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ON A FAMILY OF COHOMOLOGICAL DEGREES

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Abstract. Cohomological degrees (or extended degrees) were introduced by Doering, Gunston and Vasconcelos as measures for the complexity of structure of finitely generated modules over a Noetherian ring. Until now only very few examples of such functions have been known. Using a Cohen-Macaulay obstruction defined earlier, we construct an infinite family of cohomological degrees.

1. Introduction

A homological degree is a function from the category of finitely generated modules over a local or a graded Noetherian ring to integers. This notion of degree has been given by Vasconcelos [18] as a measure for the complexity of the algebraic structure of the ring and modules over it. Using homological degree Vasconcelos obtained upper bounds for several numerical invariants of a finitely generated module such as the minimal number of generators, Hilbert coefficients, Betti numbers, Castelnuovo-Mumford regularity, etc (see [12, 18–21]). This notion has been extended by Doering-Gunston-Vasconcelos [12] to a more abstract notion of cohomological degree.

Definition 1.1 (Cohomological degree)**.** Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and Mod_R be the category of finitely generated R-modules. A cohomological degree (or extended degree) of R is a function

$$\mathrm{Deg} : \mathrm{Mod}_R \to \mathbb{R}_{\geq 0}$$

such that for a finitely generated R-module M,

(a) $\mathrm{Deg}(M) = \mathrm{Deg}(M/H^0_{\mathfrak{m}}(M)) + \ell(H^0_{\mathfrak{m}}(M))$, where $H^i_{\mathfrak{m}}(M)$ is the i-th local cohomology module of M with respect to $\mathfrak{m}$;

(b) (Bertini's rule) If $\mathrm{depth}(M) > 0$ and $x \in \mathfrak{m}$ is a generic hyperplane section of M, then

$$\mathrm{Deg}(M) \geq \mathrm{Deg}(M/xM);$$

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(c) (The calibration rule) If M is Cohen-Macaulay, then

$$\operatorname{Deg}(M) = e(M),$$

where $e(M)$ is the ordinary multiplicity of M relative to the maximal ideal.

So if M is Artinian, then $\operatorname{Deg}(M) = \ell(H^0_{\mathfrak{m}}(M))$. If $\dim(M) = 1$, then $\operatorname{Deg}(M) = e(M) + \ell(H^0_{\mathfrak{m}}(M))$ which is unique.

The first example of a cohomological degree is the homological degree given by Vasconcelos [18, Theorem 2.13].

Definition 1.2 (Homological degree)**.** Suppose that R is a quotient of a Gorenstein local ring S. The homological degree of a finitely generated R-module M, denoted by $\operatorname{hdeg}(M)$, is defined recursively by

$$\operatorname{hdeg}(M) = e(M) + \sum_{j=0}^{d-1} \binom{d-1}{j} \operatorname{hdeg}(\operatorname{Ext}_S^{s-d+1+j}(M, S)),$$

where $s = \dim S$ and $d = \dim M$.

If in addition S is a complete local ring of the same dimension as M and $E = E_S(k)$ is the injective envelop of the residue field, then

$$\operatorname{hdeg}(M) = e(M) + \sum_{j=0}^{d-1} \binom{d-1}{j} \operatorname{hdeg}(\operatorname{Hom}_S(H^j_{\mathfrak{m}}(M), E)).$$

Two other known examples of cohomological degrees consist of extremal degree bdeg defined by Gunston [13] and unmixed degree udeg defined by N. T. Cuong-P. H. Quy [11].

The main aim of this paper is to construct an infinite family of cohomological degrees for a local ring by using the idea in the construction of N. T. Cuong-P. H. Quy. For this purpose we use the colon modules associated to a so-called almost p-standard system of parameters of a module given in [4]. These colon modules are obstruction for the Cohen-Macaulayness of the module (see Section 2). On the other hand, almost p-standard systems of parameters always exist if the ground ring is a quotient of a Cohen-Macaulay ring. So the family of cohomological degrees obtained in our construction exists for a quite large class of rings and modules.

Throughout this paper, $(R, \mathfrak{m}, k)$ is a quotient of a Cohen-Macaulay local ring with an infinite residue field. For unexplained terminologies and basic results in commutative algebra and local cohomology, we refer to the books of Matsumura [14] and Brodmann-Sharp [2].

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2. A Cohen-Macaulay obstruction

There are several well-known obstructions for a module M to be Cohen-Macaulay such as the local cohomology module $H^i_{\mathfrak{m}}(M)$, $i < \dim(M)$, or the non-Cohen-Macaulay locus $\mathrm{nCM}(M)$ of M. In [4, Proposition 3.9] another kind of Cohen-Macaulay obstruction has been defined as a set of several subquotients of the module itself. In this section we investigate further these obstructions.

Let M be a finitely generated R-module. Let $d = \dim(M)$. We denote $\mathfrak{a}_i(M) = \mathrm{Ann}_R(H^i_{\mathfrak{m}}(M))$ and put $\mathfrak{a}(M) = \mathfrak{a}_0(M)\mathfrak{a}_1(M)\cdots\mathfrak{a}_{d-1}(M)$. We have $\dim(R/\mathfrak{a}(M)) < d$ as the base ring is a quotient of a Cohen-Macaulay local ring. Hence M admits a system of parameters $x_1, \ldots, x_d$ such that $x_d \in \mathfrak{a}(M)$ and $x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M)$ for $i < d$ (see [9, Theorem 1.3]). Then $x_1, \ldots, x_d$ is called a p-standard system of parameters of the module M. This notion was first introduced by N. T. Cuong in [5]. It plays a key role in the study of singularity of Cohen-Macaulay type of Noetherian rings and modules in the works of T. Kawasaki, N. T. Cuong-D. T. Cuong, N. T. Cuong-P. H. Quy, D. T. Cuong-P. H. Nam. In [4] the authors defined the notion of almost p-standard system of parameters extending slightly this notion.

Definition 2.1. Let M be a finitely generated R-module. A system of parameters $x_1, \ldots, x_d$ of M is almost p-standard if

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \cdots n_i$$

for given numbers $\lambda_0, \ldots, \lambda_d$ and for all $n_1, \ldots, n_d > 0$.

We collect some important properties of almost p-standard systems of parameters from [3, 6, 7] for later usage.

Remark 2.2. Let M be a finitely generated R-module of dimension d.

(a) [6, Corollary 3.6] Actually the coefficient λ_i of the length function in Definition 2.1 is the multiplicity of certain subquotient modules, namely,

$$\lambda_i = e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2}, \ldots, x_d)M}).$$

(b) [6, Corollary 3.6] A system of parameters $x_1, \ldots, x_d$ of M is almost p-standard if and only if $x_1^{n_1}, \ldots, x_i^{n_i}$ is a d-sequence on $M/(x_{i+1}^{n_{i+1}}, \ldots, x_d^{n_d})M$ for $i = 1, \ldots, d$ and for all $n_1, \ldots, n_d > 0$. We recall that due to Huneke, a sequence $x_1, \ldots, x_s \in \mathfrak{m}$ is a d-sequence on M if $(x_1, \ldots, x_{i-1})M : x_j = (x_1, \ldots, x_{i-1})M : x_i x_j$ for $i = 1, \ldots, s$, and $j \geq i$.

(c) [6, Corollary 3.5] Let $x_1, \ldots, x_d$ be an almost p-standard system of parameters of M and $0 < i_1 < \cdots < i_r \leq d$ be some indexes. Then $x_{i_1}, \ldots, x_{i_r}$ is an almost p-standard system of parameters of M/IM where $I = (x_j : j \neq i_1, \ldots, i_r)$.

(d) [6, Corollary 3.9] A p-standard system of parameters is almost p-standard. Conversely, if $x_1, \ldots, x_d$ is an almost p-standard system of parameters, then $x_1^{n_1}, \ldots, x_d^{n_d}$ is p-standard for all $n_1 \geq 1, n_2 \geq 2, \ldots, n_d \geq d$.

