\dagger}_{M_1N}), \tag{23}$$

which is invertible in $\mathcal{L}(H)$ *if and only if*

$$\mathcal{R}\Big((I-AA^{\dagger}_{M_1N})^*\cdot M_1M_2^{-1}M_1\cdot(I-AA^{\dagger}_{M_1N})\Big)=\mathcal{R}\big((I-AA^{\dagger}_{M_1N})^*\big). \tag{24}$$

Proof. Note that $A^{\dagger}_{M_2N}=A^{\dagger}_{M_2N}AA^{\dagger}_{M_2N}$, so from (18) we have

$$A^{\dagger}_{M_2N}M_2^{-1}M_1(I-AA^{\dagger}_{M_1N})=0.$$

The equation above, together with (23) and Lemma 3.2, yields

$$A^{\dagger}_{M_2N}\cdot L_{M_1,M_2;N}=(A^{\dagger}_{M_2N}A)A^{\dagger}_{M_1N}=(A^{\dagger}_{M_1N}A)A^{\dagger}_{M_1N}=A^{\dagger}_{M_1N}.$$

As in the proof of Theorem 3.4, it can be shown that $L_{M_1,M_2;N}$ is invertible in $\mathcal{L}(H)$ if and only if (24) is satisfied. □

Based on Theorems 3.4 and 3.5, we can obtain the following result.

Corollary 3.6. *The following equation for the weighted M-P inverse is valid*:

$$A^{\dagger}_{M_1N_1}=R_{M_2;N_1,N_2}\cdot A^{\dagger}_{M_2N_2}\cdot L_{M_1,M_2;N_2}, \tag{25}$$

where $R_{M_1;N_1,N_2}$ *and* $L_{M_1,M_2;N_1}$ *are defined by* (19) *and* (23), *respectively.*

Proof. By (19), (23) and Lemma 3.2, we have

$$R_{M_1;N_1,N_2}=R_{M_2;N_1,N_2}\quad\text{and}\quad L_{M_1,M_2;N_1}=L_{M_1,M_2;N_2}.$$

Thus we can apply Lemmas 3.4 and 3.5 to get

$$\begin{aligned}A^{\dagger}_{M_1N_1}&=R_{M_1;N_1,N_2}\cdot A^{\dagger}_{M_1N_2}=R_{M_1;N_1,N_2}\cdot A^{\dagger}_{M_2N_2}\cdot L_{M_1,M_2;N_2}\\&=R_{M_2;N_1,N_2}\cdot A^{\dagger}_{M_2N_2}\cdot L_{M_1,M_2;N_2}.\end{aligned}$$

□

Remark 3.7. Suppose that $M_2\in\mathcal{L}(K)$ and $N_2\in\mathcal{L}(H)$ are both positive definite. Let

$$P=A^{\dagger}_{MN_1}A,\quad T=(I-P)^*N_2^{\frac{1}{2}},\quad Q=AA^{\dagger}_{M_1N}\quad\text{and}\quad S=(I-Q)^*M_1M_2^{-\frac{1}{2}}.$$

Then by Lemma 2.1, we have

$$\mathcal{R}\Big((I-P)^*N_2(I-P)\Big) = \mathcal{R}(TT^*) = \mathcal{R}(T) = \mathcal{R}\big((I-P)^*\big)$$

and

$$\mathcal{R}\Big((I-Q)^*M_1M_2^{-1}M_1(I-Q)\Big) = \mathcal{R}(SS^*) = \mathcal{R}(S) = \mathcal{R}\big((I-Q)^*\big).$$

Thus, by Theorems 3.4 and 3.5 we know that both $R_{M;N_1,N_2}$ and $L_{M_1,M_2;N}$ are invertible in $\mathcal{L}(H)$.

Remark 3.8. There certainly exist $M, M_1, M_2 \in \mathcal{L}(K)$ and $N, N_1, N_2 \in \mathcal{L}(H)$ such that neither $R_{M;N_1,N_2}$ nor $L_{M_1,M_2;N}$ is invertible. Such an example is as follows.

Example 3.9. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, M = N = I_2, M_1 = N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $M_2 = N_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It is easy to verify that both $A^\dagger_{MN_1}$ and $A^\dagger_{M_1N}$ are equal to $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$. So according to (19) and (23), we have

$$R_{M;N_1,N_2} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L_{M_1,M_2;N} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

both of which are singular.

In the above presentation, it is assumed that all the weighted M-P inverses exist. In view of Theorems 3.4 and 3.5, a slight generalization can be made as follows.

