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GENERALIZED CAMERON-STORVICK TYPE THEOREM VIA THE BOUNDED LINEAR OPERATORS

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Abstract. In this paper, we establish the generalized Cameron-Storvick type theorem on function space. We then give relationships involving the generalized Cameron-Storvick type theorem, modified generalized integral transform and modified convolution product. A motivation of studying the generalized Cameron-Storvick type theorem is to generalize formulas and results with respect to the modified generalized integral transform on function space. From the some theories and formulas in the functional analysis, we can obtain some formulas with respect to the translation theorem of exponential functionals.

1. Introduction

The concept of the integral transform $\mathcal{F}_{\alpha,\beta}$ was introduced by Lee on abstract Wiener space (B, H, m) [21]. Since then many mathematicians have studied the integral transform $\mathcal{F}_{\alpha,\beta}$, the Fourier-Gauss transform $\mathcal{G}_{\alpha,\beta}$ and the generalized Fourier-Gauss transform $\mathcal{G}_{S,T}$ of functionals on (B, H, m) [1,2,9,12–19,21]. The function space $C_{a,b}[0, T]$, induced by generalized Brownian motion, was introduced by J. Yeh in [23] and studied extensively in [4, 5, 7, 8, 10, 11]. In [6–8, 11], the authors studied the generalized integral transform $\mathcal{F}_{\gamma,\beta}$ of functionals on $C_{a,b}[0, T]$. The space $(C_{a,b}[0, T], C'_{a,b}[0, T], \mu)$ as an example of abstract Wiener space was initiated by Chang et al in [4]. The classical Wiener space and $(C_{a,b}[0, T], C'_{a,b}[0, T], \mu)$ are the most important examples of an abstract Wiener space, for more details see [4, 5, 7, 14, 20, 23].

The Cameron-Storvick type theorem is that the function space integrals involving the first variation can be expressed by the ordinary forms without concept the first variation. For this reason, it is also called the integration

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by parts formula. Numerous constructs and theories regarding the Cameron-Storvick type theorem have been studied and applied in many papers [3, 4, 6, 8, 10, 11].

In this paper we establish the most generalized Cameron-Storvick type theorem via the bounded linear operators. We then establish some relationships between the modified generalized integral transforms and the modified convolution products involving the first variations for the exponential functionals via the generalized Cameron-Storvick type theorem. We are going to work in the framework of general Gaussian space $(C_{a,b}[0,T], C'_{a,b}[0,T], \mu)$. Our transform and the Cameron-Storvick type theorem are more general than various transforms and the Cameron-Storvick type theorem considered in previous papers.

The results in this paper are quite a lot more complicated because the generalized Brownian motion used in this paper is nonstationary in time and is subject to a drift $a(t)$. The generalized Brownian motion can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [22].

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\mathbf{\Omega}, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\mathbf{\Omega}, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)$=0 almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \ldots, Y(t_n,\omega))$ is normally distributed with density function

$$W_n(\vec{t},\vec{u}) = \Big((2\pi)^n \prod_{j=1}^{n}(b(t_j) - b(t_{j-1}))\Big)^{-1/2} \times \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}\frac{((u_j - a(t_j)) - (u_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\bigg\},$$

where $\vec{u} = (u_1, \ldots, u_n)$, $u_0 = 0$, $\vec{t} = (t_1, \ldots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$ and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

In [24], Yeh showed that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$, and that the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$. We then complete this function space to obtain $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$.

Let

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\},$$

where $|a|(t)$ denotes the total variation of the function $a(\cdot)$ on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} \equiv \int_0^T u(t)v(t)d[b(t) + |a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space, where $\|\cdot\|_{a,b} = \sqrt{(\cdot,\cdot)_{a,b}}$. We note that $\|v\|_{a,b} = 0$ if and only if $v(t) = 0$ almost everywhere on $[0,T]$. In addition, for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$; see [4, 6–8, 11]. Then the PWZ stochastic integral $\langle v, x\rangle$ is a Gaussian random variable with mean $\int_0^T v(t)da(t)$ and variance $\int_0^T v^2(t)db(t)$.

Let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$D_t w = z(t) = \frac{w'(t)}{b'(t)}.$$

Then $C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T D_t w_1 D_t w_2 db(t)$$

is a separable Hilbert space. Note that the two separable Hilbert spaces, $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are homeomorphic. Furthermore, we note that

$$\widetilde{C_{a,b}[0,T]} \equiv \widetilde{C_{a,b}} \subset C'_{a,b}[0,T] \approx C'_{a,b}[0,T] \subset C_{a,b}[0,T],$$

where $\widetilde{C_{a,b}}$ is the topological dual space of $C_{a.b}[0,T]$.