The next results give a set of subquotients of a module which can be regarded as an obstruction to the Cohen-Macaulayness of the module. These subquotients will be essential in our construction of cohomological degrees later.

Proposition 2.3 ([4, Proposition 3.2 and Corollary 3.5])**.** *Let M be a finitely generated R-module of dimension $d > 0$. Let $x_1, \dots, x_d$ be an almost p-standard system of parameters of M and let $0 \le i < j \le d$.*

(a) *The subquotient*

$$(x_{i+2}^{n_{i+2}}, \dots, x_j^{n_j})M : x_{i+1}/(x_{i+2}^{n_{i+2}}, \dots, x_j^{n_j})M$$

is independent (up to an isomorphism) of the choice of the almost p-standard system of parameters and of the exponents $n_{i+2}, \dots, n_j > 1$. This module is denoted by U_M^{ij}.

(b) *Suppose $j > i+1$. There is an injective homomorphism $\varphi : U_M^{i,j-1} \to U_M^{ij}$ such that $\operatorname{Im}(\varphi)$ is a direct summand of U_M^{ij}. We denote $\operatorname{Coker}(\varphi)$ by $\overline{U}_M^{ij}$. So there is a direct sum decomposition*

$$U_M^{ij} \simeq \overline{U}_M^{ij} \oplus \overline{U}_M^{i,j-1} \oplus \cdots \oplus \overline{U}_M^{i,i+2} \oplus\ U_M^{i,i+1}.$$

We denote $\overline{U}_M^{i,i+1} := U_M^{i,i+1}$.

The subquotients U_M^{ij} and $\overline{U}_M^{ij}$ are new invariants of M. Moreover, we have

$$\dim \overline{U}_M^{ij} \le \dim U_M^{ij} \le \dim U_M^{id} \le \dim M/(x_{i+1}^2, \dots, x_d^2)M = i,$$

where $x_1, \dots, x_d$ is an almost p-standard system of parameters of M. It is clear that M is Cohen-Macaulay if and only if $U_M^{ij} = 0$ for all $i < j$, if and only if $\overline{U}_M^{ij} = 0$ for all $i < j$. Therefore they are obstructions for M to be Cohen-Macaulay. These obstructions are simpler in the case of generalized Cohen-Macaulay modules.

Recall that a finitely generated module M is a generalized Cohen-Macaulay module if the local cohomology module $H_{\mathfrak{m}}^i(M)$ is of finite length for $i = 0, 1, \dots, \dim(M) - 1$. On a generalized Cohen-Macaulay module, the almost p-standard systems of parameters are exactly the standard system of parameters in the sense of [17]. The Cohen-Macaulay obstructions U_M^{ij} and $\overline{U}_M^{ij}$ of generalized Cohen-Macaulay modules are simply computed in terms of local cohomology as follows.

Lemma 2.4. *Let M be a generalized Cohen-Macaulay module with $\dim(M) = d > 0$. Then*

$$U_M^{ij} \simeq U_M^{0,j-i} \simeq \bigoplus_{t=0}^{j-i-1} H_{\mathfrak{m}}^t(M)^{\oplus\binom{j-i-1}{t}},$$

and

$$\overline{U}_M^{ij} \simeq \overline{U}_M^{0,j-i} \simeq \bigoplus_{t=1}^{j-i-1} H_{\mathfrak{m}}^t(M)^{\oplus\binom{j-i-2}{t-1}}$$

for all $0 \leq i < j \leq d$.

Proof. The second conclusion on decomposition of $\overline{U}_M^{ij}$ is a consequence of the first conclusion.

On a generalized Cohen-Macaulay module, almost p-standard systems of parameters are the same as standard systems of parameters in the sense of [17]. In particular, any permutation of an almost p-standard system of parameters is also an almost p-standard system of parameters. Hence $U_M^{ij} \simeq U_M^{0,j-i}$. So it suffices to prove that

$$U_M^{0,j} \simeq \bigoplus_{t=0}^{j-1} H_{\mathfrak{m}}^t(M)^{\oplus\binom{j-1}{t}}$$

for $j > 0$. Let $x_1, \ldots, x_d$ be a standard system of parameters of M. In particular, $x_1, \ldots, x_d$ is a d-sequence. If $j = 1$, then $U_M^{0,1} = 0 :_M x_1^2 = 0 :_M (x_1, \ldots, x_d) = H_{\mathfrak{m}}^0(M)$. Let $j > 1$ and denote $N = M/x_j^5 M$. By induction on j, we have

$$U_M^{0,j} \simeq U_N^{0,j-1} \simeq \bigoplus_{t=0}^{j-2} H_{\mathfrak{m}}^t(N)^{\oplus\binom{j-2}{t}}.$$

Now using the splitting property of local cohomology in [3, Corollary 2.8], we have

$$\begin{aligned} H_{\mathfrak{m}}^t(N) &\simeq H_{\mathfrak{m}}^t(M) \oplus H_{\mathfrak{m}}^{t+1}(M/0 :_M x_j^5) \\ &\simeq H_{\mathfrak{m}}^t(M) \oplus H_{\mathfrak{m}}^{t+1}(M/H_{\mathfrak{m}}^0(M)) \\ &\simeq H_{\mathfrak{m}}^t(M) \oplus H_{\mathfrak{m}}^{t+1}(M). \end{aligned}$$

Therefore,

$$U_M^{0,j} \simeq \bigoplus_{t=0}^{j-1} H_{\mathfrak{m}}^t(M)^{\oplus\binom{j-1}{t}}.$$

□

In order to obtain numerical Cohen-Macaulay obstructions, we use the multiplicity of the subquotients U_M^{ij} or $\overline{U}_M^{ij}$. Firstly, for a finitely generated module N and an integer $i \geq \dim(N)$, we denote $e(N)_i = e(N)$ if $\dim N = i$ and $e(N)_i = 0$ if $\dim N < i$. A consequence of Proposition 2.3 asserts that the multiplicity $e(U_M^{ij})_i$ and $e(\overline{U}_M^{ij})_i$ are numerical invariants of M.

Proposition 2.5. *Let M be a finitely generated R-module of dimension $d > 0$. The following statements are equivalent*:

(a) *M is Cohen-Macaulay*;
(b) $e(U_M^{ij})_i = 0$ *for all* $0 \leq i < j \leq d$;
(c) $e(\overline{U}_M^{ij})_i = 0$ *for all* $0 \leq i < j \leq d$;
(d) $e(U_M^{id})_i = 0$ *for all* $0 \leq i < d$.

Proof. The implications (a)⇒(b)⇒(c) are obvious while (c)⇒(d) and (d)⇒(b) are consequence of the direct sum decomposition in Proposition 2.3(b).

We now show that (b) implies (a). Let $x_1, \dots, x_d$ be an almost p-standard system of parameters of M. Replace $x_1, \dots, x_d$ by $x_1^2, \dots, x_d^2$ if necessary, we assume that $U_M^{ij} \simeq (x_{i+2}, \dots, x_j)M : x_{i+1}/(x_{i+2}, \dots, x_j)M$. The assumption $e(U_M^{ij})_i = 0$ implies in particular that $\dim(U_M^{ij}) < i$ for all $0 \leq i < j \leq d$. Now using Remark 2.2(a), we get

$$\begin{aligned}\ell(M/(x_1, \dots, x_d)M) &= e(x_1, \dots, x_d; M) + \sum_{i=0}^{d-1} e(x_1, \dots, x_i; U_M^{id}) \\ &= e(x_1, \dots, x_d; M),\end{aligned}$$

where $e(x_1, \dots, x_d; M)$ is the multiplicity of M with respect to $x_1, \dots, x_d$. Therefore M is Cohen-Macaulay (see, for example, [14, Theorem 17.11]). □

Now we use the second half of this section and the next section to construct an infinite family of cohomological degrees as being announced. The numerical Cohen-Macaulay obstructions in Proposition 2.5 will play a central role in our construction.

