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WEIGHTED HARDY INEQUALITIES WITH SHARP CONSTANTS

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Abstract. In the paper, we establish the validity of the weighted discrete and integral Hardy inequalities with periodic weights and find the best possible constants in these inequalities. In addition, by applying the established discrete Hardy inequality to a certain second–order difference equation, we discuss some oscillation and nonoscillation results.

1. Introduction

Let us consider the following classical Hardy inequalities [6, Theorems 326 and 327]:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}f_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} f_k^p,\ f=\{f_k\},\ f\geq 0,\ f\not\equiv 0, \tag{1}$$

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(t)dt,\ f\geq 0,\ f\not\equiv 0, \tag{2}$$

where $1<p<\infty$ and $\left(\frac{p}{p-1}\right)^p$ is the best or sharp constant in (1) and (2). Here and in the sequel, $f=\{f_k\}$, $f\geq 0$, means $f_k\geq 0$, $\forall k\geq 1$.

Proposition 1.1. *Let $1<p<\infty$. Then for any fixed integer $m\geq 0$ the inequality*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+m}\sum_{k=1}^{n}f_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} f_k^p,\ f=\{f_k\},\ f\geq 0,\ f\not\equiv 0, \tag{3}$$

holds with the best constant $\left(\frac{p}{p-1}\right)^p$.

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The statement of Proposition 1.1 seems to be well–known, however, the authors find it difficult to provide relevant references. Let us note that in view of (1) the fulfillment of the inequality (3) is obvious, and the constant $\left(\frac{p}{p-1}\right)^p$ is also the best possible. Moreover, (3) has a proof similar to the proof of (1) (see [6, Theorem 326]).

Intensive studies of the weighted Hardy inequalities

$$\left(\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\sum_{k=1}^{n}f_k\right)^q\right)^{\frac{1}{q}} \le C\left(\sum_{k=1}^{\infty}(v_k f_k)^p\right)^{\frac{1}{p}}, \ f=\{f_k\}, \ f\ge 0, \tag{4}$$

$$\left(\int_0^\infty\left(\frac{u(x)}{x}\int_0^x f(t)dt\right)^q dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (v(t)f(t))^p dt\right)^{\frac{1}{p}}, \ f\ge 0, \tag{5}$$

have been conducted during the past half century. Here $0 < q \le \infty$ and $1 \le p \le \infty$; $\{u_n\}$ and $\{v_n\}$ are sequences of real numbers such that $u_n \ge 0$ and $v_n \ge 0$, $\forall n \ge 1$; $u(\cdot)$ and $v(\cdot)$ are functions measurable on semiaxis such that $u(t) \ge 0$ and $v(t) \ge 0$, $\forall t \ge 0$. The history of the development of the Hardy inequalities (4), (5) and relative results are collected in monographs [10,11,15]. The most known step in this development is a characterization of the inequality (5) for the case $1 \le p = q \le \infty$ given in the work by B. Muckenhoupt [13]. This characterization is frequently called "Muckenhoupt condition" since it was B. Muckenhoupt who presented a simple and direct proof for this range of parameters.

However, the best constants in (4) and (5) have not been found yet. In the book [10, p. 26], the best possible constants are presented only for particular cases of the inequality (5). For example, the known weighted Hardy inequality [6]

$$\int_0^\infty\left(\frac{x^\alpha}{x}\int_0^x f(t)dt\right)^p dx < \left(\frac{p}{p-\alpha p-1}\right)^p \int_0^\infty (t^\alpha f(t))^p dt, \ f\ge 0, \ f\not\equiv 0, \tag{6}$$

has the best constant $\left(\frac{p}{p-\alpha p-1}\right)^p$, where $\alpha < 1-\frac{1}{p}$.

Some attempts to establish a discrete analogue of the inequality (6) are known. For example, in the works [2,3] (see also [4,5]), the following inequalities

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^{1-\alpha}}\sum_{k=0}^{n}\left[k^{\alpha-1}-(k-1)^{\alpha-1}\right]a_k\right]^p \le \left[\frac{1-\alpha}{p-\alpha p-1}\right]^p\sum_{n=1}^{\infty}a_n^p, \quad a_n\ge 0,$$

and

$$\sum_{n=1}^{\infty}\left[\frac{1}{\sum\limits_{i=1}^{n} i^{-\alpha}}\sum_{k=1}^{n}k^{-\alpha}a_k\right]^p \le \left[\frac{1-\alpha}{p-\alpha p-1}\right]^p\sum_{n=1}^{\infty}a_n^p, \quad a_n\ge 0,$$

have been established with the best constant $\left[\frac{1-\alpha}{p-\alpha p-1}\right]^p$ for $\alpha > 0$, $p > 1$, and $\alpha p > 1$.

