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EXTENSION OF BLOCK MATRIX REPRESENTATION OF THE GEOMETRIC MEAN

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Abstract. To extend the well-known extremal characterization of the geometric mean of two $n \times n$ positive definite matrices A and B, we solve the following problem:

$$\max \left\{ X : X = X^*, \begin{pmatrix} A & V & X \\ V & B & W \\ X & W & C \end{pmatrix} \geq 0 \right\}.$$

We find an explicit expression of the maximum value with respect to the matrix geometric mean of Schur complements.

1. Introduction

The geometric mean of two $n \times n$ positive definite matrices A and B is given by an explicit formula [1, 16]:

$$A\#B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

It has an extremal characterization as follows:

$$A\#B = \max \left\{ X : X = X^*, \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0 \right\}. \tag{1}$$

Here $\leq$ denotes the Loewner ordering between Hermitian matrices.

A multivariable extension of this characterization arises naturally according to recent developments of multivariable geometric means on the Cartan-Hadamard-Riemannian manifold of positive definite matrices [2, 5, 6]. However, the extremal characterization of the geometric mean of two positive definite matrices does not seem to be easily generalized to multivariable geometric means. An extremal characterization of geometric mean of three positive definite matrices are still open in the context of matrix analysis.

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A *partial block matrix* is an $m \times m$ array $A = [A_{ij}]_{i,j=1}^{m}$ of matrices A_{ij} of fixed size, some of these matrices being specified, and others being unspecified (or missing), i.e., free variables over the matrices. An example is

$$\begin{pmatrix} A_{11} & A_{12} & ? & ? \\ A_{21} & A_{22} & A_{23} & ? \\ ? & A_{32} & A_{33} & A_{34} \\ ? & ? & A_{43} & A_{44} \end{pmatrix},$$

where ? denote the unspecified matrices.

We denote the unspecified entries of a partial matrix typically by X, Y, Z, and so on. A *completion* of the partial matrix is a specification of its unspecified entries, resulting in a conventional matrix. In this paper we focus on positive (semi)definite completions for partial block matrices. Gohberg et al. considered the positive definite completions of banded block matrix [8] and further developed in [9]. They also developed the band method for non-band block matrix completion [10]. Moreover, a completion with maximum entropy was studied in [11]. For information about positive semidefinite completions with scalar entries, see [12, 15].

The key observation is that the block matrix in (1)

$$H = \begin{pmatrix} A & X \\ X & B \end{pmatrix}$$

is a 2×2 partial block matrix with only one missing block X. It is easy to check that it has infinitely many positive semidefinite completions. There are various well-known positive semidefinite completions. One particular completion can be a matrix which has the maximum determinant of H, called a *maximum entropy property.* The only $X = 0$ has the maximum determinant. The geometric mean can be considered as one of particular completions, which is the maximum among all completions with respect to the Loewner order.

Now we consider the following 3×3 block matrix with only one missing block X,

$$H(X) = \begin{pmatrix} A & V & X \\ V^* & B & W \\ X & W^* & C \end{pmatrix}.$$

Also, it has infinitely manly positive semidefinite completions. There are various well-known positive semidefinite completions. For example, $X = VB^{-1}W$ is the unique choice for (1,3) entry of $H(X)$ for which the completion F satisfies $(F^{-1})_{13} = 0$.

For another example, if $G_1 : \operatorname{range} B \to \operatorname{range} A$ and $G_2 : \operatorname{range} C \to \operatorname{range} B$ are contractions such that $V = A^{\frac{1}{2}} G_1 B^{\frac{1}{2}}$ and $W = B^{\frac{1}{2}} G_2 C^{\frac{1}{2}}$, where $\operatorname{range} A$ means the range of linear operator A, then $A^{\frac{1}{2}} G_1 G_2 C^{\frac{1}{2}}$ is a positive semidefinite completion, which is *the central completion.* For more information about positive matrix completions, see [3] and references therein.

However, to the best of our knowledge, there are no existing results for the maximum positive definite completion with respect to the Loewner order. This observation motivated us to investigate the maximum matrix X with respect to the Loewner order for which the block matrix $H(X)$ is positive semidefinite. We show that such X can be written as the geometric mean and Schur complements.