As the first step of the construction, we note that a cohomological degree satisfies the Bertini's rule, namely, it does not increase when passing to a generic hyperplane section. This leads to a notion of genericity for a cohomological degree Deg, that is, an open subset of $\mathfrak{m}/\mathfrak{m}^2$ consisting of hyperplanes h such that $\mathrm{Deg}(M) \geq \mathrm{Deg}(M/hM)$ (see [21, Definition 1.5.1]). In the next proposition, we define for each R-module an open subset of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ which will help to determines a notion of genericity for the expected family of cohomological degrees.

Proposition 2.6. *Let M be a finitely generated R-module of dimension $d > 0$. We denote by H_M the set of all elements $h \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $h^t, x_2, \dots, x_d$ is an almost p-standard system of parameters of M for some $x_2, \dots, x_d \in \mathfrak{m}$ and some $t > 0$. The set $(H_M + \mathfrak{m}^2)/\mathfrak{m}^2$ is a non-empty open subset in $\mathfrak{m}/\mathfrak{m}^2$ with respect to the Zariski topology.*

Proof. First we show that $(H_M + \mathfrak{m}^2)/\mathfrak{m}^2$ is non-empty. Take any p-standard system of parameters $x_1, \dots, x_d$ of M, that is, $x_i \in \mathfrak{a}(M/(x_{i+1}, \dots, x_d)M)$ for $i = d, d-1, \dots, 2, 1$. Since the module $M/(x_2, \dots, x_d)M$ has dimension one, we can choose generically $h \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that h is a parameter element of $M/(x_2, \dots, x_d)M$. Clearly $h^t \in \mathfrak{a}(M/(x_2, \dots, x_d)M)$ for some $t > 0$, as the latter ideal is $\mathfrak{m}$-primary. Thus $h^t, x_2, \dots, x_d$ is an almost p-standard system of parameters of M (see Remark 2.2(d)). So $h \in H_M$ and therefore H_M is not empty.

For the openness, take a hyperplane section $h \in H_M$. Then there are some $t > 0$ and some $x_2, \dots, x_d \in \mathfrak{m}$ such that $h^t, x_2, \dots, x_d$ is an almost p-standard system of parameters of M. Using Remark 2.2(d) and the notations there, we may assume that $h^t, x_2, \dots, x_d$ is a p-standard system of parameters, that is, $x_i \in \mathfrak{a}(M/(x_{i+1}, \dots, x_d)M)$, $i = 2, \dots, d$ and $h^t \in \mathfrak{a}(M/(x_2, \dots, x_d)M)$.

Now let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be all minimal associated prime ideals of

$$M/(x_2, \ldots, x_d)M.$$

Set

$$U := \{g \in \mathfrak{m} \setminus \mathfrak{m}^2 : g \notin \cup_{i=1}^r \mathfrak{p}_i\}.$$

Then $h \in U$ and $U \subseteq H_M$. Indeed, for each element $g \in U$,

$$g^s \in \mathfrak{a}(M/(x_2, \ldots, x_d)M)$$

for some $s > 0$. This shows that $g^s, x_2, \ldots, x_d$ is a p-standard system of parameters of M and is particularly almost p-standard (see Remark 2.2(d) again). Hence $g \in H_M$ and consequently $U \subseteq H_M$. Therefore $(U + \mathfrak{m}^2)/\mathfrak{m}^2$ is an open neighborhood of $\bar{h}$ in $(H_M + \mathfrak{m}^2)/\mathfrak{m}^2$. □

Lemma 2.7. *Let M be a finitely generated R-module of dimension $d > 0$ and let $h \in H_M$ be a hyperplane as in Proposition 2.6. There are $x_2, \ldots, x_d \in \mathfrak{m}$ such that $h^t, x_2, \ldots, x_d$ is an almost p-standard system of parameters of M for some $t > 0$ and $x_2, \ldots, x_d$ is an almost p-standard system of parameters of M/hM.*

Proof. We will prove a stronger statement: Let $h \in H_M$ and $x_2, \ldots, x_d \in \mathfrak{m}$ such that $h^t, x_2, \ldots, x_d$ is an almost p-standard system of parameters of M for some $t > 0$. Then $x_2, \ldots, x_d$ is an almost p-standard system of parameters of M/hM.

Replace $h^t, x_2, \ldots, x_d$ by $h^{2t}, x_2^2, \ldots, x_d^2$ if it is necessary, we assume that

$$U_M^{id} \simeq (0 : x_{i+1}^{n_{i+1}})_{M/(x_{i+2}^{n_{i+2}}, \ldots, x_d^{n_d})M},$$

$$U_M^{0d} \simeq (0 : h^t)_{M/(x_2^{n_2}, \ldots, x_d^{n_d})M},$$

for all $i > 0$ and $n_2, \ldots, n_d > 0$. In particular,

$$(0 : h)_{M/(x_2^{n_2}, \ldots, x_d^{n_d})M} = (0 : h)_{H^0_{\mathfrak{m}}(M/(x_2^{n_2}, \ldots, x_d^{n_d})M)} \simeq (0 : h)_{U_M^{0d}}.$$

Denote $N = M/hM$. Combining the above isomorphisms with Auslander-Buchsbaum's formula relating length and multiplicity [1, Corollary 4.3], for any $n_2, \ldots, n_d > 0$, we have

$$\begin{aligned}
&\ell(N/(x_2^{n_2}, \ldots, x_d^{n_d})N)\\
&= \ell(M/(h, x_2^{n_2}, \ldots, x_d^{n_d})M)\\
&= e(h, x_2^{n_2}, \ldots, x_d^{n_d}; M) + e(h, x_2^{n_2}, \ldots, x_{d-1}^{n_{d-1}}; 0 :_M x_d^{n_d}) + \cdots\\
&\quad + e(h, x_2^{n_2}; (0 : x_3^{n_3})_{M/(x_4^{n_4}, \ldots, x_d^{n_d})M}) + e(h; (0 : x_2^{n_2})_{M/(x_3^{n_3}, \ldots, x_d^{n_d})M})\\
&\quad + \ell((0 : h)_{M/(x_2^{n_2}, \ldots, x_d^{n_d})M})\\
&= n_2 \cdots n_d e(h, x_2, \ldots, x_d; M) + n_2 \cdots n_{d-1} e(h, x_2, \ldots, x_{d-1}; U_M^{d-1,d}) + \cdots\\
&\quad + n_2 e(h, x_2; U_M^{2d}) + e(h; U_M^{1d}) + \ell((0 : h)_{U_M^{0d}}).
\end{aligned}$$

So $x_2, \ldots, x_d$ is an almost p-standard system of parameters of $N = M/hM$ by definition. □

Now comes a technical property on the Bertini's rule for the modules $\overline{U}^{ij}_M$'s.