Remark 2.1. Recall that the function $a : [0,T] \to \mathbb{R}$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$. But the function $a(t) = t^{2/3}$, $t \in [0,T]$, is not an element in $L^2_{a,b}[0,T]$ even though $a'(t) = \frac{2}{3}t^{-\frac{1}{3}} \in L^2[0,T]$. In order to apply the translation theorem, we have to add a condition for the function a. Our conditions on $b : [0,T] \to \mathbb{R}$ imply that $0 < M_1 < b'(t) < M_2$ for all $t \in [0,T]$ and some positive real numbers M_1 and M_2. Now throughout this paper we add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < \infty. \tag{1}$$

Let $w_a(t) = \frac{a'(t)}{b'(t)}$. Then we see that

$$a(t) = \int_0^t \frac{a'(s)}{b'(s)} db(s) = \int_0^t w_a(s) db(s)$$

and

$$\int_0^T w_a^2(t) d[b(t) + |a|(t)] \le \frac{1}{M_1} \|a'\|_{L^2[0,T]} + \frac{1}{M_1^2} \int_0^T |a'(t)|^2 d|a|(t) < \infty$$

by condition (1), and so w_a is an element of $L^2_{a,b}[0,T]$. Thus a is an element of $C'_{a,b}[0,T]$.

We denote the function space integral of a $\mathcal{W}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] \equiv E_x[F(x)] = \int_{C_{a,b}[0,T]} F(x) d\mu(x)$$

whenever the integral exists.

For the purposes of this paper, we define a complexification of $C_{a,b}[0,T]$ by

$$\begin{aligned} K_{a,b}[0,T] =& \{x : [0,T] \to \mathbb{C} \mid x(0) = 0,\ \mathrm{Re}(x) \in C_{a,b}[0,T] \\ & \text{and } \mathrm{Im}(x) \in C_{a,b}[0,T]\}. \end{aligned}$$

We also let $K'_{a,b}[0,T] \equiv K'_{a,b}$ denote the complexification of $C'_{a,b}[0,T]$.

For $x \in C_{a,b}[0,T]$ and $w \in C'_{a,b}[0,T]$ with $w(t) = \int_0^t z(s) db(s)$ for $t \in [0,T]$, $(w,x)^\sim \equiv \langle D_t w, z\rangle = \langle z, x\rangle$ is a well-defined Gaussian random variable with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. Then we have the following observations:

(i) For each w in $C'_{a,b}[0,T]$, $(w,a)^\sim = (w,a)_{C'_{a,b}}$ and $(w,w)^\sim = \|w\|^2_{C'_{a,b}}$.

(ii) For $x \in K_{a,b}[0,T]$ and $w \in C'_{a,b}[0,T]$, let $(w,x)^\sim = (w, \mathrm{Re}(x))^\sim + i(w, \mathrm{Im}(x))^\sim$.

(iii) For $x \in C_{a,b}[0,T]$ and $w \in K_{a,b}[0,T]$, let $(w,x)^\sim = (\mathrm{Re}(w), x)^\sim + i(\mathrm{Im}(w), x)^\sim$.

(iv) In views of (ii) and (iii), for $x \in K_{a,b}[0,T]$ and $w \in K'_{a,b}$,

$$\begin{aligned} (w,x)^\sim = & (\mathrm{Re}(w), \mathrm{Re}(x))^\sim + i(\mathrm{Im}(w), \mathrm{Re}(x))^\sim \\ & + i(\mathrm{Re}(w), \mathrm{Im}(x))^\sim - (\mathrm{Im}(w), \mathrm{Im}(x))^\sim. \end{aligned}$$

In this case $(\cdot,\cdot)^\sim$ is the complex bilinear form on $\widetilde{K_{a,b}} \times K_{a,b}[0,T]$.

We next state the following useful formula for function space integrals; namely that for $w \in K'_{a,b}$ and $x \in C_{a,b}[0,T]$,

$$E_x[\exp\{(w,x)^\sim\}] = \exp\left\{\frac{1}{2}(w,w)^\sim + (w,a)^\sim\right\}. \tag{2}$$

If also $w \in C'_{a,b}[0,T]$, then

$$E_x[\exp\{(w,x)^\sim\}] = \exp\left\{\frac{1}{2}\|w\|^2_{C_{a,b}} + (w,a)_{C_{a,b}}\right\}.$$

We finish this section by stating the definitions of the modified generalized integral transform (MGIT), the modified convolution product (MCP) and the first variation of functionals on $K_{a,b}[0,T]$.

