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ON COMPLETE CONVERGENCE FOR EXTENDED INDEPENDENT RANDOM VARIABLES UNDER SUB-LINEAR EXPECTATIONS

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Abstract. In this paper, we establish complete convergence for sequences of extended independent random variables and arrays of rowwise extended independent random variables under sub-linear expectations in Peng's framework. The results obtained in this paper extend the corresponding ones of Baum and Katz [1] and Hu and Taylor [11] from classical probability space to sub-linear expectation space.

1. Introduction

Additive probabilities and linear expectations are basic assumptions in classical probability theory. However, in fact, the additivity of probabilities and expectations has been abandoned in some areas because many uncertain phenomena cannot be well modeled by using additive probabilities and additive expectations. Recently, motivated by some problems in statistics, measures of risk, mathematical economics, super-hedging in finance and non-linear stochastic calculus, more and more researchers adopted non-additive probability and non-linear expectation to describe and interpret some uncertain phenomena in these fields which cannot be modeled exactly by classical probability theory. We refer the readers to Chen and Epstein [2], Huber [12], Huber and Strassen [13], Denis and Martini [5], Gilboa [6] and Marinacci [14] for instance.

A new notion of sub-linear expectation and related general theoretical framework of the sub-linear expectation space were proposed in Peng [15–20], and carefully studied by many scholars. For example, Hu [9] obtained Cramér's upper bound for capacities induced by sub-linear expectations, Hu and Zhang [10] established the central limit theorem for capacities, Chen and Hu [3] developed

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a law of the iterated logarithm for capacities with the notion of independent and identically distributed random variables under sub-linear expectations, Zhang [24] introduced the concept of negative dependence of random variables and established Kolmogorov's and Rosenthal's inequalities for the maximum partial sums of negatively dependent random variables under the sub-linear expectations, Zhang [25] studied the strong law of large numbers and the law of the iterated logarithm for extended independent and extended negatively dependent random variables under non-linear expectations, Wu and Jiang [21] established general strong law of large numbers and Chover's law of the iterated logarithm for a sequence of random variables under sub-linear expectations, Wu et al. [22] investigated the approximations of inverse moments for double-indexed weighted sums of random variables under sub-linear expectations, Zhong and Wu [26] studied complete convergence and complete moment convergence for weighted sums of extended negatively dependent random variables under sub-linear expectations, Xi et al. [23] obtained some results on complete convergence for arrays of rowwise extended negatively dependent random variables under sub-linear expectations and gave its statistical applications to nonparametric regression models. However, under sub-linear expectations, there are few available results related to complete convergence introduced by Hsu and Robbins [7]. This work aims to give complete convergence results of random variables under two cases in the sub-linear expectation space. As we all know, strong laws of large numbers can be also deduced from complete convergence by the corresponding Borel-Cantelli lemma (see Zhang [24]) in the sub-linear expectation space. Therefore, establishing complete convergence results under sub-linear expectations is of great interest.

We use the framework and notations of Peng [18]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C\left(1 + |\mathbf{x}|^m + |\mathbf{y}|^m\right)|\mathbf{x} - \mathbf{y}|, \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables".

Definition 1.1. A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}}$: $\mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: if $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;
(b) Constant preserving: $\hat{\mathbb{E}}[c] = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$, whenever $\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, $\lambda \geq 0$.

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The tripe $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\hat{\mathbb{E}}$, denote the conjugate expectation $\hat{\varepsilon}$ of

$\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}[X] \doteq -\hat{\mathbb{E}}[-X],\ \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that $\hat{\varepsilon}[X] \leq \hat{\mathbb{E}}[X]$, $\hat{\mathbb{E}}[X+c] = \hat{\mathbb{E}}[X] + c$ and $\hat{\mathbb{E}}[X-Y] \geq \hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]$ for all $c \in \mathbb{R}$ and all $X, Y \in \mathcal{H}$ with $\hat{\mathbb{E}}[Y]$ being finite. Further, if $\hat{\mathbb{E}}[|X|]$ is finite, then $\hat{\varepsilon}[X]$ and $\hat{\mathbb{E}}[X]$ are both finite.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G}$ be some subset of $\mathcal{F}$.