Proposition 2.8. *Let M be a finitely generated R-module of dimension $d > 0$ and $h \in H_M$ be a hyperplane section. For $1 \le i < j \le d$, there are exact sequences*

$$0 \to U^{ij}_M / hU^{ij}_M \to U^{i-1,j-1}_{M/hM} \to L \to 0,$$

$$0 \to \overline{U}^{ij}_M / h\overline{U}^{ij}_M \to \overline{U}^{i-1,j-1}_{M/hM} \to N \to 0,$$

where L and N are modules of finite length such that there is an isomorphism

$$(0 : h)_{U^{i-1,j}_M} \simeq L \oplus (0 : h)_{U^{ij}_M},$$

and

$$\ell(N) = \begin{cases} \ell((0 : h)_{\overline{U}^{i-1,j}_M}) - \ell((0 : h)_{\overline{U}^{ij}_M}) & \text{if } j > i + 1, \\ \ell((0 : h)_{\overline{U}^{i-1,i+1}_M}) & \text{if } j = i + 1. \end{cases}$$

Proof. By Lemma 2.7, there are $x_2, \ldots, x_d \in \mathfrak{m}$ such that $h^t, x_2, \ldots, x_d$ is an almost p-standard system of parameters of M for some $t > 0$ and $x_2, \ldots, x_d$ is an almost p-standard system of parameters of M/hM. Replacing $x_2, \ldots, x_d$ by $x_2^2, \ldots, x_d^2$ if necessary, we assume that

$$U^{ij}_M = (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_j)M} \text{ and } U^{ij}_{M/hM} = (0 : x_{i+2})_{M/(h,x_{i+3},\ldots,x_{j+1})M}.$$

At the first step, we prove the conclusion for $j = i + 1$ by showing an exact sequence

$$0 \to U^{i,i+1}_M / hU^{i,i+1}_M \to U^{i-1,i}_{M/hM} \to (0 : h)_{\overline{U}^{i-1,i+1}_M} \to 0.$$

We have $\overline{U}^{i,i+1}_M = U^{i,i+1}_M = 0 :_M x_{i+1}$ and $\overline{U}^{i-1,i}_{M/hM} = U^{i-1,i}_{M/hM} \simeq (0 : x_{i+1})_{M/hM}$. By Remark 2.2(b), the system of parameters $h^t, x_2, \ldots, x_d$ is a d-sequence on M and thus $0 :_M h \subseteq 0 :_M h^t \subseteq 0 :_M x_{i+1}$. Then from the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M/0 :_M h \arrow[r, "*h"] \arrow[d, "*x_{i+1}"] & M \arrow[r] \arrow[d, "*x_{i+1}"] & M/hM \arrow[r] \arrow[d, "*x_{i+1}"] & 0 \\ 0 \arrow[r] & M/0 :_M h \arrow[r, "*h"] & M \arrow[r] & M/hM \arrow[r] & 0 \end{tikzcd}$$

we get an exact sequence

$$0 \longrightarrow 0 :_M x_{i+1}/0 :_M h \xrightarrow{*h} 0 :_M x_{i+1} \longrightarrow (0 : x_{i+1})_{M/hM}$$
$$\longrightarrow M/(x_{i+1}M + 0 :_M h) \xrightarrow{\psi} M/x_{i+1}M.$$

It gives rise to an exact sequence

$$0 \to (0 :_M x_{i+1})/h(0 :_M x_{i+1}) \to (0 : x_{i+1})_{M/hM} \to \mathrm{Ker}(\psi) \to 0.$$

Let $L = \mathrm{Ker}(\psi)$. We have

$$\begin{aligned} L &= x_{i+1}M :_M h/(x_{i+1}M + 0 :_M h) \\ &\simeq \left((0:h)_{M/x_{i+1}M}\right) / \left((x_{i+1}M + 0 :_M h)/x_{i+1}M\right) \\ &= \Big((0:h)_{U_M^{i-1,i+1}}\Big) \Big/ \Big((x_{i+1}M + (0:h)_{U_M^{i-1,i}})/x_{i+1}M\Big). \end{aligned}$$

The last equality is due to the fact that $0 :_M h \subseteq 0 :_M x_i = U_M^{i-1,i}$, in particular, $0 :_M h = (0:h)_{U_M^{i-1,i}}$ and

$$(0:h)_{M/x_{i+1}M} = (0:h)_{U^{i-1,i}_{M/x_{i+1}M}} = (0:h)_{U_M^{i-1,i+1}}.$$

On the other hand, by Proposition 2.3(b), there is a direct sum decomposition

$$U_M^{i-1,i+1} \simeq U_M^{i-1,i} \oplus \overline{U}_M^{i-1,i+1}.$$

Hence

$$(0:h)_{U_M^{i-1,i+1}} \simeq (0:h)_{U_M^{i-1,i}} \oplus (0:h)_{\overline{U}_M^{i-1,i+1}}.$$

Through this isomorphism one have $(x_{i+1}M + (0:h)_{U_M^{i-1,i}})/x_{i+1}M \simeq (0:h)_{U_M^{i-1,i}}$. Furthermore, since $h^t, x_2, \ldots, x_d$ is a d-sequence on M, there are inclusions

$$0 :_M h \subseteq 0 :_M (h^t, x_2, \ldots, x_d) \subseteq H^0_{\mathfrak{m}}(M) \subseteq 0 :_M x_i \subseteq 0 :_M x_{i+1}.$$

In particular, $0 :_M h = (0:h)_{U_M^{i-1,i}} = (0:h)_{U_M^{i,i+1}}$ which induces an isomorphism

$$(0:h)_{U_M^{i-1,i+1}} \simeq (0:h)_{U_M^{i,i+1}} \oplus (0:h)_{\overline{U}_M^{i-1,i+1}}.$$

Denote $L := \mathrm{Ker}(\psi) \simeq (0:h)_{\overline{U}_M^{i-1,i+1}}$ and it completes the proof for the case $j = i+1$.

Now let $j > i+1$. Applying the first part of the proof to the almost p-standard system of parameters $h^t, x_2, \ldots, x_{i+1}, x_{j+1}, \ldots, x_d$ of the module $P = M/(x_{i+2}, \ldots, x_j)M$ (see Remark 2.2(c)), we obtain an exact sequence

$$0 \to U_P^{i,i+1}/hU_P^{i,i+1} \to U_{P/hP}^{i-1,i} \to (0:h)_{\overline{U}_P^{i-1,i+1}} \to 0,$$

where the last module satisfies

$$(0:h)_{U_P^{i-1,i+1}} \simeq (0:h)_{U_P^{i,i+1}} \oplus (0:h)_{\overline{U}_P^{i-1,i+1}}.$$

Note that $U_P^{i,i+1} \simeq U_M^{ij}$, $U_{P/hP}^{i-1,i} \simeq U_{M/hM}^{i-1,j-1}$ and $U_P^{i-1,i+1} \simeq U_M^{i-1,j}$. Denote $L = (0:h)_{\overline{U}_P^{i-1,i+1}}$, then

$$(0:h)_{U_M^{i-1,j}} \simeq (0:h)_{\overline{U}_P^{i-1,i+1}} \oplus (0:h)_{U_M^{ij}} = L \oplus (0:h)_{U_M^{ij}}.$$

We obtain the conclusion on the first exact sequence.

For the second exact sequence, following Proposition 2.3(b) we have an isomorphism

$$U_M^{ij} \simeq \overline{U}_M^{ij} \oplus U_M^{i,j-1},$$

which leads to an isomorphism of quotients

$$U_M^{ij}/hU_M^{ij} \simeq (\overline{U}_M^{ij}/h\overline{U}_M^{ij}) \oplus (U_M^{i,j-1}/hU_M^{i,j-1}).$$

Then there is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U_M^{i,j-1}/hU_M^{i,j-1} \arrow[r, "\varphi"] \arrow[d] & U_{M/hM}^{i-1,j-2} \arrow[r] \arrow[d] & (0:h)_{U_M^{i-1,j-1}}/(0:h)_{U_M^{i,j-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U_M^{ij}/hU_M^{ij} \arrow[r, "\psi"] \arrow[d] & U_{M/hM}^{i-1,j-1} \arrow[r] \arrow[d] & (0:h)_{U_M^{i-1,j}}/(0:h)_{U_M^{ij}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{U}_M^{ij}/h\overline{U}_M^{ij} \arrow[r, "\overline{\psi}"] \arrow[d] & \overline{U}_{M/hM}^{i-1,j-1} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the rows and the columns are exact, φ, ψ are the canonical projections and $\overline{\psi}$ is induced from ψ (see also [4, Lemma 3.4(a)]). Therefore,

$$\begin{aligned}
\ell(N) &= \ell\left((0:h)_{U_M^{i-1,j}}/(0:h)_{U_M^{ij}}\right) - \ell\left((0:h)_{U_M^{i-1,j-1}}/(0:h)_{U_M^{i,j-1}}\right)\\
&= \ell((0:h)_{\overline{U}_M^{i-1,j}}) - \ell((0:h)_{\overline{U}_M^{ij}}).
\end{aligned}$$