In this paper, we will mainly consider positive semidefinite completions for the 3×3 block matrix based on the Loewner order. In Section 2, we will review some basic facts for 2×2 block matrices. Also, a known result for 3×3 partial block matrix completions is shown. In Section 3, we show that the maximum and the minimum of completions for a certain 3×3 partial block matrix with respect to the Loewner order can be expressed as the matrix geometric mean of Schur complements. Moreover, completions for various special cases and examples are considered.

2. Preliminaries

In this section, we review well-known facts for positive definite block matrices. For more details on positive matrix completions, see the book [3] and references therein.

A partial block matrix $A = [A_{ij}]_{i,j=1}^{n}$ is said to be *partially positive semidefinite* if the following conditions hold:

(i) All diagonal entries A_{ii} are specified and positive definite matrices;
(ii) A_{ij} is specified if and only if A_{ji} is specified and $A_{ij}^* = A_{ji}$;
(iii) All fully specified principal minors of A are positive semidefinite.

Note that every principal submatrix of a positive semidefinite matrix is positive semidefinite. Thus, being partially positive semidefinite is a necessary condition for the existence of positive completion.

Let $\mathbb{M}_{m\times n}(\mathbb{F})$ be the ring of all $m\times n$ matrices with entries in the field $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$. We simply denote $M_{m\times n} := M_{m\times n}(\mathbb{F})$ for $\mathbb{F}=\mathbb{C}$ and also denote $\mathbb{M}_n := \mathbb{M}_{n\times n}$. Let $\mathbb{H}_n$ be the real vector space of all $n\times n$ Hermitian matrices and let $\mathbb{P}_n \subset \mathbb{H}_n$ be the open convex cone of all positive definite matrices. For any $A, B \in \mathbb{H}_n$ we write $A \le B$ if $B-A$ is positive semi-definite, and $A < B$ if $B-A$ is positive definite. This is indeed a partial order on $\mathbb{H}_n$, known as the Loewner order.

Here, we recall some known results for the partial block matrix to have a positive semidefinite completion.

Theorem 2.1 ([13, Theorem 7.7.9]). *Let $A \in \mathbb{M}_m, B \in \mathbb{M}_n, C \in \mathbb{M}_{m\times n}$ and let*

$$T = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}.$$

Then $T \ge 0$ if and only if

(i) $A \ge 0$,

(ii) $B \geq 0$, *and*
(iii) $C = A^{1/2}EB^{1/2}$ *for some contraction* E (*that is,* $||E|| \leq 1$ *for the operator norm*).

In fact, this statement holds for bounded operators on Hilbert spaces with infinite dimension [7].

Definition. For the 2×2 block matrix

$$M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

the *Schur complement* of A_{22}, denoted by M/A_{22}, is

$$A_{11} - A_{12}A_{22}^{-1}A_{21},$$

provided that A_{22} is nonsingular. Similarly, the Schur complement of A_{11}, denoted by M/A_{11}, is

$$A_{22} - A_{21}A_{11}^{-1}A_{12},$$

provided that A_{11} is nonsingular.

Theorem 2.2 (Schur complement condition for positive semidefiniteness [4, 13])**.** *Let* M *be a* 2×2 *block matrix partitioned as*

$$M = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}.$$

If A *is invertible, then the following are true*:
(i) $M > 0$ *if and only if* $A > 0$ *and* $M/A > 0$.
(ii) *If* $A > 0$, *then* $M \geq 0$ *if and only if* $M/A \geq 0$.

If B *is invertible, then the following are true*:
(iii) $M > 0$ *if and only if* $B > 0$ *and* $M/B > 0$.
(iv) *If* $B > 0$, *then* $M \geq 0$ *if and only if* $M/B \geq 0$.

The following are some known results for the special types of 3×3 partial block matrix to have the positive definite completion.