Definition. Let $\mathcal{L} \equiv \mathcal{L}(K_{a,b}[0,T])$ be the set of all bounded linear operators on $K_{a,b}[0,T]$. Let Ψ and Φ be functionals on $K_{a,b}[0,T]$ and let $S, R, A, B, C, D \in \mathcal{L}$. Then the MGIT $\mathcal{F}_{S,R}(\Psi)$ of Ψ is defined by the formula (if it exists)

$$\mathcal{F}_{S,R}(\Psi)(y) = \int_{C_{a,b}[0,T]} \Psi(Sx + Ry)d\mu(x), \qquad y \in K_{a,b}[0,T]. \tag{3}$$

Also, the MCP $(\Psi * \Phi)_{ABCD}$ of Ψ and Φ is defined by the formula (if it exists)

$$\begin{aligned} &(\Psi * \Phi)_{ABCD}(y) \\ &= \int_{C_{a,b}[0,T]} \Psi(Ax + By)\Phi(Cx + Dy)d\mu(x), \quad y \in K_{a,b}[0,T] \end{aligned} \tag{4}$$

and the first variation of Ψ is defined by the formula (if it exists)

$$\delta\Psi(x|u) = \frac{\partial}{\partial k}\Psi(x + ku)\bigg|_{k=0}, \quad x, u \in K_{a,b}[0,T]. \tag{5}$$

Remark 2.2. (1) When $S = \gamma$ and $T = \beta$, for nonzero complex numbers γ and β, $\mathcal{F}_{S,T}$ is the generalized integral transform used in [7, 8].

(2) When $A = \frac{\gamma}{\sqrt{2}}, B = \frac{1}{\sqrt{2}}, C = -\frac{\gamma}{\sqrt{2}}$ and $D = \frac{1}{\sqrt{2}}$, for a nonzero complex number γ, $(\Psi * \Phi)_{ABCD}$ is the convolution product used in [8].

(3) The first variation $\delta F(x|u)$ acts like a directional derivative in the direction of u.

3. Fundamental formulas

In this section we list various fundamental formulas with respect to the MGIT, the MCP and the first variation for the exponential functionals.

For the study of our MGIT and MCP of functionals on $K_{a,b}[0,T]$, we will use a fundamental set of $L^2(C_{a,b}[0,T])$. We then introduce a meaningful class of functionals on $K_{a,b}[0,T]$. Let $\mathcal{A}$ be the class of all functionals which have the form

$$\Psi_w(x) = \exp\bigg\{(w, x)^\sim - \frac{1}{2}(w, w)^\sim - (w, a)^\sim\bigg\} \tag{6}$$

for each $w \in \widetilde{K_{a,b}}$ and $x \in C_{a,b}[0,T]$. Hence $E[\Psi_n] = 1$. The functionals given by Equation (6) are called the generalized exponential type functionals on $C_{a,b}[0,T]$. We also note that each Ψ_w is an element of $L^2(C_{a,b}[0,T])$. Then we have the following observation to understand this paper. For each $w \in \widetilde{K_{a,b}}, x \in C_{a,b}[0,T]$ and $S \in \mathcal{L}$

$$(w, Sx)^\sim = (S^*w, x)^\sim.$$

Because $S \in \mathcal{L}$, $S^* \in \mathcal{L}(\widetilde{K_{a,b}}, \widetilde{K_{a,b}})$ and hence $S^*w \in \widetilde{K_{a,b}}$ is well-defined and so $(S^*w, x)^\sim$ is also well-defined.

Our goal in the remainder of this paper is to establish various formulas and relationships for the MGIT, the MCP and the first variation of the generalized exponential type functionals defined by (6) above which we proceed to do in Sections 4, 5 and 6 below.

For notational convenience we adopt the following notation: for $R_1, R_2, \ldots, R_n \in \mathcal{L}$ and $w \in \widetilde{K_{a,b}}$, let

$$M(R_1,\ldots,R_n;w) \equiv \exp\Big\{\frac{1}{2}\Big(\big(\sum_{j=1}^{n} R_jR_j^* - I\big)w, w\Big)^{\sim} + \Big(\big(\sum_{j=1}^{n} R_j^* - I\big)w, a\Big)^{\sim}\Big\}, \tag{7}$$

where R_j^* is the Hilbert-adjoint operator of R_j, $j = 1, 2, \ldots, n$. Note that the symmetric property for $M(\cdot; w)$. That is to say,

$$M(R_1, R_2, \ldots, R_n; w) = M(R_{\pi(1)}, R_{\pi(2)}, \ldots, R_{\pi(n)}; w)$$

for any permutation π of $\{1, \ldots, n\}$.

To obtain simple expressions for our formulas and results, we use the following lemma which plays a key role in this paper.

Lemma 3.1. *Let $\Psi_{w_n} \equiv \Psi_n \in \mathcal{S}(C_{a,b}[0,T])$ be a generalized exponential type functional of the form (6) and let $R_1, R_2 \in \mathcal{L}$. Then*

$$M(R_1; w_n) \exp\{(R_2^* w_n, y)^{\sim}\} = M(R_1, R_2; w_n)\Psi_{R_2^* w_n}(y). \tag{8}$$

The following theorem was established in [11, Theorems 4.3, 4.4 and 4.5].