Moreover, in the article [12], there are some other analogues of the inequality (6) in the forms

$$\sum_{n=-\infty}^{\infty}\left[\frac{1}{q^{n\lambda}}\sum_{k=0}^{n} q^{k\lambda}a_k\right]^p \le \frac{1}{(1-q^\lambda)^p}\sum_{n=-\infty}^{\infty} a_n^p, \quad a_n \ge 0,$$

and

$$\sum_{n=1}^{\infty}\left[\frac{1}{q^{n\lambda}}\sum_{k=0}^{n} q^{k\lambda}a_k\right]^p \le \frac{1}{(1-q^\lambda)^p}\sum_{n=1}^{\infty} a_n^p, \quad a_n \ge 0,$$

where $0 < q < 1$, $p \ge 1$, $\alpha < 1 - 1/p$ and $\lambda := 1 - 1/p - \alpha$.

For $p \ge 1$ and $\alpha < 1 - 1/p$ an analogue of the inequality (6) in h-calculus is recently established in the article [16]. This inequality leads to another more precise discrete analogue of the inequality (6):

$$\sum_{n=0}^{\infty}\left(n^{(\alpha-1)}\sum_{k=0}^{n}\frac{a_k}{k^{(\alpha)}}\right)^p \le \left(\frac{p}{p-\alpha p-1}\right)^p\sum_{n=0}^{\infty} a_k^p, \quad a_k \ge 0,$$

where

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \qquad t \in \mathbb{R}.$$

Inequalities with the best constants play an important role in many problems of Analysis. The main aim of this article is to establish the validity of the weighted discrete and integral Hardy inequalities (4) and (5) with periodic weights and find their best constants.

The article is organized as follows: in Section 2, we present and prove our main result concerning the discrete Hardy inequality. Applications of the obtained inequality to the oscillatory properties of a certain difference equation are formulated and proved in Section 3. Finally, in Section 4 we state and prove the integral Hardy inequality.

2. Weighted discrete Hardy inequality with the best constant

Let $w \ge 2$ be an integer number. We denote by P_w the class of sequences of nonnegative real numbers $u = \{u_k\}_{k=1}^{\infty}$ such that

$$\sum_{i=1}^{w} u_i = \sum_{i=(k-1)w+1}^{kw} u_i, \ \forall k \ge 1. \tag{7}$$

It is obvious that the nonnegative w-periodic sequence u such that $u_k \ge 0$, $u_k = u_{k+w}$, $\forall k \ge 1$, belongs to P_w. Indeed, for $i = j + (k-1)w$ we have

$$\sum_{i=(k-1)w+1}^{kw} u_i = \sum_{j=1}^{w} u_{j+(k-1)w} = \sum_{j=1}^{w} u_j.$$

Let us also note that the P_w class is wider than the class of just w-periodic sequences.

Define

$$T_p = \left(\frac{1}{w}\sum_{i=1}^{w} u_i^p\right)\left(\frac{1}{w}\sum_{j=1}^{w} v_j^{-p'}\right)^{p-1}.$$

Let $v = \{v_k\}_{k=1}^{\infty}$ be a nontrivial nonnegative sequence of real numbers. Denote by $l_{p,v}$, $1 \le p < \infty$, the space of sequences of real numbers $f = \{f_k\}_{k=1}^{\infty}$, for which

$$\sum_{k=1}^{\infty} |v_k f_k|^p < \infty.$$

Theorem 2.1. *Let $1 < p < \infty$ and $u_k \ge 0$, $v_k > 0$, $\forall k \ge 1$. Let the sequences $u^p = \{u_k^p\}_{k=1}^{\infty}$ and $v^{-p'} = \{v_k^{-p'}\}_{k=1}^{\infty}$ belong to the class P_w. Then for any integer number $m \ge 1$ the inequality*

$$\sum_{n=1}^{\infty}\left(\frac{u_n}{n+mw}\sum_{i=1}^{n} f_i\right)^p \le C\sum_{j=1}^{\infty}(v_j f_j)^p, \;\; f = \{f_i\}_{i=1}^{\infty} \in l_{p,v}, f \ge 0, \tag{8}$$

holds with the best constant

$$C = \left(\frac{p}{p-1}\right)^p T_p. \tag{9}$$

Moreover, in (8) *the equality is reached only for the trivial sequence, i.e., when $f_i = 0$, $\forall i \ge 1$.*