Definition 1.2. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0,\ V(\Omega) = 1 \ \text{ and } \ V(A) \leq V(B) \ \text{ for any } \ A \subset B,\ A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \bigcup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \bigcup B \in \mathcal{G}$. It is called to be countably sub-additive if for any $A_n \in \mathcal{G}$, $n = 1, 2, \ldots,$

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n). \tag{1.1}$$

In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \nu)$ of capacities by

$$\mathbb{V}(A) \doteq \inf\left\{\hat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\right\},\ \nu(A) \doteq 1 - \mathbb{V}(A^c),\ \forall\ A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\hat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}[g],\ \hat{\varepsilon}[f] \leq \nu(A) \leq \hat{\varepsilon}[g] \text{ if } f \leq I_A \leq g \text{ and } f, g \in \mathcal{H}.$$

It is obvious that $\mathbb{V}$ is sub-additive.

Also, we define the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\nu})$ by

$$C_V(X) = \int_0^{\infty} V(X \geq t)dt + \int_{-\infty}^{0} [V(X \geq t) - 1]dt$$

with V being replaced by $\mathbb{V}$ and ν, respectively. It can be verified that if $\lim_{c\to\infty} \hat{\mathbb{E}}[(|X|-c)^+] = 0$, then $\hat{\mathbb{E}}[|X|] \leq C_{\mathbb{V}}(|X|)$, which can be referred to Lemma 3.9 of Zhang [24].

The concepts of independence and identical distribution under sub-linear expectations were introduced by Peng [16] and [18].

Definition 1.3 (cf. Peng [16] and [18])**.**

(i) (Identical distribution) Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $\mathbf{X}_1 \overset{d}{=} \mathbf{X}_2$, if

$$\hat{\mathbb{E}}_1[\varphi(\mathbf{X}_1)] = \hat{\mathbb{E}}_2[\varphi(\mathbf{X}_2)],\ \forall \varphi \in C_{l,Lip}(\mathbb{R}^n),$$

whenever the sub-linear expectations above are finite. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $\mathbf{X} = (X_1, X_2, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$,

$$\hat{\mathbb{E}}[\varphi(\mathbf{X}, \mathbf{Y})] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})]|_{\mathbf{x}=\mathbf{X}}\right],$$

whenever $\bar{\varphi}(\mathbf{x}) \doteq \hat{\mathbb{E}}[|\varphi(\mathbf{x}, \mathbf{Y})|]$ for all $\mathbf{x}$ and $\hat{\mathbb{E}}[|\bar{\varphi}(\mathbf{X})|] < \infty$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be independent, if X_{i+1} is independent to $(X_1, X_2, \ldots, X_i)$ for each $i \geq 1$.

Zhang [25] introduced a concept of extended independence, which is much weaker and easier to verify than the above independence structure.

Definition 1.4 (Extended Independence, cf. Zhang [25])**.** In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a sequence $\{X_n, n \geq 1\}$ of random variables is said to be extended independent (EI, for short), if for any $n \geq 1$,

$$\hat{\mathbb{E}}\left[\prod_{i=1}^{n} \psi_i(X_i)\right] = \prod_{i=1}^{n} \hat{\mathbb{E}}\left[\psi_i(X_i)\right], \tag{1.2}$$

whenever $\psi_i \in C_{l,Lip}(\mathbb{R})$ $(i = 1, 2, \ldots, n)$ are all non-negative functions.

An array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ of random variables is said to be rowwise extended independent, if for any fixed $n \geq 1$, $\{X_{ni}, 1 \leq i \leq n\}$ are extended independent random variables.

It is easy to check that the independence implies extended independence. The independence in the sense of (1.2) was proposed by Chen et al. [4]. But their function space of ψ_i's is assumed to be the family of all non-negative Borel functions. Here Zhang [25] used the same function space as Peng's.