Theorem 3.2. *Let $\Psi_n, \Psi_m \in \mathcal{S}(C_{a,b}[0,T])$ be generalized exponential type functionals of the form (6) and let u be an element of $C'_{a,b}[0,T]$. Then for $S, R, A, B, C, D \in \mathcal{L}$, the MGIT $\mathcal{F}_{S,R}(\Psi_n)$ of Ψ_n, the MCP $(\Psi_n * \Psi_m)_{ABCD}$ of Ψ_n and Ψ_m, and the first variation $\delta\Psi_n(\cdot|u)$ of Ψ_n exist, and are given by the formulas*

$$\mathcal{F}_{S,R}(\Psi_n)(y) = M(S, R; w_n)\Psi_{R^* w_n}(y), \tag{9}$$

$$(\Psi_n * \Psi_m)_{ABCD}(y) = M(A, B; w_n)M(C, D; w_m) \exp\Big\{(CA^* w_n, w_m)^{\sim}\Big\}\Psi_{B^* w_n}(y)\Psi_{D^* w_m}(y) \tag{10}$$

and

$$\delta\Psi_n(x|u) = (w_n, u)^{\sim}\Psi_n(x). \tag{11}$$

Furthermore, they are also generalized exponential type functionals.

In our next theorem, we obtain the composition formula for the MGIT.

Theorem 3.3. *Let $\Psi_n \in \mathcal{S}(C_{a,b}[0,T])$ be a generalized exponential type functional of the form (6) and let S_1, S_2, R_1 and R_2 be elements of $\mathcal{L}$. Then we have*

$$\mathcal{F}_{S_1,R_1}(\mathcal{F}_{S_2,R_2}\Psi_n)(y) = \mathcal{F}_{S_3,R_3}(\Psi_n)(y), \tag{12}$$

where $R_3 = R_1R_2$ if and only if the following condition

$$M(S_2, S_1R_2, R_1R_2; w_n) = M(S_3, R_3; w_n)$$

holds.

Proof. Equation (12) can be obtained immediately from the definition of MGIT and Equation (9) repeatedly. □

4. Cameron-Storvick type theorem

In this section we are going to establish the generalized Cameron-Storvick type theorem with respect to the MGIT. In order to do this, we need the following Lemma 4.1 below.

The following lemma was established in [10, p. 379].

Lemma 4.1 (Translation theorem)**.** *Let x_0 be an element of $C'_{a,b}[0,T]$. If F is μ-integrable on $C_{a,b}[0,T]$, then*

$$E_x[F(x+x_0)] = \exp\bigg\{-\frac{1}{2}\|x_0\|^2_{C'_{a,b}} - (x_0, a)_{C'_{a,b}}\bigg\}E_x[F(x)\exp\{(x_0, x)^{\sim}\}]. \tag{13}$$

Using Equation (13), we establish a more generalized translation theorem to obtain the generalized Cameron-Storvick type theorem.

Theorem 4.2 (Translation theorem with respect to the operators)**.** *Let S_1 be an elements of $\mathcal{L}$ with $S_1^*S_1 = I$, where I is the identity operator and let $S_2 \in \mathcal{L}(C'_{a,b}, \widetilde{C_{a,b}})$. Let F be a integrable functional on $C_{a,b}[0,T]$ and let $x_0 \in C'_{a,b}[0,T]$. Then*

$$\begin{aligned} E_x[F(S_1x + S_2x_0)] = {} & \exp\bigg\{-\frac{1}{2}\|S_1^*S_2x_0\|^2_{C'_{a,b}} - (S_1^*S_2x_0, a)_{C'_{a,b}}\bigg\} \\ & \times E_x[F(S_1x)\exp\{(S_1^*S_2x_0, x)^{\sim}\}]. \end{aligned} \tag{14}$$

Proof. We first note that for $x_0 \in C'_{a,b}[0,T]$, $S_2x_0 \in \widetilde{C_{a,b}} \subset C'_{a,b}[0,T]$. Since $S_1 \in \mathcal{L}$, $S_1^* \in \mathcal{L}(\widetilde{K_{a,b}}, \widetilde{K_{a,b}})$ and hence $S_1^*S_2x_0$ is well-defined and it is an element of $\widetilde{C_{a,b}} \subset C'_{a,b}[0,T]$. Thus, Equation (14) immediately follows from Equation (13) by replacing F_{S_1} by F, where $F_{S_1}(x) = F(S_1x)$. Because we note that

$$F(S_1x + S_2x_0) = F(S_1(x + S_1^*S_2x_0)) = F_{S_1}(x_0 + \theta_0)$$

with $\theta_0 = S_1^*S_2x_0$. □

Corollary 4.3. *Theorem 4.2 tells us that our translation theorem is the most generalized theorem to date. All version of the translation theorem is a corollary of Theorem 4.2. Furthermore, if we take the Gaussian process $Z_h(x,t)$ as operators S_1 and S_2, then Theorem* 3.2 *established and used in* [3] *is a corollary of Theorem 4.2 above.*

From now on, we are going to establish the generalized Cameron-Strovick type theorem.