Theorem 3.10. *Let $M \in \mathcal{L}(K)$ and $N_1, N_2 \in \mathcal{L}(H)$ be weights. If $A \in \mathcal{L}(H,K)$ is given such that $A^\dagger_{MN_1}$ exists and the operator $R_{M;N_1,N_2}$ defined by* (19) *is invertible, then $A^\dagger_{MN_2}$ exists and is of the form*

$$A^\dagger_{MN_2} = R^{-1}_{M;N_1,N_2} \cdot A^\dagger_{MN_1}.$$

Proof. Let $X = R^{-1}_{M;N_1,N_2} \cdot A^\dagger_{MN_1}$. Then X is adjointable, since both $R^{-1}_{M;N_1,N_2}$ and $A^\dagger_{MN_1}$ are adjointable. It follows from (19) that $AR_{M;N_1,N_2} = A$, hence $AR^{-1}_{M;N_1,N_2} = A$, which gives directly $AXA = A$, $XAX = X$ and $MAX = MAA^\dagger_{MN_1}$, therefore $(MAX)^* = MAX$. Furthermore, by (19) and (2), we have

$$N_1R_{M;N_1,N_2} = N_1A^\dagger_{MN_1}A + N_1(I - A^\dagger_{MN_1}A)N_1^{-1}N_2,$$

$$(N_1R_{M;N_1,N_2})^* = N_1A^\dagger_{MN_1}A + N_2(I - A^\dagger_{MN_1}A),$$

which lead to

$$\begin{aligned}(N_1R_{M;N_1,N_2})N_2^{-1}N_1A^\dagger_{MN_1}A &= N_1A^\dagger_{MN_1}AN_2^{-1}N_1A^\dagger_{MN_1}A \\ &= N_1A^\dagger_{MN_1}AN_2^{-1}(N_1R_{M;N_1,N_2})^*.\end{aligned}$$

It follows that

$$\begin{aligned} N_2^{-1}N_1A^{\dagger}_{MN_1}A\cdot\big((N_1R_{M;N_1,N_2})^{-1}\big)^* &= (N_1R_{M;N_1,N_2})^{-1}\cdot N_1A^{\dagger}_{MN_1}AN_2^{-1}\\ &= R^{-1}_{M;N_1,N_2}\cdot A^{\dagger}_{MN_1}AN_2^{-1}, \end{aligned}$$

and thus

$$(N_1A^{\dagger}_{MN_1}A)^*\cdot\big((N_1R_{M;N_1,N_2})^{-1}\big)^*N_2^* = N_2R^{-1}_{M;N_1,N_2}A^{\dagger}_{MN_1}A,$$

which can obviously be simplified to $(N_2XA)^* = N_2XA$. □

Similarly, we have the following result.

Theorem 3.11. *Let $M_1, M_2 \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be weights. If $A \in \mathcal{L}(H,K)$ is given such that $A^{\dagger}_{M_1N}$ exists and the operator $L_{M_1,M_2;N}$ defined by* (23) *is invertible, then $A^{\dagger}_{M_2N}$ exists and is of the form*

$$A^{\dagger}_{M_2N} = A^{\dagger}_{M_1N}\cdot L^{-1}_{M_1,M_2;N}.$$

Proof. The proof is so similar to that of Theorem 3.10 that we omit it. □

We finish this section by applying our results to obtain norm estimations for the weighted M–P inverse. In the sequel, $M\in\mathcal{L}(K)$ and $N\in\mathcal{L}(H)$ are two weights, and $A\in\mathcal{L}(H,K)$ is given such that $A^{\dagger}_{MN}$ exists. By (2) both $M(I-AA^{\dagger}_{MN})\in\mathcal{L}(K)$ and $(I-A^{\dagger}_{MN}A)N^{-1}\in\mathcal{L}(H)$ are self-adjoint, which mean that r_1 and r_2 are the spectral radii of $M(I-AA^{\dagger}_{MN})$ and $(I-A^{\dagger}_{MN}A)N^{-1}$ respectively, where

$$r_1 = \big\|M(I-AA^{\dagger}_{MN})\big\| \quad\text{and}\quad r_2 = \big\|(I-A^{\dagger}_{MN}A)N^{-1}\big\|.$$

Now let $\delta_{M^{-1}}\in\mathcal{L}(K)$ and $\delta_N\in\mathcal{L}(H)$ be self-adjoint such that

$$\|\delta_{M^{-1}}\|\cdot\max\{\|M\|, r_1\} < 1 \quad\text{and}\quad \|\delta_N\|\cdot\max\{\|N^{-1}\|, r_2\} < 1. \tag{26}$$

Under the above conditions, two weights $\widehat{M}$ and $\widehat{N}$ can be induced as

$$\widehat{M} = ((M)^{-1}+\delta_{M^{-1}})^{-1} \quad\text{and}\quad \widehat{N} = N+\delta_N. \tag{27}$$

By (19), (27) and Lemma 3.2, we have

$$R_{M;N,\widehat{N}} = A^{\dagger}_{MN}A + (I-A^{\dagger}_{MN}A)N^{-1}\widehat{N} = I + (I-A^{\dagger}_{MN}A)N^{-1}\delta_N, \tag{28}$$

which is invertible since by assumption we have

$$\|(I-A^{\dagger}_{MN}A)N^{-1}\delta_N\| \le \|(I-A^{\dagger}_{MN}A)N^{-1}\|\cdot\|\delta_N\| = r_2\cdot\|\delta_N\| < 1. \tag{29}$$