If the inequality (8) *holds for $m = 0$ with the best constant $C > 0$, then*

$$C \ge \left(\frac{p}{p-1}\right)^p T_p. \tag{10}$$

Proof. Let $m \ge 1$. Assume $w_k = (k-1)w+1$, $k \ge 1$. Let $f = \{f_i\}_{i=1}^{\infty}$, $f \in l_{p,v}$, $f \ge 0$, $f \not\equiv 0$. Using the relation (7), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{u_n}{n+mw}\sum_{j=1}^{n} f_j\right)^p &= \sum_{k=1}^{\infty}\sum_{n=w_k}^{kw}\left(\frac{u_n}{n+mw}\sum_{j=1}^{n} f_j\right)^p \\ &\le \sum_{k=1}^{\infty}\left(\frac{1}{w_k+mw}\sum_{j=1}^{kw} f_j\right)^p \sum_{n=w_k}^{kw} u_n^p \\ &\le \frac{1}{w^p}\sum_{n=1}^{w} u_n^p \sum_{k=1}^{\infty}\left(\frac{1}{k+m-1}\sum_{i=1}^{k} F_i\right)^p,\end{aligned} \tag{11}$$

where $F_i = \sum\limits_{j=w_i}^{iw} f_j$, $i \ge 1$.

From $f \geq 0$, $f \not\equiv 0$, it follows that $F = \{F_i\}_{i=1}^{\infty} \geq 0$, $F \not\equiv 0$. Therefore, in view of (3), we get

$$\sum_{k=1}^{\infty} \left(\frac{1}{k+m-1} \sum_{i=1}^{k} F_i \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{i=1}^{\infty} F_i^p. \tag{12}$$

On the basis of Hölder's inequality, we have

$$\begin{aligned} F_i^p = \left(\sum_{j=w_i}^{iw} f_j \right)^p &\leq \left(\sum_{j=w_i}^{iw} v_j^{-p'} \right)^{p-1} \sum_{j=w_i}^{iw} (v_j f_j)^p \\ &= \left(\sum_{j=1}^{w} v_j^{-p'} \right)^{p-1} \sum_{j=w_i}^{iw} (v_j f_j)^p . \end{aligned}$$

Then

$$\sum_{i=1}^{\infty} F_i^p = \left(\sum_{j=1}^{w} v_j^{-p'} \right)^{p-1} \sum_{i=1}^{\infty} \sum_{j=w_i}^{iw} (v_j f_j)^p = \left(\sum_{j=1}^{w} v_j^{-p'} \right)^{p-1} \sum_{i=1}^{\infty} (v_i f_i)^p . \tag{13}$$

From (11), (12) and (13) it follows that

$$\sum_{n=1}^{\infty} \left(\frac{u_n}{n+mw} \sum_{j=1}^{n} f_j \right)^p < \left(\frac{p}{p-1} \right)^p T_p \sum_{i=1}^{\infty} (v_i f_i)^p ,$$

i.e., the inequality (8) holds with the estimate

$$C \leq \left(\frac{p}{p-1} \right)^p T_p \tag{14}$$

for the best constant C in (8).