The proof is complete. □

3. A family of cohomological degrees

The first example of a cohomological degree is the homological degree hdeg given by Vasconcelos [18]. The second example of a cohomological degree is the extremal degree bdeg given by Gunston [13]. It is simply defined for a module M by

$$\operatorname{bdeg}(M) = \inf\{\operatorname{Deg}(M): \text{ all cohomological degrees } \operatorname{Deg}\}.$$

Recently N. T. Cuong-P. H. Quy introduced a notion of unmixed degree udeg and showed that it is also a cohomological degree (see [11, Theorem 5.18]). The unmixed degree is defined in terms of the subquotients U_M^{id} as

$$\operatorname{udeg}(M) := e(M) + \sum_{i=0}^{d-1} e(U_M^{id})_i.$$

In this section we will construct an infinite family of cohomological degrees on Mod_R.

A cohomological degree can be actually defined case by case. For examples, let M be a finitely generated R-module such that $0 < \operatorname{depth}(M) < \dim M =$

$\dim R$. We define a map $\mathrm{Deg} : \mathrm{Mod}_R \to \mathbb{R}$ by letting $\mathrm{Deg}(N) = \mathrm{bdeg}(N)$ for any $N \neq M$ and assigning $\mathrm{Deg}(M)$ to any number bigger than $\mathrm{bdeg}(M)$. So $\mathrm{Deg} \neq \mathrm{bdeg}$ and Deg is a cohomological degree. By this way we can construct many cohomological degrees from bdeg. However, these functions are not really what we expect as they are not given by "formula".

Using the Cohen-Macaulay obstruction defined in Proposition 2.3 we construct a family of cohomological degrees.

Theorem 3.1. *Let R be a quotient of a Cohen-Macaulay local ring with* $\dim(R) = n > 0$. *Let $\Lambda = \{\lambda_{ijk} \in \mathbb{R} : 0 \leq i < j \leq k \leq n\}$ be a set of real numbers such that*

$$\lambda_{01k} = 1 \textit{ for } 1 \leq k \leq n,$$

$$\lambda_{0jk} \leq \lambda_{0,j+1,k+1} \textit{ and } \lambda_{ijk} \leq \lambda_{i+1,j+1,k+1} \textit{ for } 0 \leq i < j \leq k < n.$$

Define a function $\mathrm{Deg}_\Lambda : \mathrm{Mod}_R \to \mathbb{R}$ by assigning a finitely generated R-module M of dimension d to the number

$$\mathrm{Deg}_\Lambda(M) := e(M) + \sum_{0 \leq i < j \leq d} \lambda_{ijd} e\left(\overline{U}_M^{ij}\right)_i.$$

Then Deg_Λ is a cohomological degree.

Proof. Let M be a finitely generated R-module of dimension $d > 0$. We denote $\overline{M} = M/H^0_{\mathfrak{m}}(M)$.

We first show that $\mathrm{Deg}_\Lambda(M) = \mathrm{Deg}_\Lambda(\overline{M}) + \ell(H^0_{\mathfrak{m}}(M))$. Let $x_1, \ldots, x_d$ be an almost p-standard system of parameters of the direct sum $M \oplus \overline{M}$. By [4, Proposition 2.10], it is an almost p-standard system of parameters of M and of $\overline{M}$. Replace it by $x_1^2, \ldots, x_d^2$ if necessary, we assume that

$$U_M^{ij} \simeq (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_j)M}, \quad U_{\overline{M}}^{ij} \simeq (0 : x_{i+1})_{\overline{M}/(x_{i+2},\ldots,x_j)\overline{M}}$$

for all $0 \leq i < j \leq d$. Now, since $x_1, \ldots, x_d$ is a d-sequence on M (see Remark 2.2(b)), $H^0_{\mathfrak{m}}(M) = 0 :_M x_1$ and $H^0_{\mathfrak{m}}(M) \cap (x_{i+2}, \ldots, x_j)M = 0$ for $0 \leq i < j \leq d$. We then have a short exact sequence

$$(1) \qquad 0 \to H^0_{\mathfrak{m}}(M) \to M/(x_{i+2}, \ldots, x_j)M \to \overline{M}/(x_{i+2}, \ldots, x_j)\overline{M} \to 0.$$

Suppose $i > 0$. Applying the functor $\mathrm{Hom}_R(R/(x_{i+1}), -)$ to the exact sequence (1) and notice that $\mathrm{Hom}_R(R/(x_{i+1}), H^0_{\mathfrak{m}}(M)) \simeq H^0_{\mathfrak{m}}(M)$ and

$$U_M^{ij} \simeq (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_j)M} \simeq \mathrm{Hom}_R(R/(x_{i+1}), M/(x_{i+2}, \ldots, x_j)M),$$

$$U_{\overline{M}}^{ij} \simeq (0 : x_{i+1})_{\overline{M}/(x_{i+2},\ldots,x_j)\overline{M}} \simeq \mathrm{Hom}_R(R/(x_{i+1}), \overline{M}/(x_{i+2}, \ldots, x_j)\overline{M}),$$

we get an exact sequence for Ext modules, namely,

$$0 \to H^0_{\mathfrak{m}}(M) \to U_M^{ij} \to U_{\overline{M}}^{ij} \to \mathrm{Ext}^1_R(R/(x_{i+1}), H^0_{\mathfrak{m}}(M)).$$

Since $H^0_{\mathfrak{m}}(M)$ and $\mathrm{Ext}^1_R(R/(x_{i+1}), H^0_{\mathfrak{m}}(M))$ are of finite length, we obtain

$$e(U_M^{ij})_i = e(U_{\overline{M}}^{ij})_i.$$

From the direct sum decompositions $U_M^{ij} \simeq U_M^{i,j-1} \oplus \overline{U}_M^{ij}$ and $U_{\overline{M}}^{ij} \simeq U_{\overline{M}}^{i,j-1} \oplus \overline{U}_{\overline{M}}^{ij}$, we get

$$e(\overline{U}_M^{ij})_i = e(\overline{U}_{\overline{M}}^{ij})_i. \tag{2}$$

For $i = 0$, it is worth mentioning that $x_1, x_{j+1}, \ldots, x_d$ is an almost p-standard system of parameters of $M/(x_{i+2}, \ldots, x_j)M$ (see Remark 2.2(c)), so it is a d-sequence on $M/(x_{i+2}, \ldots, x_j)M$ and we have

$$U_M^{0,j} \simeq (0 : x_1)_{M/(x_2,\ldots,x_j)M} \simeq H_{\mathfrak{m}}^0(M/(x_2, \ldots, x_j)M),$$
$$U_{\overline{M}}^{0,j} \simeq (0 : x_1)_{\overline{M}/(x_2,\ldots,x_j)\overline{M}} \simeq H_{\mathfrak{m}}^0(\overline{M}/(x_2, \ldots, x_j)\overline{M}).$$

Now applying the functor $\Gamma_{\mathfrak{m}}(-)$ to the short exact sequence (1), we get an exact sequence

$$0 \to H_{\mathfrak{m}}^0(M) \to U_M^{0,j} \to U_{\overline{M}}^{0,j} \to 0.$$

Hence $\ell(U_M^{0,j}) = \ell(U_{\overline{M}}^{0,j}) + \ell(H_{\mathfrak{m}}^0(M))$. Consequently,

$$\ell(\overline{U}_M^{0,j}) = \ell(\overline{U}_{\overline{M}}^{0,j}), \tag{3}$$

for $j > 1$ and

$$\ell(\overline{U}_M^{0,1}) = \ell(\overline{U}_{\overline{M}}^{0,1}) + \ell(H_{\mathfrak{m}}^0(M)). \tag{4}$$

Combining the equalities (2), (3), (4), we obtain $\mathrm{Deg}_\Lambda(M) = \mathrm{Deg}_\Lambda(\overline{M}) + \ell(H_{\mathfrak{m}}^0(M))$.