In the paper, we aim to establish the complete convergence for sequences of EI random variables and arrays of rowwise EI random variables under sub-linear expectations in Peng's framework. The results of the paper mainly extend the corresponding ones of Baum and Katz [1] and Hu and Taylor [11] from classical probability space to sub-linear expectation space, and make some improvements.

This work is organized as follows. In next section, we give some preliminary lemmas, which are useful to prove our main results. Main results on complete convergence for sequences of EI random variables and arrays of rowwise EI random variables are provided in Section 3. Detailed proofs of main results are put in the last section.

Throughout the paper, C stands for a positive constant whose value may vary at each occurrence.

2. Preliminaries

In this section, we will present some important lemmas which will be used to prove the main results of the paper. In addition, Höder's inequality, Markov's inequality and Jensen's inequality are still true under sub-linear expectations, see Zhang [24] and Hu [8] for instance.

The first one is a basic property for EI random variables, which can be found in Zhang [25].

Lemma 2.1 (cf. Zhang [25]). *Let $\{X_n, n \geq 1\}$ be a sequence of EI random variables. Then $\{f_n(X_n), n \geq 1\}$ are still EI random variables if $f_1(x), f_2(x), \ldots \in C_{l,Lip}(\mathbb{R})$.*

The next one is the triangle inequality for two random variables in sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

Lemma 2.2. *For any random variables $X,\ Y \in \mathcal{H}$, the following inequalities hold*:

(i) $\left|\hat{\mathbb{E}}[X-Y]\right| \leq \left|\hat{\mathbb{E}}[X]\right| + \hat{\mathbb{E}}[|Y|]$;

(ii) $\left|\hat{\mathbb{E}}[X+Y]\right| \leq \left|\hat{\mathbb{E}}[X]\right| + \hat{\mathbb{E}}[|Y|]$.

Proof. (i) It follows from the sub-additivity of $\hat{\mathbb{E}}$ that

$$\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X-Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[-Y].$$

Thus by the monotonicity of $\hat{\mathbb{E}}$, we have

$$\left|\hat{\mathbb{E}}[X-Y]\right| \leq \max\left\{\left|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]\right|, \left|\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[-Y]\right|\right\} \leq \left|\hat{\mathbb{E}}[X]\right| + \hat{\mathbb{E}}[|Y|].$$

(ii) Replacing Y by $-Y$ in (i), we can obtain (ii) immediately.

The proof is completed. □

The following one is the exponential inequality for EI random variables, whose detailed proof is similar to that of Theorem 3.1 in Zhang [25].

Lemma 2.3 (Exponential inequality). *Assume that $\{X_n, n \geq 1\}$ is a sequence of EI random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_i] \leq 0$ for each $i \geq 1$, $S_n = \sum_{i=1}^n X_i$, and $B_n = \sum_{i=1}^n \hat{\mathbb{E}}[X_i^2]$. Then for all $x, y > 0$,*

$$\mathbb{V}(S_n \geq x) \leq \mathbb{V}\left(\max_{1 \leq i \leq n} |X_i| > y\right) + e^{\frac{x}{y}} \left(\frac{B_n}{xy}\right)^{\frac{x}{y}}. \tag{2.1}$$

In particular, taking $y = x > 0$, we have for any $x > 0$ that

$$\mathbb{V}(S_n \geq x) \leq (1+e)\frac{B_n}{x^2}. \tag{2.2}$$

The next one is the Borel-Cantelli Lemma under sub-additive capacity, which can be found in Zhang [24].

Lemma 2.4 (Borel-Cantelli Lemma, cf. Zhang [24]). *Let $\{A_n, n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^\infty V(A_n) < \infty$, then*

$$V(A_n, i.o.) = 0,$$

where $(A_n, i.o.) = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m$.

The last one is the Rosenthal's inequality for independent random variables, which can be referred to Zhang [24].