The first version of the generalized Cameron-Strovick type theorem is that the MGIT of the first variation for the exponential functional.

Theorem 4.4. *Let $S, R \in \mathcal{L}$ with $S^*S = I$ and let $\Psi_n \in \mathcal{S}(C_{a,b}[0,T])$ be a generalized exponential type functional of the form* (6). *Also, let $u \in \widetilde{C_{a,b}} \subset C'_{a,b}[0,T]$. Then we have*

$$\begin{aligned}\mathcal{F}_{S,R}(\delta\Psi_n(\cdot|u))(y) = \mathcal{F}_{S,R}((u,\cdot)^\sim \Psi_n(\cdot))(y) \\ - ((R^*u, y)^\sim + (S^*u, a)^\sim)\mathcal{F}_{S,R}(\Psi_n)(y).\end{aligned} \tag{15}$$

Proof. The existence of Equation (15) is obtained from Theorem 3.2. We left to show that the equality in Equation (15) holds. This equality (15) are immediate consequence of Lemma 4.1 and Theorem 4.2 by replacing S_1 and S_2 by S and I respectively. □

Equation (15) tells us that

$$\begin{aligned}\mathcal{F}_{S,R}((u,\cdot)^\sim \Psi_n(\cdot))(y) = \mathcal{F}_{S,R}(\delta\Psi_n(\cdot|u))(y) \\ + ((R^*u, y)^\sim - (S^*u, a)^\sim)\mathcal{F}_{S,R}(\Psi_n)(y).\end{aligned} \tag{16}$$

In fact, it is not easy to calculate the MGIT involving polynomial weight. That is to say, a calculation of the following function space integral

$$\int_{C_{a,b}[0,T]} (u_1, x)^\sim \exp\{(u_2, x)^\sim\} d\mu(x)$$

is not easy unless u_1 and u_2 are orthogonal. In these cases, we have to use the concept of the Gram-Schmidt process and the usual function space integration formulas. From Equation (16), we note that the MGIT of exponential functionals with polynomial weight can be calculated very easily from the MGIT of exponential functionals.

From an example, we explain the usefulness of the Cameron-Storvick type theorem.

Example 4.5. Let $\Psi_n \in \mathcal{S}(C_{a,b}[0,T])$ be a generalized exponential type functional of the form (6). Also, let $u \in C'_{a,b}[0,T]$. Using Equations (10) and (12), we obtain that

$$\mathcal{F}_{S,R}(\Psi_n)(y) = M(S, R; w_n)\Psi_{R^*w_n}(y)$$

and

$$\Psi_n(x|u) = (w_n, u)^\sim \Psi_n(x).$$

Hence using Equation (16), we have

$$\begin{aligned}&\mathcal{F}_{S,R}((u,\cdot)^\sim \Psi_n(\cdot))(y) \\ &= (w_n, u)^\sim M(S, R; w_n)\Psi_{R^*w_n}(y) \\ &\quad + ((R^*u, y)^\sim - (S^*u, a)^\sim)M(S, R; w_n)\Psi_{R^*w_n}(y) \\ &= ((w_n, u)^\sim + (R^*u, y)^\sim - (S^*u, a)^\sim)M(S, R; w_n)\Psi_{R^*w_n}(y).\end{aligned}$$

In Theorem 4.6, we give the second version of the Cameron-Storvick type theorem. This is that the first variation of the MGIT for the exponential functional.

Theorem 4.6. *Let $S \in \mathcal{L}$ with $S^*S = I$ and $R \in \mathcal{L}(C'_{a,b}, \widetilde{C_{a,b}})$. Let $\Psi_n \in \mathcal{S}(C_{a,b}[0,T])$ be a generalized exponential type functional of the form* (6). *Also, let u be an element of $C'_{a,b}[0,T]$. Then*

$$\begin{aligned}\delta\mathcal{F}_{S,R}(\Psi_n)(y|u) = {}& \mathcal{F}_{S,R}((Ru,\cdot)^{\sim}\Psi_n(\cdot))(y) \\ & - ((Ru,Ry)^{\sim} + (S^*Ru,a)^{\sim})\mathcal{F}_{S,R}(\Psi_n)(y).\end{aligned} \tag{17}$$

Proof. The existence of Equation (17) is obtained from Theorem 3.2. We left to show that the equality in Equation (17) holds. In order to do this, we shall use Equations (9), (11), (14) with replacing S_1 and S_2 by S and R respectively. Then the equality in Equation (17) is immediate obtained by the integration by parts formula. □

Remark 4.7. From Equations (15) and (17) in Theorems 4.4 and 4.6 respectively, we can conclude that

$$\delta\mathcal{F}_{S,R}(\Psi_n)(y|u) = \mathcal{F}_{S,R}(\delta\Psi_n(\cdot|Ru))(y).$$

5. Relationships

In this section we establish various relationships among the MGIT, the MCP and the first variation for generalized exponential type functionals via the generalized Cameron-Storvick type theorems in Theorems 4.4 and 4.6.