Let now the inequality (8) hold for any $m \geq 0$. Let $g = \{g_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $g_i = v_i^{-p'} \varphi_k$ for $w_k \leq i \leq kw$, $k \geq 1$, where $\varphi_k \geq 0$ and $0 < \sum\limits_{k=1}^{\infty} \varphi_k^p < \infty$. Then

$$\sum_{i=1}^{\infty} (v_i g_i)^p = \sum_{k=1}^{\infty} \sum_{i=w_k}^{kw} (v_i g_i)^p = \sum_{k=1}^{\infty} \varphi_k^p \sum_{i=w_k}^{kw} v_i^{-p'} = \sum_{i=1}^{w} v_i^{-p'} \sum_{k=1}^{\infty} \varphi_k^p < \infty. \tag{15}$$

Hence, $g \in l_{p,v}$ and $g \geq 0$. We have that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{u_n}{n+mw} \sum_{j=1}^{n} g_j \right)^p &= \sum_{k=1}^{\infty} \sum_{n=w_k}^{kw} \left(\frac{u_n}{n+mw} \sum_{j=1}^{n} g_j \right)^p \\ &\geq \sum_{k=1}^{\infty} \left(\frac{1}{kw+mw} \sum_{j=1}^{w_k} g_j \right)^p \sum_{n=w_k}^{kw} u_n^p \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{w^p}\sum_{n=1}^{w} u_n^p \sum_{k=2}^{\infty}\left(\frac{1}{k+m}\sum_{j=1}^{(k-1)w} g_j\right)^p\\
&= \frac{1}{w^p}\sum_{n=1}^{w} u_n^p \sum_{k=1}^{\infty}\left(\frac{1}{k+m+1}\sum_{j=1}^{kw} g_j\right)^p\\
&= \frac{1}{w^p}\sum_{n=1}^{w} u_n^p \sum_{k=1}^{\infty}\left(\frac{1}{k+m+1}\sum_{i=1}^{k}\sum_{j=w_i}^{iw} g_j\right)^p\\
&= \frac{1}{w^p}\sum_{n=1}^{w} u_n^p \sum_{k=1}^{\infty}\left(\frac{1}{k+m+1}\sum_{i=1}^{k}\varphi_i\sum_{j=w_i}^{iw} v_j^{-p'}\right)^p\\
&= \sum_{j=1}^{w} v_j^{-p'} T_p \sum_{k=1}^{\infty}\left(\frac{1}{k+m+1}\sum_{i=1}^{k}\varphi_i\right). \qquad (16)
\end{aligned}
$$

Let C be the best constant in the inequality (8). Then, on the basis of (8), (15), (16) and (3), we have

$$
C=\sup_{f\geq 0}\frac{\sum\limits_{n=1}^{\infty}\left(\frac{u_n}{n+mw}\sum\limits_{j=1}^{n} f_j\right)^p}{\sum\limits_{i=1}^{\infty}(v_i f_i)^p}\geq T_p\sup_{\varphi\geq 0}\frac{\sum\limits_{k=1}^{\infty}\left(\frac{1}{k+m+1}\sum\limits_{i=1}^{k}\varphi_i\right)^p}{\sum\limits_{k=1}^{\infty}\varphi_k^p}=\left(\frac{p}{p-1}\right)^p T_p.
$$

Consequently,

$$
C\geq\left(\frac{p}{p-1}\right)^p T_p \qquad (17)
$$

for $m \geq 0$, i.e., for $m \geq 1$ and $m = 0$. Then for $m \geq 1$ from (14) and (17) we get (9), and for $m = 0$ we get (10). The proof of Theorem 2.1 is complete. ☐

Since the inequality (8) is equivalent to the inequality

$$
\sum_{n=w_{m+1}}^{\infty}\left(\frac{u_n}{n}\sum_{k=w_{m+1}}^{n} f_k\right)^p\leq C\sum_{i=w_{m+1}}^{\infty}(v_i f_i)^p,\ f=\{f_i\}_{i=1}^{\infty}\in l_{p,v}, f\geq 0, \qquad (18)
$$

then from Theorem 2.1 we have the following corollary.

Corollary 2.2. *Let $1 < p < \infty$ and $u_k \geq 0$, $v_k > 0$, $\forall k \geq 1$. Let the sequences $u^p = \{u_k^p\}_{k=1}^{\infty}$ and $v^{-p'} = \{v_k^{-p'}\}_{k=1}^{\infty}$ belong to the class P_w. Then for any integer number $m \geq 1$ the inequality* (18) *holds with the best constant* (9). *Moreover, in* (18) *the equality is reached only for the trivial sequence.*

If the inequality (18) *holds for $m = 0$ with the best constant $C > 0$, then* (10) *holds.*

3. Oscillatory properties of a class of second order difference equations

Here we consider an application of Theorem 2.1 to the problem of oscillation and nonoscillation of the second order difference equation

$$\Delta(v_i|\Delta y_i|^{p-2}\Delta y_i) + \lambda\frac{u_i}{i^p}|y_{i+1}|^{p-2}y_{i+1} = 0,\ i \geq 1, \tag{19}$$

with coefficients $v_i > 0$, $u_i \geq 0$, $i \geq 1$, where $1 < p < \infty$ and $\lambda > 0$.