Lemma 2.5 (Rosenthal's inequality, cf. Zhang [24])**.** *Assume that* $\{X_n, n \geq 1\}$ *is a sequence of independent random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *with* $\hat{\mathbb{E}}[X_i] \leq 0$ *for each* $i \geq 1$*, and* $S_n = \sum_{i=1}^n X_i$*. Then for each* $n \geq 1$*,*

$$\hat{\mathbb{E}}[(S_n^+)^p] \leq C_p \left(\sum_{i=1}^n \hat{\mathbb{E}}[|X_i|^p] + \left(\sum_{i=1}^n \hat{\mathbb{E}}[X_i^2] \right)^{p/2} \right) \text{ for } p \geq 2. \tag{2.3}$$

3. Main results

In this section, we will present the main results of the paper, including complete convergence for sequences of EI random variables and arrays of rowwise EI random variables.

3.1. Complete convergence for sequences of EI random variables

Theorem 3.1. *Let* $\alpha p > 1$ *and* $\alpha > 1/2$*. Assume that* $\{X_n, n \geq 1\}$ *is a sequence of identically distributed EI random variables with* $C_{\mathbb{V}}(|X_1|^p) < \infty$ *and* $\lim_{c\to\infty} \hat{\mathbb{E}}\left[(|X_1|^p - c)^+\right] = 0$*.*

(i) *If* $0 < p < 1$*, then for any* $\epsilon > 0$*,*

$$\sum_{n=1}^\infty n^{\alpha p-2} \mathbb{V}\left(\left| \sum_{i=1}^n X_i \right| > \epsilon n^\alpha \right) < \infty. \tag{3.1}$$

(ii) *If* $p \geq 1$*, then for any* $\epsilon > 0$*,*

$$\sum_{n=1}^\infty n^{\alpha p-2} \mathbb{V}\left(\left(\sum_{i=1}^n (X_i - \hat{\mathbb{E}}[X_i]) > \epsilon n^\alpha \right) \bigcup \left(\sum_{i=1}^n (X_i - \hat{\varepsilon}[X_i]) < -\epsilon n^\alpha \right) \right) < \infty. \tag{3.2}$$

Furthermore, if $\hat{\mathbb{E}}[X_1] = \hat{\varepsilon}[X_1] = 0$*, then* (3.1) *holds for any* $\epsilon > 0$*.*

Taking $\alpha = 1$ and $p = 2$ in Theorem 3.1, we can obtain the following corollary.

Corollary 3.1. *Assume that* $\{X_n, n \geq 1\}$ *is a sequence of identically distributed EI random variables with* $C_{\mathbb{V}}(X_1^2) < \infty$ *and* $\lim_{c\to\infty} \hat{\mathbb{E}}\left[(X_1^2 - c)^+\right] = 0$*. If* $\mathbb{V}$ *is countably sub-additive, then*

$$\mathbb{V}\left(\left(\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_i > \hat{\mathbb{E}}[X_1] \right) \bigcup \left(\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_i < \hat{\varepsilon}[X_1] \right) \right) = 0 \tag{3.3}$$

and

$$\nu\left(\hat{\varepsilon}[X_1] \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_i \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_i \leq \hat{\mathbb{E}}[X_1] \right) = 1. \tag{3.4}$$

For $\alpha p = 1$, we have the following result.

Theorem 3.2. *Let $\alpha p = 1$ and $\alpha > 1/2$. Assume that $\{X_n, n \geq 1\}$ is a sequence of identically distributed EI random variables with*

$$\lim_{c\to\infty} \hat{\mathbb{E}}\left[(|X_1|^p - c)^+\right] = 0.$$

(i) *If $0 < p < 1$, then $C_{\mathbb{V}}(|X_1|) < \infty$ implies that* (3.1) *holds for any $\epsilon > 0$.*
(ii) *If $p \geq 1$, then $C_{\mathbb{V}}(|X_1|^p) < \infty$ implies that* (3.2) *holds for any $\epsilon > 0$.*