Theorem 2.3 ([15, Proposition 1])**.** *Consider the following partial matrix with the only one missing entry*:

$$H(x) = \begin{pmatrix} a & v^\top & x \\ v & C & w \\ x & w^\top & b \end{pmatrix}, \tag{2}$$

where $a, b \in \mathbb{R}$, $v, w \in \mathbb{R}^n$, $C \in M_n(\mathbb{R})$, *and* $x \in \mathbb{R}$ *is the missing entry. If* $H(x)$ *is partial positive definite,* $H(x)$ *has a positive definite completion. Indeed, the set of all such completions is given by the inequality*

$$|x - v^\top C^{-1}w|^2 < \frac{\det A \det B}{(\det C)^2},$$

where

$$A = \begin{pmatrix} a & v^\top \\ v & C \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} C & w \\ w^\top & b \end{pmatrix}.$$

Two endpoints of this interval give singular positive semidefinite completions of $H(x)$. When $x = v^\top C^{-1} w$, the positive definite completion has the maximum determinant

$$\frac{\det(A)\det(B)}{\det(C)}.$$

3. Main results

In this section we show the sufficient and necessary condition for positive semidefinite completion of special partial block matrices with respect to Schur complement. Moreover, we give a necessary condition for such a positive semidefinite completion with respect to the geometric mean.

By Theorem 2.3 it is easy to see the following.

Corollary 3.1. *Consider* 3×3 *partial positive semidefinite matrix as follows*:

$$H(x) = \begin{pmatrix} a & v & x \\ v & b & w \\ x & w & c \end{pmatrix}, \tag{3}$$

where a, b, c, v, w are given, and $a, b, c > 0$ and x is the missing entry. Then the following are satisfied:

(i) *The set of positive semidefinite completions is given by the inequality*

$$\left| x - vb^{-1}w \right| \leq \frac{\sqrt{(ab - v^2)(bc - w^2)}}{b};$$

(ii) *The maximum value is*

$$x = \frac{vw + \sqrt{(ab - v^2)(bc - w^2)}}{b},$$

and such x gives the determinant 0;

(iii) *When $x = vb^{-1}w$, the positive definite completion has the maximum determinant.*

Note that if the matrix $H(x)$ given in (3) is a partial positive semidefinite matrix, then it follows that $v^2 \leq ab$ and $w^2 \leq bc$. If $v = \sqrt{ab}$, then by (i) it follows that

$$x = \sqrt{ab^{-1}}w \quad \text{if and only if} \quad H(x) \geq 0.$$

If $w = \sqrt{bc}$, then by (i) it follows that

$$x = v\sqrt{b^{-1}c} \quad \text{if and only if} \quad H(x) \geq 0.$$

If both $v = \sqrt{ab}$ and $w = \sqrt{bc}$, then by (i) we have

$$x = \sqrt{ac} \quad \text{if and only if} \quad H(x) \geq 0.$$

Now we generalize these results for 3×3 block matrix.

Proposition 3.2. *Consider the following partial block matrix with the only one missing block matrix*:

$$H(X) = \begin{pmatrix} A & V & X \\ V^* & B & W \\ X^* & W^* & C \end{pmatrix},$$

where $A \in \mathbb{P}_k$, $B \in \mathbb{P}_m$ *and* $C \in \mathbb{P}_n$ *are given, and* X *is a missing block. Assume that*

$$E = \begin{pmatrix} A & V \\ V^* & B \end{pmatrix} > 0.$$

Then $H(X) \geq 0$ *if and only if*

$$G/B \geq (F/B)^*(E/B)^{-1}(F/B),$$

where

$$F = \begin{pmatrix} X & V \\ W & B \end{pmatrix}, \quad G = \begin{pmatrix} B & W \\ W^* & C \end{pmatrix}.$$

Proof. Since $H(X)$ is partial positive semidefinite, the 2×2 block matrices E and G are positive semidefinite. Since $E > 0$, by the Schur complement inequality it holds that $H(X) \geq 0$ if and only if

$$C - \begin{pmatrix} X^* & W^* \end{pmatrix} E^{-1} \begin{pmatrix} X \\ W \end{pmatrix} \geq 0. \tag{4}$$