Let us list notions and definitions required for the equation (19). Let $m \geq 1$ be an integer number.

- If there exists a nontrivial solution $y = \{y_i\}_{i=1}^{\infty}$ of the equation (19) such that $y_m \neq 0$ and $y_m y_{m+1} < 0$ or $y_m = 0$, then the solution y has a generalized zero on the interval $(m, m+1]$.

- A nontrivial solution y of the equation (19) is called oscillatory, if it has infinite number of generalized zeros, otherwise it is called nonoscillatory.

- The equation (19) is called oscillatory if all its nontrivial solutions are oscillatory, otherwise it is called nonoscillatory.

- For the equation (19) the Sturm theorem on the separation of zeroes is valid, thus the equation (19) is oscillatory, if its one nontrivial solution is oscillatory.

- The equation (19) is called disconjugate on the interval $[m, n]$, $1 \leq m < n < \infty$, if its any nontrivial solution has no more than one zero on $(m, n+1]$, otherwise it is called conjugate on $[m, n]$.

- The equation (19) is called disconjugate on the interval $[m, \infty)$, if for any $n > m$ it is disconjugate on the interval $[m, n]$.

Let $y = \{y_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Let *supp* $y := \{i \geq 1 : y_i \neq 0\}$ and $1 \leq m < n \leq \infty$. Denote by $\overset{\circ}{Y}(m, n)$ the set of all nontrivial sequences of real numbers $y = \{y_i\}_{i=1}^{\infty}$ such that *supp* $y \subset [m+1, n]$, $n < \infty$. When $n = \infty$, we assume that for any y there exists an integer number $k = k(y)$ and *supp* $y \subset [m+1, k]$.

The basic properties of the equation (19) are given in so–called "roundabout theorem" [17, Theorem 1]. This Theorem gives the equivalence of some four statements (i)–(iv) concerning solutions of the equation (19), and the equivalence of the statements (i) and (iv) implies a criterion of disconjugality of the equation (19) in the given discrete interval $[m, n]$, $1 \leq m < n \leq \infty$. In [9] it is shown that this criterion is equivalent to the following Lemma.

Lemma 3.1. *Let $1 \leq m < n \leq \infty$. The equation* (19) *is disconjugate on the interval $[m, n]$ ($[m, \infty)$) if and only if*

$$\lambda\sum_{i=m}^{n} u_{i-1}\left|\frac{y_i}{i}\right|^p \leq \sum_{i=m}^{n} v_i|\Delta y_i|^p \tag{20}$$

for all $y \in \overset{\circ}{Y}(m, n)$, where $u_0 = 0$.

Consider the inequality

$$\lambda\sum_{i=m}^{n} u_{i-1}\left|\frac{y_i}{i}\right|^p \le C\sum_{i=m}^{n} v_i|\Delta y_i|^p,\ y\in\overset{\circ}{Y}(m,n). \tag{21}$$

If (20) holds, then (21) holds with the best constant

$$0<C\le 1. \tag{22}$$

Inversely, if (21) holds with the best constant (22), then (20) holds.

Let

$$\sum_{j=1}^{\infty} v_j^{1-p'} = \infty. \tag{23}$$

Then, as shown in [1, 9], the following Lemma holds.

Lemma 3.2. *Let $n=\infty$ and (23) hold. Then the inequality (21) is equivalent to the Hardy inequality*

$$\lambda\sum_{k=m}^{\infty} u_k\left(\frac{1}{k}\sum_{i=m}^{k} f_i\right)^p \le C\sum_{k=m}^{\infty} v_k f_k^p,\ f_k\ge 0, \forall i\ge m. \tag{24}$$

Moreover, the best constants in (21) and (24) coincide.

Condition A. *Suppose that the coefficients of the equation (19) satisfy the condition $u\in P_w$ $v^{1-p'}=\{v_i^{1-p'}\}\in P_w$.*

Condition A gives that (23) holds. Therefore, on the basis of Lemma 3.2, the equation (19) is disconjugate on the interval $[m,\infty)$ if and only if the equation (24) holds with the best constant (22).