The calibration rule follows from Proposition 2.5 and the definition of Deg_Λ. If M is Cohen-Macaulay, then $\overline{U}_M^{ij} = 0$ for all $0 \le i < j \le d$, so $\mathrm{Deg}_\Lambda(M) = e(M)$.

Now we prove the Bertini's rule. Suppose $\operatorname{depth}(M) > 0$. Let $h \in H_M$ be a hyperplane as being defined in Proposition 2.6. Using Proposition 2.8, we have

$$e\left(\overline{U}_{M/hM}^{ij}\right)_i = e\left(\overline{U}_M^{i+1,j+1}/h\overline{U}_M^{i+1,j+1}\right)_i = e\left(\overline{U}_M^{i+1,j+1}\right)_{i+1}$$

for $0 < i < j < d$, and for $0 = i < j < d$,

$$\begin{aligned}\ell(\overline{U}_{M/hM}^{0,j}) &= \ell(\overline{U}_M^{1,j+1}/h\overline{U}_M^{1,j+1}) + \ell((0:h)_{\overline{U}_M^{0,j+1}}) - \ell((0:h)_{\overline{U}_M^{1,j+1}}) \\ &= e(h; \overline{U}_M^{1,j+1}) + \ell((0:h)_{\overline{U}_M^{0,j+1}}).\end{aligned}$$

If we choose h generically, then $e(h; \overline{U}_M^{1,j+1}) = e(\overline{U}_M^{1,j+1})_1$, which induces

$$\ell(\overline{U}_{M/hM}^{0,j}) = e(\overline{U}_M^{1,j+1})_1 + \ell((0:h)_{\overline{U}_M^{0,j+1}}) \le e(\overline{U}_M^{1,j+1})_1 + \ell(\overline{U}_M^{0,j+1}).$$

Combining these inequalities with the assumption $\lambda_{0jk} \le \lambda_{0j+1,k+1}$ and $\lambda_{ijk} \le \lambda_{i+1,j+1,k+1}$ for $0 \le i < j \le k < n$, we obtain

$$\mathrm{Deg}_\Lambda(M/hM) = e(M/hM) + \sum_{0 \le i < j \le d-1} \lambda_{i,j,d-1} e\left(\overline{U}_{M/hM}^{ij}\right)_i$$

$$\leq e(M) + \sum_{0<i<j\leq d} \lambda_{i-1,j-1,d-1} e\left(\overline{U}_M^{ij}\right)_i + \sum_{j=2}^{d} \lambda_{0j-1,d-1} \ell(\overline{U}_M^{0,j})$$
$$\leq e(M) + \sum_{0<i<j\leq d} \lambda_{ijd} e\left(\overline{U}_M^{ij}\right)_i + \sum_{j=2}^{d} \lambda_{0jd} \ell(\overline{U}_M^{0,j})$$
$$= \mathrm{Deg}_\Lambda(M). \qquad \square$$

Remark 3.2. From the assumption on the number λ_{ijk} in Theorem 3.1, we have

$$\lambda_{ijk} \geq \lambda_{0,j-i,k-i} \geq \lambda_{0,1,k-j+1} = 1.$$

So all coefficients λ_{ijk}'s of Deg_Λ are at least 1.

Theorem 3.1 applies to give rise to many cohomological degrees by assigning concrete values to the coefficients λ_{ijk}'s.

Example 3.3. (a) Given real numbers $1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. Let $\lambda_{ijk} = \lambda_i$ for $0 \leq i < j \leq k \leq n$ and $\Lambda := \{\lambda_{ijk} : 0 \leq i < j \leq k \leq n\}$. By Theorem 3.1, we get a cohomological degree

$$\mathrm{Deg}_\Lambda(M) = e(M) + \sum_{0\leq i<j\leq d} \lambda_{ijd} e\left(\overline{U}_M^{ij}\right)_i$$
$$= e(M) + \lambda_{d-1} e(U_M^{d-1,d})_{d-1} + \cdots + \lambda_1 e(U_M^{1d})_1 + \ell(U_M^{0d}),$$

where M is a finitely generated R-module of dimension d and the second equality follows from Proposition 2.3(b). In particular, if $\lambda_0 = \cdots = \lambda_n = 1$, then one gets the unmixed degree udeg.

(b) Let $\lambda_{ijk} = \binom{k-1}{i}$ for any integers $0 \leq i < j \leq k$. Then $\lambda_{0jk} = 1$ and $\lambda_{ijk} \leq \lambda_{i+1,j+1,k+1}$ for all $0 \leq i < j \leq k$. By Theorem 3.1, the following function is a cohomological degree

$$\mathrm{Deg}_b(M) := e(M) + \sum_{0\leq i<j\leq d} \binom{d-1}{i} e(\overline{U}_M^{ij})_i,$$

where $d = \dim(M)$.

Let $\mathcal{D}(R)$ be the family of cohomological degrees constructed in Theorem 3.1. This is a convex set and we can compute its dimension as follows.

Corollary 3.4. *Let R be a quotient of a Cohen-Macaulay local ring with $\dim(R) = n > 0$. The set $\mathcal{D}(R)$ is a convex set in a real vector space and satisfies*

$$\dim(\mathcal{D}(R)) = \binom{n+2}{3} - n.$$

Proof. The convexity is obvious.

For the dimension, we notice that in the construction of the family, the coefficients λ_{ijk}, $0 \leq i < j \leq k \leq n$, are only required to satisfy the relations

$\lambda_{01k} = 1$, $\lambda_{0jk} \le \lambda_{0,j+1,k+1}$ and $\lambda_{ijk} \le \lambda_{i+1,j+1,k+1}$. At each dimension k, we have $\binom{k+1}{2} - 1$ free parameters. Therefore

$$\dim(\mathcal{D}(R)) = \sum_{k=1}^{n} (\binom{k+1}{2} - 1) = \binom{n+2}{3} - n. \qquad \square$$

In the family of cohomological degrees constructed in Theorem 3.1, the unmixed degree udeg is the lowest member (see Remark 3.2 and Example 3.3(a)). It is particularly interesting to know how the homological degree hdeg relates to this family. We discuss this question in the last part of this section. We will show that in some special cases the homological degree coincides with the degree Deg_b obtained in Example 3.3(b). Recall that

$$\mathrm{Deg}_b(M) := e(M) + \sum_{0 \le i < j \le \dim M} \binom{\dim(M) - 1}{i} e(\overline{U}_M^{ij})_i.$$

From now on, R is a quotient of a Gorenstein local ring S of the same dimension $\dim(R) = \dim(S)$. Let M be a finitely generated R-module. For simplicity, we may assume that $\dim(M) = d = \dim(R)$. The i-th module of deficiency of M is

$$K_M^i := \mathrm{Ext}_S^{d-i}(M, S)$$

for $i = 0, 1, \ldots, d$. The top module K_M^d is the canonical module of M. Then we have

$$\mathrm{hdeg}(M) = e(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \mathrm{hdeg}(K_M^i).$$