The following relationships were established in [11]. These relationships are called fundamental formulas with respect to the MGIT, the MCP and the first variation for the exponential functionals.

Theorem 5.1. *We list various relationships between the MGIT and the MCP as follows*:

(i) *The MGIT is a commutative operator, that is to say,*

$$\mathcal{F}_{S_2,R_2}(\mathcal{F}_{S_1,R_1}(\Psi_n))(y) = \mathcal{F}_{S_1,R_1}(\mathcal{F}_{S_2,R_2}(\Psi_n))(y)$$

if and only if

$$S_1 = S_2, \quad R_1R_2 = R_2R_1 \quad \text{and} \quad R_1S_2 = R_2S_1.$$

(ii) *The MCP is a commutative operator, that is to say, $(\Psi_n * \Psi_m)_{ABCD} = (\Psi_m * \Psi_n)_{ABCD}$ if and only if*

$$A = C \quad \text{and} \quad B = D.$$

(iii) *Fundamental formula* 1:

$$\mathcal{F}_{S_1,R_1}(\Psi_n * \Psi_m)_{A_1B_1C_1D_1}(y) = (\mathcal{F}_{S_2,R_2}(\Psi_n) * \mathcal{F}_{S_3,R_3}(\Psi_m))_{A_2B_2C_2D_2}(y) \tag{18}$$

if and only if the following conditions hold:

(a) $B_1R_1 = R_2B_2$ *and* $D_1R_1 = R_3D_2$;

(b) $M(A_1, B_1S_1; w_n)M(C_1, D_1S_1; w_m)$
$= M(S_2, R_2A_2; w_n)M(S_3, R_3C_2; w_m)$;
(c) $C_1A_1^* + (D_1S_1)(B_1S_1)^* = (R_3C_2)(R_2A_2)^*$.

(iv) *Fundamental formula* 2:

$$\begin{aligned}&\delta(\Psi_n * \Psi_m)_{ABCD}(y|u)\\ &= ((\delta\Psi_n(\cdot|Bu) * \Psi_m)_{ABCD} + (\Psi_n * \delta\Psi_m(\cdot|Du))_{ABCD})(y).\end{aligned} \tag{19}$$

In our next theorem, we establish the fundamental formula with respect to the MGIT and the MCP. Equation (20) is called the Fubini formula or the Bearman formula.

Theorem 5.2. *Let S_i, R_i, A, B, C and D be elements of $\mathcal{L}$ for $i = 1, 2, 3$. Let $\Psi_n, \Psi_m \in \mathcal{S}(C_{a,b}[0,T])$ be generalized exponential type functionals of the form* (6) *and let u be an element of $C'_{a,b}[0,T]$. Then*

$$\mathcal{F}_{S_1,T_1}(\Psi_n * \Psi_m)_{ABCD}(y) = \mathcal{F}_{S_2,T_2}(\Psi_n)(y)\mathcal{F}_{S_3,T_3}(\Psi_m)(y) \tag{20}$$

if and only if the following conditions hold:

(a) $B = D = I$;
(b) $M(A, S_1; w_n) = M(S_2; w_n)$ *and* $M(C, S_1; w_m) = M(S_3; w_m)$;
(c) $CA_1^*S_1S_1^* = 0$.

Proof. First using Equations (3), (9) and (10), we obtain that

(21)
$$\begin{aligned}&\mathcal{F}_{S_1,R_1}(\Psi_n * \Psi_m)_{ABCD}(y)\\ &= M(A; w_n)M(C; w_m)\int_{C_{a,b}[0,T]} \exp\bigg\{((BR_1)^*w_n + (DR_1)^*w_m, y)^\sim\\ &\qquad + ((BS_1)^*w_n + (DS_1)^*w_m, x)^\sim + (CA_1^*w_n, w_m)^\sim\\ &\qquad + ((A^* - I)w_n, a)^\sim + ((C^* - I)w_n, a)^\sim\bigg\}d\mu(x)\\ &= M(A, BS_1; w_n)M(C, DS_1; w_m)\\ &\quad \times \exp\bigg\{((BR_1)^*w_n + (DR_1)^*w_m, y)^\sim + ((CA^* + (DS_1)(BS_1)^*)w_n, w_m)^\sim\bigg\}.\end{aligned}$$