In (24) we replace the value m by the value w_{m+1}, then (24) will have the following form

$$\lambda\sum_{k=w_{m+1}}^{\infty}\left(\frac{u_k^{\frac{1}{p}}}{k}\sum_{i=w_{m+1}}^{k} f_i\right)^p \le C\sum_{k=w_{m+1}}^{\infty}\left(v_k^{\frac{1}{p}} f_k\right)^p,\ f_k\ge 0,\ \forall i\ge m. \tag{25}$$

On the basis of Corollary 2.2, the best constant in (25) is the value

$$C=\lambda\left(\frac{p}{p-1}\right)^p M_p, \tag{26}$$

where

$$M_p = T_p(u^{\frac{1}{p}}, v^{\frac{1}{p}}) = \frac{1}{w}\sum_{i=1}^{w} u_i\left(\frac{1}{w}\sum_{j=1}^{w} v_j^{1-p'}\right)^{p-1}.$$

Due to (26), the condition (22) is equivalent to the condition

$$\lambda\le\left(\frac{p-1}{p}\right)^p M_p^{-1}. \tag{27}$$

Since the condition (27) does not depend on $m \geq 1$, then on the basis of (22) we have the following Theorem.

Theorem 3.3. *Let $1 < p < \infty$ and Condition A hold. Then*
(i) *the equation* (19) *is nonoscillatory* (*disconjugate on the interval* $[1, \infty)$) *if and only if* (27) *holds*;
(ii) *the equation* (19) *is oscillatory if and only if*

$$\lambda > \left(\frac{p-1}{p}\right)^p M_p^{-1}$$

holds.

Two parts (i) and (ii) of Theorem 3.3 imply that the number $\left(\frac{p-1}{p}\right)^p M_p^{-1}$ is the threshold value such that (19) is oscillatory for λ exceeding this number and nonoscillatory otherwise. Thus, $\lambda = \left(\frac{p-1}{p}\right)^p M_p^{-1}$ is called the critical oscillation constant for the equation (19). The critical oscillation constants play an important role in the investigation of the oscillatory properties of equations. However, it is not always possible to find such constants, so that they are still undefined in many cases.

Remark 3.4. If the sequences $u = \{u_i\}_{i=1}^\infty$ and $v = \{v_i\}_{i=1}^\infty$ of the equation (19) are w-periodic, then by Theorem 3.3 its critical oscillation constant is the following

$$\lambda = \left(\frac{p-1}{p}\right)^p \left(\frac{1}{w}\sum_{i=1}^{w} u_i\right)^{-1} \left(\frac{1}{w}\sum_{j=1}^{w} v_j^{1-p'}\right)^{1-p},$$

and, in particular, for the linear case $p = 2$ it is the following

$$\lambda = \frac{1}{4}\left(\frac{1}{w}\sum_{i=1}^{w} u_i\right)^{-1} \left(\frac{1}{w}\sum_{j=1}^{w} v_j^{-1}\right)^{-1}.$$

In the works [7, 8, 18] the finding of these critical constants is noted as an open question.

4. Weighted integral Hardy inequality with best constant

Let $w > 0$ be a real number. Here by P_w we denote the class of nonnegative functions summable on $(0, T]$ for any $T > 0$ satisfying the following condition

$$\int_0^w u(t)dt = \int_{(k-1)w}^{kw} u(t)dt, \;\; k = 1, 2, 3, \ldots. \tag{28}$$

It is obvious that the nonnegative w-periodic function u belongs to the class P_w. Indeed, for $k \geq 1$ we have

$$\int_{(k-1)w}^{kw} u(t)dt = \int_0^w u(t + (k-1)w)dt = \int_0^w u(t)dt.$$

Let $m \geq 0$ and v be a function measurable on $(0, \infty)$. By $L_{p,v}$ we denote the space of all measurable on $(0, \infty)$ functions f, for which

$$\int_{mw}^{\infty} |v(t)f(t)|^p dt < \infty$$

holds. Define

$$G_p = \left(\frac{1}{w} \int_0^w u^p(t)dt \right) \left(\frac{1}{w} \int_0^w v^{-p'}(s)ds \right)^{p-1}.$$