Note that

$$\begin{pmatrix} A & V \\ V^* & B \end{pmatrix} = \begin{pmatrix} I & VB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ B^{-1}V^* & I \end{pmatrix},$$

where $S = A - VB^{-1}V^*$ is the Schur's complement of the block B of the matrix E. Note that $S > 0$ by assumption. Then we have

$$\begin{aligned} &\begin{pmatrix} X^* & W^* \end{pmatrix} E^{-1} \begin{pmatrix} X \\ W \end{pmatrix} \\ &= \begin{pmatrix} X^* & W^* \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^{-1}V^* & I \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} I & -VB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} X \\ W \end{pmatrix} \\ &= \begin{pmatrix} X^* - W^*B^{-1}V^* & W \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} X - VB^{-1}W \\ W \end{pmatrix} \\ &= \begin{pmatrix} Y^* & W^* \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} Y \\ W \end{pmatrix} \\ &= Y^*S^{-1}Y + W^*B^{-1}W, \end{aligned}$$

where $Y = X - VB^{-1}W$. Then the inequality (4) can be expressed as

$$C - W^*B^{-1}W \geq Y^*S^{-1}Y,$$

equivalently,

$$C - W^*B^{-1}W \geq (X^* - W^*B^{-1}V^*)(A - VB^{-1}V^*)^{-1}(X - VB^{-1}W). \quad \square$$

Proposition 3.3. *Let $H(X)$ be the partial block matrix appeared as in Proposition 3.2, where $A \in \mathbb{P}_k$, $B \in \mathbb{P}_m$ and $C \in \mathbb{P}_n$ are given, and X is a missing block. Assume that*

$$G = \begin{pmatrix} B & W \\ W^* & C \end{pmatrix} > 0.$$

Then $H(X) \geq 0$ if and only if

$$E/B \geq (F/B)(G/B)^{-1}(F/B)^*,$$

where E and F are defined in Proposition 3.2.

Proof. By applying the Schur's complement inequality to $H(X)/G$, we have that $H(X) \geq 0$ if and only if

$$A - \begin{pmatrix} V & X \end{pmatrix} G^{-1} \begin{pmatrix} V^* \\ X^* \end{pmatrix} \geq 0.$$

By the similar argument of Proposition 3.2, we obtain the required inequality. □

Let $H \in \mathbb{M}_m(\mathbb{M}_n)$. For index sets $\alpha, \beta \subset \{1, \ldots, m\}$, we denote the (sub)matrix that lies in the rows of H indexed by α and the columns indexed by β as $H(\alpha, \beta)$. For example,

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} (\{1,3\}, \{2,3\}) = \begin{pmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{pmatrix}.$$

If $\alpha = \beta$, the submatrix $H(\alpha, \alpha)$ is abbreviated to $H(\alpha)$.

Corollary 3.4. *Let $H = [H_{ij}] \in \mathbb{M}_m(\mathbb{M}_n)$ $(m \geq 3)$ be a partially positive semidefinite block matrix whose diagonal block entry is A_i and (k, l)-block and (l, k)-block are missing blocks for some given $k < l$. Then the following are true*:

(i) *if $H(\{l\}^c) > 0$, then*

$$H \geq 0 \Longleftrightarrow G/B \geq (F/B)^*(E/B)^{-1}(F/B);$$

(ii) *if $H(\{k\}^c) > 0$, then*

$$H \geq 0 \Longleftrightarrow E/B \geq (F/B)(G/B)^{-1}(F/B)^*;$$

where

$$E = \begin{pmatrix} A_k & V \\ V^* & B \end{pmatrix}, \quad F = \begin{pmatrix} X & V \\ W & B \end{pmatrix}, \quad G = \begin{pmatrix} B & W \\ W^* & A_l \end{pmatrix}$$

and

$$V = H(\{k\}, \{k,l\}^c), \quad W = H(\{k,l\}^c, \{l\}), \quad B = H(\{k,l\}^c).$$

Proof. Consider a permutation matrix $P = J_k^T \oplus I_{n(l-k)} \oplus J_{m-l}$ where

$$J_k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & I \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix} \in \mathbb{M}_k(\mathbb{M}_n).$$

Then

$$P^T H P = \begin{pmatrix} A_k & V & X \\ V^* & B & W \\ X^* & W^* & A_l \end{pmatrix}.$$

By Propositions 3.2 and 3.3, we obtain the required inequalities. □

It is well-known in [14, Theorem 3.4] and [4, Theorem 4.1.3] that the geometric mean $A\#B$ of two positive definite matrices A and B has an extremal property such that

$$A\#B = \max\left\{X \in \mathbb{H}_n : \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\}.$$

However, Corollary 3.5 in [14] is not correct. Alternatively, we give a necessary condition for a missing Hermitian matrix X such that $\left(\begin{smallmatrix} A & X \\ X & B \end{smallmatrix}\right)$ is positive semidefinite.