On the other hand, using Equation (9) repeatedly, we have

$$\begin{aligned}&\mathcal{F}_{S_2,T_2}(\Psi_n)(y)\mathcal{F}_{S_3,T_3}(\Psi_m)(y)\\ &= M(S_2, R_2; w_n)M(S_3, R_3; w_m)\Psi_{R_2^*w_n}(y)\Psi_{R_3^*w_m}(y)\\ &= M(S_2; w_n)M(S_3; w_m)\exp\{(R_2^*w_n + R_3^*w_m, y)^\sim\}.\end{aligned} \tag{22}$$

By comparing two Equations (21) and (22), the proof of Theorem 5.2 is completed as desired. □

From Theorem 3.2 thru 5.2, we gave some conditions for the operators to establish various results and formulas. From now on, we shall omit these conditions to avoid being expressed in complexity in the statements.

We recall the basic property of the first variation. For exponential functionals Ψ_n and Ψ_m, $\delta(\Psi_n\Psi_m)(x|u) = \delta\Psi_n(x|u)\Psi_m + \Psi_n\delta\Psi_m(x|u)$. It looks like that the derivative of the product for the exponential functionals. Using this basic property with relationships which were obtained in Theorems 5.1 and 5.2, we have the following theorem.

Theorem 5.3. *Let $\Psi_n, \Psi_m \in \mathcal{S}(C_{a,b}[0,T])$ be generalized exponential type functionals of the form* (6) *and let u be an element of $C'_{a,b}[0,T]$. Then we have*

$$\begin{aligned}
&\delta(\mathcal{F}_{S_1,R_1}(\Psi_n * \Psi_m)_{ABCD})(y|u)\\
&= \mathcal{F}_{S_2,R_2}((R_2u,\cdot)^{\sim}\Psi_n(\cdot))(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y)\\
&\quad + \mathcal{F}_{S_3,R_3}((R_3u,\cdot)^{\sim}\Psi_m(\cdot))(y)\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\\
&\quad - ((R_2^*R_2u + R_3^*R_3u, y)^{\sim} + (S_2^*R_2u + S_3^*R_3u, a)^{\sim})\\
&\quad \times \mathcal{F}_{S_2,R_2}(\Psi_n)(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y).
\end{aligned}\tag{23}$$

Proof. Using Equation (20) together with the basic property with respect to the first variation to $\mathcal{F}_{S_2,R_2}(\Psi_n)\mathcal{F}_{S_3,R_3}(\Psi_m)$ instead of FG, we have

$$\begin{aligned}
&\delta(\mathcal{F}_{S_1,R_1}(\Psi_n * \Psi_m)_{ABCD})(y|u)\\
&= \delta(\mathcal{F}_{S_2,R_2}(\Psi_n)\mathcal{F}_{S_3,R_3}(\Psi_m))(y|u)\\
&= \delta\mathcal{F}_{S_2,R_2}(\Psi_n)(y|u)\mathcal{F}_{S_3,R_3}(\Psi_m)(y) + \mathcal{F}_{S_2,R_2}(\Psi_n)(y)\delta\mathcal{F}_{S_3,R_3}(\Psi_m)(y|u).
\end{aligned}\tag{24}$$

Applying Equation (17) to the two terms in the last expression of Equation (24), we have

$$\begin{aligned}
&\delta(\mathcal{F}_{S_1,R_1}(\Psi_n * \Psi_m)_{ABCD})(y|u)\\
&= \mathcal{F}_{S_2,R_2}((R_2u,\cdot)^{\sim}\Psi_n(\cdot))(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y)\\
&\quad - ((R_2u, R_2y)^{\sim} + (S_2^*R_2u, a)^{\sim})\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y)\\
&\quad + \mathcal{F}_{S_3,R_3}((R_3u,\cdot)^{\sim}\Psi_m(\cdot))(y)\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\\
&\quad - ((R_3u, R_3y)^{\sim} + (S_3^*R_3u, a)^{\sim})\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y),
\end{aligned}$$

which yields Equation (23) as desired. □

In our next theorem, we give a relationship between the MGIT and the MCP.

Theorem 5.4. *Let $\Psi_n, \Psi_m \in \mathcal{S}(C_{a,b}[0,T])$ be generalized exponential type functionals of the form* (6) *and let u be an element of $C'_{a,b}[0,T]$. Then we have*

$$\begin{aligned}
&\mathcal{F}_{S_1,R_1}(\delta(\Psi_n * \Psi_m)_{ABCD}(\cdot|u))(y)\\
&= \mathcal{F}_{S_2,R_2}((Bu,\cdot)^{\sim}\Psi_n(\cdot))(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y)\\
&\quad + \mathcal{F}_{S_3,R_3}((Du,\cdot)^{\sim}\Psi_m(\cdot))(y)\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\\
&\quad - ((R_2^*Bu + R_3^*Du, y)^{\sim} - (S_2^*Bu + S_3^*Du, a)^{\sim})\\
&\quad \times \mathcal{F}_{S_2,R_2}(\Psi_n)(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y).
\end{aligned}\tag{25}$$