Recall that a finitely generated R-module M is a generalized Cohen-Macaulay if $H_{\mathfrak{m}}^i(M)$ is of finite length for any $i = 0, 1, \ldots, \dim(M) - 1$. An interesting extension of Cohen-Macaulayness and generalized Cohen-Macaulayness for modules with associated prime ideals of different heights are the notions of sequentially Cohen-Macaulay modules due to Stanley and Schenzel and sequentially generalized Cohen-Macaulay modules due to N. T. Cuong-L. T. Nhan. In order to give the definition of these modules, it is worth mentioning that any finitely generated R-module M has the so-called dimension filtration

$$D_0 = H_{\mathfrak{m}}^0(M) \subset D_1 \subset \cdots \subset D_t = M,$$

where D_i is the unique maximal submodule of smaller dimension of D_{i+1} for $i = t-1, \ldots, 0$. In particular, $\dim(M) = \dim(D_t) > \dim(D_{t-1}) > \cdots > \dim(D_1) > \dim(D_0)$. Actually, if $d_i := \dim(D_i)$, then $D_i = U_M^{d_i, d_i+1}$ (see [7, Lemma 3.5]). The module M is sequentially (generalized) Cohen-Macaulay if D_{i+1}/D_i is (generalized) Cohen-Macaulay for $i = 0, 1, \ldots, t-1$. Sequential Cohen-Macaulayness has an interesting characterization by means of the Cohen-Macaulayness of the modules of deficiency due to Stanley [16] and Schenzel [15]. Namely, M is sequentially Cohen-Macaulay if and only if K_M^i is either

Cohen-Macaulay of dimension i or zero for $i = 0, 1, \ldots, \dim(M) - 1$. An analogue for sequentially generalized Cohen-Macaulay modules is obtained by N. T. Cuong-L. T. Nhan [10, Theorem 5.3]. They showed that M is sequentially generalized Cohen-Macaulay if and only if K_M^i is either generalized Cohen-Macaulay of dimension i or of finite length for $i = 0, 1, \ldots, \dim(M) - 1$.

In the next, we compute the homological degree and the value of Deg_b for sequentially generalized Cohen-Macaulay modules and compare them in some special cases. To ease the presentation, we stipulate that the zero module has dimension -1 and $\binom{a}{b} = 0$ if either $a < b$ or $b < 0$ for integers a, b.

Proposition 3.5. *Let M be a finitely generated R-module of dimension d.*

(a) *If M is a generalized Cohen-Macaulay module, then*

$$\mathrm{hdeg}(M) = \mathrm{Deg}_b(M) = \mathrm{udeg}(M) = e(M) + I(M),$$

where

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H_{\mathfrak{m}}^i(M)),$$

is the Buchsbaum invariant.

(b) *More generally, suppose that M is a sequentially generalized Cohen-Macaulay module with the dimension filtration*

$$D_0 = H_{\mathfrak{m}}^0(M) \subset D_1 \subset \cdots \subset D_t = M.$$

Denote $d_0 = 0$, $d_j = \dim(D_j)$ for $j = 1, 2, \ldots, t-1$, and set $\Delta := \{d_0, d_1, \ldots, d_{t-1}\}$. Then

$$\mathrm{hdeg}(M) = e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} (e(D_j) + I(K_M^{d_j})) + \sum_{i \notin \Delta} \binom{d-1}{i} \ell(K_M^i),$$

$$\mathrm{Deg}_b(M) = e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} e(D_j) + \gamma,$$

where

$$\gamma = \sum_{i=0}^{t-1} \sum_{j=0}^{d_{i+1}-1} (\binom{d_{i+1}-1}{j} - \binom{d_i - 1}{j}) \ell(H_{\mathfrak{m}}^j(M/D_i)).$$

Proof. (a) Following Vasconcelos [18, Remark 2.10], we have

$$\mathrm{hdeg}(M) = e(M) + I(M),$$

where

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H_{\mathfrak{m}}^i(M)).$$

We will show that $\mathrm{Deg}_b(M) = e(M) + I(M)$.

We have by Proposition 2.3(b),

$$U_M^{id} \simeq \overline{U}_M^{id} \oplus \overline{U}_M^{i,d-1} \oplus \cdots \oplus \overline{U}_M^{i,i+2} \oplus U_M^{i,i+1},$$

which is of finite length as M is generalized Cohen-Macaulay. Hence $e(\overline{U}^{ij}_M)_i = 0$ if $i > 0$ and we obtain by Lemma 2.4,

$$\mathrm{Deg}_b(M) = e(M) + \sum_{0<j\le d} \ell(\overline{U}^{0,j}_M) = e(M) + \ell(U^{0,d}_M) = e(M) + I(M).$$

(b) Let M be a sequentially generalized Cohen-Macaulay module. Due to [10, Theorem 5.3], the module of deficiency K^i_M, $0 \le i < d$, is generalized Cohen-Macaulay of dimension i when $i \in \Delta$ and is of finite length when $i \notin \Delta$. Suppose $i = d_j = \dim(D_j)$ for some $j > 0$. From the short exact sequence

$$0 \to D_j \to M \to M/D_j \to 0,$$

we get an exact sequence

$$0 \to K^i_{M/D_j} \to K^i_M \to K^i_{D_j} \to K^{i-1}_{M/D_j}.$$

Note that $K^i_{M/D_j}, K^{i-1}_{M/D_j}$ are of finite length (see [8, Proposition 3.5]) and $K^i_{D_j}$ is the canonical module of D_j. So $e(K^i_M) = e(K^i_{D_j}) = e(D_j)$. In this case, due to (a), we have $\mathrm{hdeg}(K^i_M) = e(K^i_M) + I(K^i_M) = e(D_j) + I(K^i_M)$. If $i \notin \Delta$, then K^i_M is of finite length and thus $\mathrm{hdeg}(K^i_M) = \ell(K^i_M)$. This implies that

$$\begin{aligned}\mathrm{hdeg}(M) =& e(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \mathrm{hdeg}(K^i_M)\\ =& e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} (e(D_j) + I(K^{d_j}_M)) + \sum_{i\notin\Delta} \binom{d-1}{i} \ell(K^i_M).\end{aligned}$$

For $\mathrm{Deg}_b(M)$, we first recall the direct sum decomposition in Proposition 2.3(b),

$$U^{ij}_M \simeq \overline{U}^{ij}_M \oplus \overline{U}^{i,j-1}_M \oplus \cdots \oplus \overline{U}^{i,i+2}_M \oplus\ U^{i,i+1}_M.$$

Following [7, Lemma 3.5], $D_s = U^{i,i+1}_M$ for any integer i such that $d_s \le i < d_{s+1}$. So by [4, Proposition 3.9(2)], $\overline{U}^{ij}_M \oplus \overline{U}^{i,j-1}_M \oplus \cdots \oplus \overline{U}^{i,i+2}_M$ is of finite length. Hence

$$e(\overline{U}^{ij}_M)_i = \begin{cases} 0 & \text{if } i > 0, j \ge i+2,\\ 0 & \text{if } j = i+1, i \ne d_0, \dots, d_t,\\ e(D_s) & \text{if } j = i+1, i = d_s.\end{cases}$$

This shows in particular that

$$\begin{aligned}\mathrm{Deg}_b(M) &= e(M) + \sum_{j=1}^{t-1} \binom{d-1}{d_j} e(D_j) + \sum_{j=1}^{d} \ell(\overline{U}^{0,j}_M)\\ &= e(M) + \sum_{j=1}^{t-1} \binom{d-1}{d_j} e(D_j) + \ell(U^{0d}_M).\end{aligned}$$

Now we choose an almost p-standard system of parameters $x_1, \ldots, x_d$ of M such that $U_M^{0d} = (0 : x_1)_{M/(x_2,\ldots,x_d)M}$. We have

$$\begin{aligned}\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = n_1 \ldots n_d e(x_1, \ldots, x_d; M) \\ + \sum_{i=1}^{d-1} n_1 \cdots n_i e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_d)M}) \\ + \ell(U_M^{0d})\end{aligned}$$

for all $n_1, \ldots, n_d > 0$ (see Remark 2.2(a)). On the other hand, by [8, Theorem 4.3] we have

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{j=0}^{t} n_1 \cdots n_{d_j} e(x_1, \ldots, x_{d_j}; D_j) + \gamma,$$

also for all $n_1, \ldots, n_d > 0$. Hence $\ell(U_M^{0d}) = \gamma + \ell(H_\mathfrak{m}^0(M))$. Therefore,

$$\mathrm{Deg}_b(M) = e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} e(D_j) + \gamma.$$

□

Corollary 3.6. *Let M be a finitely generated R-module. We have*

$$\mathrm{hdeg}(M) = \mathrm{Deg}_b(M),$$

in the following cases:

(a) *M is a sequentially Cohen-Macaulay module*;
(b) *M is a generalized Cohen-Macaulay module*;
(c) (see also [11, Corollary 5.10]) $\dim(M) = 2$.