Theorem 4.1. *Let $1 < p < \infty$ and $u(t) \geq 0$, $v(t) > 0$, $\forall t \geq 0$. Let functions u^p and $v^{-p'}$ belong to the class P_w. Then for any integer number $m \geq 1$ the inequality*

$$\int_{mw}^{\infty} \left(\frac{u(x)}{x} \int_{mw}^{x} f(t)dt \right)^p dx \leq C \int_{mw}^{\infty} (v(t)f(t))^p dt, \ f \geq 0, f \in L_{p,v}, \tag{29}$$

holds with the best constant

$$C = \left(\frac{p}{p-1} \right)^p G_p. \tag{30}$$

Moreover, in (29) *the equality is reached only for the trivial function.*

If the inequality (29) *holds for $m = 0$ with the best constant $C > 0$, then*

$$C \geq \left(\frac{p}{p-1} \right)^p G_p. \tag{31}$$

Remark 4.2. Theorem 4.1 was presented in [14] without proof in connection with its application to the oscillatory properties of certain class of Sturm–Liouville type quasilinear equations.

Proof. Let $f \not\equiv 0$ be a nonnegative function from $L_{p,v}$. Let $m \geq 1$. Using the relations (28) and (3), we have

$$\begin{aligned} &\int_{mw}^{\infty} \left(\frac{u(x)}{x} \int_{mw}^{x} f(t)dt \right)^p dx \\ &= \sum_{k=1}^{\infty} \int_{(k+m-1)w}^{(k+m)w} \left(\frac{u(x)}{x} \int_{mw}^{x} f(t)dt \right)^p dx \\ &\leq \sum_{k=1}^{\infty} \left(\frac{1}{(k+m-1)w} \int_{mw}^{(k+m)w} f(t)dt \right)^p \int_{(k+m-1)w}^{(k+m)w} u^p(x)dx \\ &= \frac{1}{w^p} \int_0^w u^p(x)dx \sum_{k=1}^{\infty} \left(\frac{1}{k+m-1} \sum_{i=1}^{k} \int_{(i+m-1)w}^{(i+m)w} f(t)dt \right)^p \\ &= \frac{1}{w^p} \int_0^w u^p(x)dx \sum_{k=1}^{\infty} \left(\frac{1}{k+m-1} \sum_{i=1}^{k} F_i \right)^p, \qquad (32) \end{aligned}$$

where

$$F_i = \int_{(i+m-1)w}^{(i+m)w} f(t)dt.$$

Due to (3), we get

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+m-1}\sum_{i=1}^{k} F_i\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{i=1}^{\infty} F_i^p. \tag{33}$$

Since by Hölder's inequality we have

$$\begin{aligned}\sum_{i=1}^{\infty} F_i^p &= \sum_{i=1}^{\infty}\left(\int_{(i+m-1)w}^{(i+m)w} f(t)dt\right)^p\\ &\le \sum_{i=1}^{\infty}\left(\int_{(i+m-1)w}^{(i+m)w} v^{-p'}(t)dt\right)^{p-1}\int_{(i+m-1)w}^{(i+m)w}|v(t)f(t)|^p dt\\ &= \left(\int_0^w v^{-p'}(t)dt\right)^{p-1}\sum_{i=1}^{\infty}\int_{(i+m-1)w}^{(i+m)w}|v(t)f(t)|^p dt\\ &= \left(\int_0^w v^{-p'}(t)dt\right)^{p-1}\int_{mw}^{\infty}|v(t)f(t)|^p dt,\end{aligned}$$

then from (32) and (33) we get

$$\int_{mw}^{\infty}\left(\frac{u(x)}{x}\int_{mw}^{x} f(t)dt\right)^p dx < \left(\frac{p}{p-1}\right)^p G_p \int_{mw}^{\infty}|v(t)f(t)|^p dt.$$

Hence,

$$C \le \left(\frac{p}{p-1}\right)^p G_p, \tag{34}$$

where C is the best constant in (29).

Now we prove that the estimate (31) holds for $m \ge 0$ and (34) implies (30) for $m \ge 1$.