It follows from Theorem 3.10 that $A^{\dagger}_{M\widehat{N}}$ exists. Similarly,

$$\begin{aligned} L_{M,\widehat{M};N} &= AA^{\dagger}_{MN} + \widehat{M}^{-1}M(I-AA^{\dagger}_{MN})\\ &= I + \delta_{M^{-1}}M(I-AA^{\dagger}_{MN})\\ &= I + \delta_{M^{-1}}M(I-AA^{\dagger}_{M\widehat{N}}) = L_{M,\widehat{M};\widehat{N}}, \end{aligned}\tag{30}$$

which is also invertible since by assumption we have

$$\|\delta_{M^{-1}}M(I-AA^\dagger_{MN})\| \leq \|\delta_{M^{-1}}\|\cdot\|M(I-AA^\dagger_{MN})\| = r_1\cdot\|\delta_{M^{-1}}\| < 1.$$

Therefore, by the existence of $A^\dagger_{M\widehat{N}}$ and the invertibility of $L_{M,\widehat{M};\widehat{N}}$ we can conclude from Theorem 3.11 that $A^\dagger_{\widehat{M}\widehat{N}}$ is also existent. Furthermore, we may combine (28) and (29) to conclude that

$$\left\|R^{-1}_{M;N,\widehat{N}}\right\| \leq \frac{1}{1-r_2\|\delta_N\|} \tag{31}$$

and

$$\left\|I-R^{-1}_{M;N,\widehat{N}}\right\| \leq \frac{r_2\|\delta_N\|}{1-r_2\|\delta_N\|}. \tag{32}$$

Similarly, we can obtain

$$\left\|L^{-1}_{M,\widehat{M};N}\right\| \leq \frac{1}{1-r_1\|\delta_{M^{-1}}\|} \tag{33}$$

and

$$\left\|I-L^{-1}_{M,\widehat{M};N}\right\| \leq \frac{r_1\|\delta_M^{-1}\|}{1-r_1\|\delta_{M^{-1}}\|}. \tag{34}$$

Based on the above observations, we have the following theorem.

Theorem 3.12. *Let $\delta_{M^{-1}}\in\mathcal{L}(K)$ and $\delta_N\in\mathcal{L}(H)$ be self-adjoint such that* (26) *is satisfied. Then*

(i) $\left\|A^\dagger_{\widehat{M}\widehat{N}}\right\| \leq \frac{\|A^\dagger_{MN}\|}{(1-r_1\|\delta_{M^{-1}}\|)\cdot(1-r_2\|\delta_N\|)}$;

(ii) $\left\|A^\dagger_{\widehat{M}\widehat{N}}-A^\dagger_{MN}\right\| \leq \frac{r_1\|\delta_{M^{-1}}\|+r_2\|\delta_N\|-r_1r_2\|\delta_{M^{-1}}\|\,\|\delta_N\|}{(1-r_1\|\delta_{M^{-1}}\|)\cdot(1-r_2\|\delta_N\|)}\|A^\dagger_{MN}\|$;

(iii) $\left\|A^\dagger_{\widehat{M}\widehat{N}}A-A^\dagger_{MN}A\right\| \leq \frac{r_2\|\delta_N\|}{1-r_2\|\delta_N\|}\|A^\dagger_{MN}A\|$;

(iv) $\left\|AA^\dagger_{\widehat{M}\widehat{N}}-AA^\dagger_{MN}\right\| \leq \frac{r_1\|\delta_{M^{-1}}\|}{1-r_1\|\delta_{M^{-1}}\|}\|AA^\dagger_{MN}\|$.

Proof. By Corollary 3.6 we have

$$A^\dagger_{\widehat{M}\widehat{N}} = R^{-1}_{\widehat{M};N,\widehat{N}}\cdot A^\dagger_{MN}\cdot L^{-1}_{M,\widehat{M};\widehat{N}} = R^{-1}_{M;N,\widehat{N}}\cdot A^\dagger_{MN}\cdot L^{-1}_{M,\widehat{M};N}, \tag{35}$$

so

$$A^\dagger_{\widehat{M}\widehat{N}}-A^\dagger_{MN} = \big(R^{-1}_{M;N,\widehat{N}}-I\big)A^\dagger_{MN}L^{-1}_{M,\widehat{M};N}+A^\dagger_{MN}\big(L^{-1}_{M,\widehat{M};N}-I\big). \tag{36}$$

It is noticed by (28) and (30) that $AR_{M;N,\widehat{N}}=A=L_{M,\widehat{M};N}A$, therefore

$$AR^{-1}_{M;N,\widehat{N}} = A = L^{-1}_{M,\widehat{M};N}A. \tag{37}$$

It follows from (35) and (37) that

$$A^\dagger_{\widehat{M}\widehat{N}}A-A^\dagger_{MN}A = \big(R^{-1}_{M;N,\widehat{N}}-I\big)A^\dagger_{MN}A, \tag{38}$$

$$AA^\dagger_{\widehat{M}\widehat{N}}-AA^\dagger_{MN} = AA^\dagger_{MN}\big(L^{-1}_{M,\widehat{M};N}-I\big). \tag{39}$$

Norm upper bounds (i)–(iv) can then be derived from (35), (36), (38), (39) and (31)–(34). □

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