Proof. Without lost of generality, we assume that

$$\dim(M) = \dim(R) = \dim(S) = d > 0.$$

(a) Suppose M is a sequentially Cohen-Macaulay module. Then K_M^i is either Cohen-Macaulay of dimension i or zero (see [15] or [16]). Using Proposition 3.5(b), we get that

$$\mathrm{hdeg}(M) = e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} e(D_j),$$

(see also [20, Example 1.5.23]) and

$$\mathrm{Deg}_b(M) = e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} e(D_j) + \gamma,$$

where

$$\gamma = \sum_{i=0}^{t-1} \sum_{j=0}^{d_{i+1}-1} (\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}) \ell(H_\mathfrak{m}^j(M/D_i)) = 0.$$

So $\operatorname{hdeg}(M) = \operatorname{Deg}_b(M)$.

(b) This is proved in Proposition 3.5(a).

(c) Suppose $\dim(M) = 2$. Following [8, Proposition 3.2], the module M is either a generalized Cohen-Macaulay module or a sequentially generalized Cohen-Macaulay module with the dimension filtration

$$D_0 = H^0_{\mathfrak{m}}(M) \subset D_1 \subset D_2 = M,$$

where $\dim(D_1) = 1$. The first case follows from (b). We consider the second case. By Proposition 3.5(b), we have

$$\operatorname{hdeg}(M) = e(M) + e(D_1) + \ell(D_0) + I(K^1_M),$$

$$\operatorname{Deg}_b(M) = e(M) + e(D_1) + \ell(D_0) + \ell(H^1_{\mathfrak{m}}(M/D_1)).$$

Hence it suffices to show that $\ell(H^1_{\mathfrak{m}}(M/D_1)) = I(K^1_M) = \ell(H^0_{\mathfrak{m}}(K^1_M))$.

Without lost of generality we assume that $\dim(R) = \dim(S) = 2$. We have an exact sequence $0 \to D_1 \to M \to M/D_1 \to 0$ where $\dim(D_1) = 1$ and M/D_1 is a generalized Cohen-Macaulay module of dimension 2 and positive depth (see [8, Proposition 3.2]). It derives a long exact sequence

$$\cdots \to \operatorname{Ext}^i_S(M/D_1, S) \to \operatorname{Ext}^i_S(M, S) \to \operatorname{Ext}^i_S(D_1, S) \to \cdots$$

in which $\operatorname{Hom}_S(D_1, S) = 0$ and $\operatorname{Ext}^2_S(M/D_1, S) \simeq K^0_{M/D_1} = 0$. It gives rise to an exact sequence

$$0 \to K^1_{M/D_1} \to K^1_M \to K^1_{D_1} \to 0.$$

Since M/D_1 is generalized Cohen-Macaulay of dimension 2, K^1_{M/D_1} is of finite length. Furthermore, since $K^1_{D_1}$ is the canonical module of D_1, it is Cohen-Macaulay of dimension 1. Hence $K^1_{M/D_1} \simeq H^0_{\mathfrak{m}}(K^1_M)$ from the above short exact sequence. Therefore,

$$\ell(H^0_{\mathfrak{m}}(K^1_M)) = \ell(K^1_{M/D_1}) = \ell(H^1_{\mathfrak{m}}(M/D_1)),$$

by duality. □

In [20, Question 1.5.63(2)], Vasconcelos asked the following question.

Question 3.7. Let R be a Cohen-Macaulay local ring of dimension d. Denote by $\mathcal{N}$ the set of all rational numbers

$$\frac{\operatorname{hdeg}(M) - \operatorname{hdeg}(M/hM)}{e(M)},$$

where $M \neq 0$ runs over the category of finitely generated R-modules and h is any generic hyperplane section with respect to M. Is $\mathcal{N}$ finite or bounded?

As an application of Proposition 3.5 and Corollary 3.6, we give an example showing that the answer to Vasconcelos' question is negative.

Example 3.8. Let $R = k[[X,Y]]$ be the ring of formal power series with coefficients in a field k. Let $\mathfrak{m} = (X,Y)$ and $L_t = \mathfrak{m}^{t+1}$ for $t \geq 0$. The short exact sequence

$$0 \to L_t \to R \to R/\mathfrak{m}^{t+1} \to 0$$

deduces that L_t is a generalized Cohen-Macaulay module with $H^0_{\mathfrak{m}}(L_t) = 0$ and $H^1_{\mathfrak{m}}(L_t) \simeq R/\mathfrak{m}^{t+1}$. Following Proposition 3.5(a),

$$\operatorname{hdeg}(L_t) = e(L_t) + \ell(R/\mathfrak{m}^{t+1}) = 1 + \binom{t+2}{2}.$$

On the other hand, we may change the variables so that $h = X$ is a generic hyperplane of L_t. As $\dim(L_t/hL_t) = 1$, we have

$$\operatorname{hdeg}(L_t/hL_t) = e(L_t) + \ell(H^0_{\mathfrak{m}}(L_t/XL_t)).$$

From the two exact sequences

$$0 \to L_t \to R \to R/\mathfrak{m}^{t+1} \to 0,$$

$$0 \to R \xrightarrow{*X} R \to R/(X) \to 0,$$

we get the exact sequences

$$0 \to \operatorname{Tor}_1^R(R/(X), R/\mathfrak{m}^{t+1}) \to L_t/XL_t \to R/XR \to R/(X, Y^{t+1}) \to 0,$$

$$0 \to \operatorname{Tor}_1^R(R/(X), R/\mathfrak{m}^{t+1}) \to R/\mathfrak{m}^{t+1} \xrightarrow{*X} R/\mathfrak{m}^{t+1} \to R/(X, Y^{t+1}) \to 0.$$

The first sequence induces an isomorphism

$$\operatorname{Tor}_1^R(R/(X), R/\mathfrak{m}^{t+1}) \simeq H^0_{\mathfrak{m}}(L_t/XL_t)$$

while the second gives rise to isomorphisms $\operatorname{Tor}_1^R(R/(X), R/\mathfrak{m}^{t+1}) \simeq (0 : X)_{R/\mathfrak{m}^{t+1}} \simeq \mathfrak{m}^t/\mathfrak{m}^{t+1}$. Hence

$$\operatorname{hdeg}(L_t/hL_t) = 1 + \ell(\mathfrak{m}^t/\mathfrak{m}^{t+1}) = t + 2.$$

We obtain

$$\frac{\operatorname{hdeg}(L_t) - \operatorname{hdeg}(L_t/hL_t)}{e(L_t)} = \binom{t+1}{2}.$$

Therefore $\mathcal{N} \supseteq \{\binom{t+1}{2} : \text{ for } t \geq 0\}$ which is neither finite nor bounded.

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