Let $m \ge 0$ and inequality (29) hold for all functions $f \in L_{p,v}$, $f \ge 0$. We introduce the function g: $g(t) = v^{-p'}(t)f_k$, $k+m-1 \le t < k+m$, $\forall k \ge 1$, where $f = \{f_k\}_{k=1}^{\infty}$ is a sequence of real numbers such that $f_k \ge 0$, $\forall k \ge 1$, and $0 < \sum\limits_{k=1}^{\infty} f_k^p < \infty$.

Let us show that $g \in L_{p,v}$:

$$\begin{aligned}\int_{mw}^{\infty}(v(t)g(t))^p dt &= \sum_{k=1}^{\infty}\int_{(k+m-1)w}^{(k+m)w}(v(t)g(t))^p dt\\ &= \sum_{k=1}^{\infty} f_k^p \int_{(k+m-1)w}^{(k+m)w} v^{-p'}(t)dt\end{aligned}$$

$$= \int_0^w v^{-p'}(t)dt \sum_{k=1}^{\infty} f_k^p < \infty. \tag{35}$$

Therefore, for g the inequality (29) holds.

Moreover, we have

$$\begin{aligned}
&\int_{mw}^{\infty} \left(\frac{u(x)}{x} \int_{mw}^{x} g(t)dt \right)^p dx \\
&= \sum_{k=1}^{\infty} \int_{(k+m-1)w}^{(k+m)w} \left(\frac{u(x)}{x} \int_{mw}^{x} g(t)dt \right)^p dx \\
&\geq \sum_{k=2}^{\infty} \left(\frac{1}{(k+m)w} \int_{mw}^{(k+m-1)w} g(t)dt \right)^p \int_{(k+m-1)w}^{(k+m)w} u^p(x)dx \\
&= \frac{1}{w^p} \int_0^w u^p(x)dx \sum_{k=1}^{\infty} \left(\frac{1}{k+m+1} \int_{mw}^{(k+m)w} g(t)dt \right)^p \\
&= \frac{1}{w^p} \int_0^w u^p(x)dx \sum_{k=1}^{\infty} \left(\frac{1}{k+m+1} \sum_{i=1}^{k} \int_{(i+m-1)w}^{(i+m)w} g(t)dt \right)^p \\
&= \frac{1}{w^p} \int_0^w u^p(x)dx \sum_{k=1}^{\infty} \left(\frac{1}{k+m+1} \sum_{i=1}^{k} f_i \right)^p \left(\int_{(i+m-1)w}^{(i+m)w} v^{-p'}(t)dt \right)^p \\
&= \int_0^w v^{-p'}(t)dt \; G_p \sum_{k=1}^{\infty} \left(\frac{1}{k+m+1} \sum_{i=1}^{k} f_i \right)^p. \qquad (36)
\end{aligned}$$

Let C be the best constant in (29), then from (29), (35), (36) and (3) we have

$$\begin{aligned}
C &= \sup_{f\geq 0} \frac{\int_{mw}^{\infty} \left(\frac{u(x)}{x} \int_{mw}^{x} f(t)dt \right)^p dx}{\int_{mw}^{\infty} (v(t)f(t))^p dt} \\
&\geq G_p \sup_{f=\{f_k\}\geq 0} \frac{\sum\limits_{k=1}^{\infty} \left(\frac{1}{k+m+1} \sum\limits_{i=1}^{k} f_k \right)^p}{\sum\limits_{k=1}^{\infty} f_k^p} = \left(\frac{p}{p-1} \right)^p G_p.
\end{aligned}$$

Therefore, the estimate (31) holds. The proof of Theorem 4.1 is complete. □

Since under the conditions of Theorem 4.1 the inequality (29) is equivalent to the inequality

$$\int_0^{\infty} \left(\frac{u(x)}{x+mw} \int_0^x f(t)dt \right)^p dx \leq C \int_0^{\infty} (v(t)f(t))^p dt, \tag{37}$$

then from Theorem 4.1 we get the following corollary.

Corollary 4.3. *Let $1 < p < \infty$ and $u(t) \geq 0$, $v(t) > 0$, $\forall t \geq 0$. Let the functions u^p and $v^{-p'}$ belong to the class P_w. Then for any integer number $m \geq 1$ the inequality* (37) *holds with the best constant* (30). *Moreover, in* (37) *the equality is reached only for the trivial function.*

If the inequality (37) *holds for $m = 0$ with the best constant $C > 0$, then* (31) *holds.*

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