Proof. In order to establish Theorem 5.4, we first use Equations (11) and (20). Then we have

(26)

$$\begin{aligned}
&\mathcal{F}_{S_1,R_1}(\delta(\Psi_n * \Psi_m)_{ABCD}(\cdot|u))(y)\\
=&\mathcal{F}_{S_1,R_1}((((\delta\Psi_n(\cdot|Bu) * \Psi_m)_{ABCD}+(\Psi_n * \delta\Psi_m(\cdot|Du))_{ABCD}))(y)\\
=&\mathcal{F}_{S_1,R_1}((((\delta\Psi_n(\cdot|Bu) * \Psi_m)_{ABCD})(y)+\mathcal{F}_{S_1,R_1}(\Psi_n * \delta\Psi_m(\cdot|Du))_{ABCD}))(y)\\
=&\mathcal{F}_{S_2,R_2}(\delta\Psi_n(\cdot|Bu))(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y)+\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\mathcal{F}_{S_3,R_3}(\delta\Psi_m(\cdot|Du))(y).
\end{aligned}$$

We now apply Equation (17) to the two terms in the last expression of Equation (26), we have

$$\begin{aligned}
&\mathcal{F}_{S_1,R_1}(\delta(\Psi_n * \Psi_m)_{ABCD}(\cdot|u))(y)\\
&= \mathcal{F}_{S_2,R_2}((Bu,\cdot)^{\sim}\Psi_n(\cdot))(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y)\\
&\quad - ((Bu,R_2y)^{\sim} + (S_2^*Bu,a)^{\sim})\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y)\\
&\quad + \mathcal{F}_{S_3,R_3}((Du,\cdot)^{\sim}\Psi_m(\cdot))(y)\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\\
&\quad - ((Du,R_3y)^{\sim} + (S_3^*Du,a)^{\sim})\mathcal{F}_{S_2,R_2}(\Psi_n)(y)\mathcal{F}_{S_3,R_3}(\Psi_m)(y),
\end{aligned}$$

which yields Equation (25) as desired. □

We finish this paper by stating a formula with respect to the Fubini theorem involving the first variation.

Theorem 5.5. *Let $\Psi_n \in \mathcal{S}(C_{a,b}[0,T])$ be a generalized exponential type functional of the form* (6) *and let u be an element of $C'_{a,b}[0,T]$. Then we have*

$$\begin{aligned}
&\mathcal{F}_{S_1,R_1}((\mathcal{F}_{S_2,R_2}(\delta\Psi_n)(\cdot|R_2R_1u)))(y)\\
&= \mathcal{F}_{S_3,R_3}((R_2R_1u,\cdot)^{\sim}\Psi_n(\cdot))(y)\\
&\quad - ((R_2R_1u,R_3y)^{\sim} + (S^*R_2R_1u,a)^{\sim})\mathcal{F}_{S_3,R_3}(\Psi_n)(y).
\end{aligned}\tag{27}$$

Proof. Using Equation (11), we have

$$\begin{aligned}
&\mathcal{F}_{S_1,R_1}((\mathcal{F}_{S_2,R_2}(\delta\Psi_n)(\cdot|R_2R_1u)))(y)\\
&= \mathcal{F}_{S_1,R_1}(\delta(\mathcal{F}_{S_2,R_2}(\Psi_n))(\cdot|R_1u))(y)\\
&= \delta(\mathcal{F}_{S_1,R_1}(\mathcal{F}_{S_2,R_2}(\Psi_n)))(y|u).
\end{aligned}\tag{28}$$

Form Equations (11) and (20) again, we have

$$\begin{aligned}
&\mathcal{F}_{S_1,R_1}((\mathcal{F}_{S_2,R_2}(\delta\Psi_n)(\cdot|R_2R_1u)))(y)\\
&= \mathcal{F}_{S_3,R_3}(\delta\Psi_n)(\cdot|R_2R_1u))(y)\\
&= \mathcal{F}_{S_3,R_3}((R_2R_1u,\cdot)^{\sim}\Psi_n(\cdot))(y)\\
&\quad - ((R_2R_1u,R_3y)^{\sim} + (S^*R_2R_1u,a)^{\sim})\mathcal{F}_{S_3,R_3}(\Psi_n)(y).
\end{aligned}$$

Hence we have the desired result. □

Remark 5.6. Form Theorems 3.3 thru 5.5 above, we obtained various relationships via the Cameron-Storvick type Theorems 4.4 and 4.6. As mentioned in

Section 1, by choice of operator, one can see that many formulas in previous papers are the corollaries of the formulas in this paper.

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