Theorem 3.5. *Let $A, B \in \mathbb{P}_n$ and $X \in \mathbb{H}_n$. If $H(X) = \left(\begin{smallmatrix} A & X \\ X & B \end{smallmatrix}\right) \geq 0$, then*

$$-(A\#B) \leq X \leq A\#B.$$

Proof. Note that by Theorem 2.2, it holds that

$$H(X) \geq 0 \iff XA^{-1}X \leq B.$$

The assertion that $XA^{-1}X \leq B$ implies $X \leq A\#B$ follows from Theorem 3.4 in [14]. Since $XB^{-1}X = (-X)B^{-1}(-X)$, by Theorem 2.2(iv),

$$\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0 \quad \text{is equivalent to} \quad \begin{pmatrix} A & -X \\ -X & B \end{pmatrix} \geq 0.$$

So we have $-X \leq A\#B$. Therefore, $-A\#B \leq X \leq A\#B$. □

Theorem 3.6. *Let $H(X)$ be the partial block matrix appeared as in Proposition 3.2, where $A \in \mathbb{P}_n$, $B \in \mathbb{P}_m$ and $C \in \mathbb{P}_n$ are given, and $X \in \mathbb{H}_n$ is a missing block. Assume that*

$$E = \begin{pmatrix} A & V \\ V^* & B \end{pmatrix} > 0,$$

and $VB^{-1}W = W^*B^{-1}V^*$. If $H(X) \geq 0$, then

$$VB^{-1}W - (E/B)\#(G/B) \leq X \leq VB^{-1}W + (E/B)\#(G/B).$$

Furthermore, $VB^{-1}W + (E/B)\#(G/B)$ (resp. $VB^{-1}W - (E/B)\#(G/B)$) is the maximum (resp. the minimum) with respect to the Loewner order for all $X \in \mathbb{H}$ for which the block matrix $H(X)$ is positive semidefinite.

Proof. By Proposition 3.2, it follows that

$$G/B \geq (F/B)(E/B)^{-1}(F/B).$$

By Theorem 3.5 it holds that

$$-(E/B)\#(G/B) \leq F/B \leq (E/B)\#(G/B).$$

It suffices to show that $H(X) \geq 0$ for $X = VB^{-1}W + (E/B)\#(G/B)$ or $VB^{-1}W - (E/B)\#(G/B)$. Since $F/B = (E/B)\#(G/B)$ for $X = VB^{-1}W + (E/B)\#(G/B)$, it follows that

$$(F/B)(E/B)^{-1}(F/B) = ((E/B)\#(G/B))(E/B)^{-1}((E/B)\#(G/B)) = G/B.$$

By Proposition 3.2, $H(X) \geq 0$ for $X = VB^{-1}W + (E/B)\#(G/B)$. In the similar way, it is easy to show that $H(X) \geq 0$ for $X = VB^{-1}W - (E/B)\#(G/B)$. □

Remark 3.7. Even though $V = A\#B$ and $W = B\#C$, it does not hold that $VB^{-1}W = W^*B^{-1}V^*$, in general. For instance, consider

$$A = \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}, \quad B = \begin{pmatrix} 13 & 11 \\ 11 & 10 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}.$$

Then it is easy to check that

$$VB^{-1}W = (A\#B)B^{-1}(B\#C) \approx \begin{pmatrix} 1.1157 & 2.6679 \\ 1.3097 & 5.8209 \end{pmatrix}$$

is not Hermitian.

On the other hand, there are several cases satisfying the condition $VB^{-1}W = W^*B^{-1}V^*$. For instance, $V = 0$ or $W = 0$, or $V = \alpha I$ and $W = \beta I$ for some $\alpha, \beta \in \mathbb{R}$. We consider these special cases in the following. For given $B \in \mathbb{P}_n$ it is questionable what the sufficient and necessary condition of V and W for $VB^{-1}W = W^*B^{-1}V^*$ is.

Corollary 3.8. *Let $A, B, C \in \mathbb{P}_n$ be given. Consider the following partial block matrix with the only one missing block matrix*:

$$H(X) = \begin{pmatrix} A & \alpha V & X \\ \alpha V & B & \beta V \\ X & \beta V & C \end{pmatrix},$$

where $V \in \mathbb{H}_n$ such that

$$E = \begin{pmatrix} A & \alpha V \\ \alpha V & B \end{pmatrix} \geq 0, \quad G = \begin{pmatrix} B & \beta V \\ \beta V & C \end{pmatrix} \geq 0.$$

If $H(X) \geq 0$, then

$$\alpha\beta VB^{-1}V - (E/B)\#(G/B) \leq X \leq \alpha\beta VB^{-1}V + (E/B)\#(G/B),$$

equivalently,

$$\alpha\beta K - (A - \alpha^2 K)\#(C - \beta^2 K) \leq X \leq \alpha\beta K + (A - \alpha^2 K)\#(C - \beta^2 K),$$

where $K = VB^{-1}V$. Especially, the following are true.

(1) *For the case when* $V = I$*, if* $H(X) \geq 0$*, then*

$$\begin{aligned}&\alpha\beta B^{-1} - (A - \alpha^2 B^{-1})\#(C - \beta^2 B^{-1})\\ &\leq X \leq \alpha\beta B^{-1} + (A - \alpha^2 B^{-1})\#(C - \beta^2 B^{-1}).\end{aligned}$$

(2) *For the case when* $\alpha = 0$*, if* $H(X) \geq 0$*, then*

$$-(A\#(G/B)) \leq X \leq A\#(G/B).$$

(3) *For the case when* $\beta = 0$*, if* $H(X) \geq 0$*, then*

$$-((E/B)\#C) \leq X \leq (E/B)\#C.$$

(4) *For the case when* $\alpha = \beta = 0$*, if* $H(X) \geq 0$*, then*

$$-(A\#C) \leq X \leq A\#C.$$

Definition. A family $\mathcal{F} \subseteq \mathbb{M}_n$ of matrices is a nonempty finite or infinite set of matrices. A *commuting family* is the family of matrices in which every pair of matrices commutes. A family $\mathcal{F}$ is called a Γ-*commuting family*, or the family $\mathcal{F}$ Γ-commutes, if there exists invertible M such that $M\mathcal{F}M^* := \{MXM^* | X \in \mathcal{F}\}$ is a commuting family.

Lemma 3.9. *Let* $B \in \mathbf{GL}_n \cap \mathbb{H}_n$ *and let* $\mathcal{F} = \{A, B, C\} \subseteq \mathbb{H}_n$ *be* Γ*-commuting family, where* $\mathbf{GL}_n$ *denotes the general linear group of all* $n \times n$ *invertible matrices. Then*

$$AB^{-1}C = CB^{-1}A.$$

Proof. Since $\mathcal{F}$ is a Γ-commuting family, there exists invertible M such that $M\mathcal{F}M^*$ is a commuting family. Then it follows that

$$\begin{aligned}MAB^{-1}CM^* &= (MAM^*)(MBM^*)^{-1}(MCM^*)\\ &= (MCM^*)(MBM^*)^{-1}(MAM^*)\\ &= MCB^{-1}AM^*.\end{aligned}$$

The second equality follows from the fact that $AB = BA$ is equivalent to $AB^{-1} = B^{-1}A$. Since M is invertible, it holds that $AB^{-1}C = CB^{-1}A$. □

Corollary 3.10. *Let* $H(X)$ *be the partial block matrix appeared as in Proposition 3.2, where* $A, B, C \in \mathbb{P}_n$ *and* $V, W \in \mathbb{H}_n$ *are given, and* $X \in \mathbb{H}_n$ *is a missing block. Assume that*

$$E = \begin{pmatrix} A & V \\ V & B \end{pmatrix} > 0,$$

and $\{V, B, W\}$ *is a* Γ*-commuting family. If* $H(X) \geq 0$*, then*

$$VB^{-1}W - (E/B)\#(G/B) \leq X \leq VB^{-1}W + (E/B)\#(G/B).$$

Proof. By Theorem 3.6 and Lemma 3.9, it is straightforward. □

The following theorems are the extension of results mentioned after Corollary 3.1.

Theorem 3.11. *For each $V \in \mathbb{H}_n$, let $H_V(X)$ be the partial block matrix*

$$H_V(X) = \begin{pmatrix} A & V & X \\ V & B & W \\ X & W & C \end{pmatrix},$$

where $A, B, C \in \mathbb{P}_n$ and $W \in \mathbb{H}_n$, and $X \in \mathbb{H}_n$ is a missing block. If $\{A\#B, B, W\}$ is a Γ-commuting family and $H_{A\#B}(X)$ is positive semidefinite, then $X = (AB^{-1})^{1/2}W$.

Proof. Since $\{A\#B, B, W\}$ is a Γ-commuting family, there exists an invertible matrix M such that $\{M(A\#B)M^*, MBM^*, MWM^*\}$ is a commuting family. Consider the sequence

$$V_k := A\#B - \frac{1}{k}(M^*M)^{-1} \quad \text{and} \quad E_k = \begin{pmatrix} A & V_k \\ V_k & B \end{pmatrix}.$$

Then $V_k \nearrow A\#B$ as $k \to \infty$ and $\{V_k, B, W, (M^*M)^{-1}\}$ is a Γ-commuting family. Note that

$$\begin{aligned} E_k/B &= A - \left(A\#B - \frac{1}{k}(M^*M)^{-1}\right) B^{-1} \left(A\#B - \frac{1}{k}(M^*M)^{-1}\right) \\ &= A - (A\#B)B^{-1}(A\#B) \\ &\quad + \frac{1}{k}\left[(M^*M)^{-1}B^{-1}(A\#B) + (A\#B)B^{-1}(M^*M)^{-1} \right. \\ &\qquad \left. - \frac{1}{k}(M^*M)^{-1}B^{-1}(M^*M)^{-1}\right] \\ &= \frac{1}{k}\left[2(M^*M)^{-1}B^{-1}(A\#B) - \frac{1}{k}(M^*M)^{-1}B^{-1}(M^*M)^{-1}\right]. \end{aligned}$$

The last equality follows from the facts that $A = (A\#B)B^{-1}(A\#B)$ and $\{B, A\#B, (M^*M)^{-1}\}$ is a Γ-commuting family.

For $k > \frac{1}{2}\lambda_{\max}((A\#B)(M^*M)^{-1})$,

$$2(M^*M)^{-1}B^{-1}(A\#B) - \frac{1}{k}(M^*M)^{-1}B^{-1}(M^*M)^{-1} > 0.$$

So $E_k/B > 0$, and hence $E_k > 0$ by Theorem 2.2(iii).

Since $\{V_k, B, W\}$ is a Γ-commuting family and $E_k > 0$, we obtain by Corollary 3.10 that if $H_{V_k}(X) \geq 0$, then

$$V_kB^{-1}W - (E_k/B)\#(G/B) \leq X \leq V_kB^{-1}W + (E_k/B)\#(G/B).$$

Taking limit as $k \to \infty$, we have $\lim_{k\to\infty} H_{V_k}(X) = H_{A\#B}(X)$, $\lim_{k\to\infty} E_k/B = 0$, and $\lim_{k\to\infty}(E_k/B)\#(G/B) = 0$. Hence, we obtain that if $H_{A\#B}(X) \geq 0$, then $X = (A\#B)B^{-1}W = (AB^{-1})^{1/2}W$. $\square$

By the similar argument of Theorem 3.11, we obtain the following.

Theorem 3.12. *For each $W \in \mathbb{H}_n$, let $\widehat{H}_W(X)$ be the partial block matrix*

$$\widehat{H}_W(X) = \begin{pmatrix} A & V & X \\ V & B & W \\ X & W & C \end{pmatrix},$$

where $A, B, C \in \mathbb{P}_n$ and $V \in \mathbb{H}_n$, and $X \in \mathbb{H}_n$ is a missing block. If $\{B\#C, B, V\}$ is a Γ-commuting family and $\widehat{H}_{B\#C}(X)$ is positive semidefinite, then $X = V(B^{-1}C)^{1/2}$.

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