3.2. Complete convergence for arrays of rowwise EI random variables

Theorem 3.3. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise EI random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\{\Psi_i(t), i \geq 1\}$ be a sequence of positive even functions such that*

$$\frac{\Psi_i(|t|)}{|t|^q} \uparrow, \ \frac{\Psi_i(|t|)}{|t|^p} \downarrow \ \text{as } |t| \uparrow \tag{3.5}$$

for some $1 \leq q < p \leq 2$ and each $i \geq 1$. Assume that $\{a_n, n \geq 1\}$ is a sequence of positive real numbers such that $0 < a_n \uparrow \infty$, and

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} \hat{\mathbb{E}}\left[\frac{\Psi_i(uX_{ni})}{\Psi_i(a_n)}\right] < \infty, \ \sum_{n=1}^{\infty}\sum_{i=1}^{n} \hat{\mathbb{E}}\left[\frac{\Psi_i(X_{ni})}{\Psi_i(ua_n)}\right] < \infty \tag{3.6}$$

for some $0 < u < 1$. Then for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left(\frac{1}{a_n}\sum_{i=1}^{n}(X_{ni} - \hat{\mathbb{E}}[X_{ni}]) > \epsilon\right) \bigcup \left(\frac{1}{a_n}\sum_{i=1}^{n}(X_{ni} - \hat{\varepsilon}[X_{ni}]) < -\epsilon\right)\right) < \infty. \tag{3.7}$$

Corollary 3.2. *Under the conditions of Theorem 3.3, if $\hat{\mathbb{E}}[X_{ni}] = \hat{\varepsilon}[X_{ni}] = 0$, then for any $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left|\frac{1}{a_n}\sum_{i=1}^{n} X_{ni}\right| > \epsilon\right) < \infty. \tag{3.8}$$

Furthermore, if $\mathbb{V}$ is countably sub-additive, then

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \to 0 \ \text{a.s. } \nu, \ n \to \infty. \tag{3.9}$$

Remark 3.1. Since it is not sure whether Rosenthal's inequality like (2.3) is true or not for EI random variables, we couldn't establish the result similarly to Theorem 3.3 when $p > 2$. This is an open question proposed in the paper.

Here, we conjecture that the result may be the following form: Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise EI random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\{\Psi_i(t), i \geq 1\}$ be a sequence of positive even functions satisfying (3.5) for some $1 \leq q < p$, $p > 2$ and each $i \geq 1$. Assume that $\{a_n, n \geq 1\}$ is a sequence of positive real numbers satisfying $0 < a_n \uparrow \infty$, (3.6) and

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} \hat{\mathbb{E}}\left[\left(\frac{X_{ni}}{a_n}\right)^2\right]\right)^{s/2} < \infty \tag{3.10}$$

for some $s \geq p$. Then for any $\epsilon > 0$, (3.7) holds. The main reason for this is that the above result holds for independent random variables, see Theorem 3.4 below.

Theorem 3.4. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\{\Psi_i(t), i \geq 1\}$ be a sequence of positive even functions satisfying* (3.5) *for some $1 \leq q < p$, $p > 2$ and each $i \geq 1$. Assume that $\{a_n, n \geq 1\}$ is a sequence of positive real numbers satisfying $0 < a_n \uparrow \infty$,* (3.6) *and* (3.10) *for some $s \geq p$. Then for any $\epsilon > 0$,* (3.7) *holds.*

Corollary 3.3. *Under the conditions of Theorem 3.4, if $\hat{\mathbb{E}}[X_{ni}] = \hat{\varepsilon}[X_{ni}] = 0$, then* (3.8) *holds. Furthermore, if $\mathbb{V}$ is countably sub-additive, then* (3.9) *holds.*

4. Proofs of main results

Proof of Theorem 3.1. When $p \geq 1$, it is easy to observe that (3.2) is equivalent to

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n}(X_i - \hat{\mathbb{E}}[X_i]) > \epsilon n^{\alpha}\right) < \infty \tag{4.1}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n}(X_i - \hat{\varepsilon}[X_i]) < -\epsilon n^{\alpha}\right) < \infty. \tag{4.2}$$

Furthermore, if (4.1) holds, then (4.2) is also true by replacing X_i by $-X_i$. Hence, to prove (3.2), we only need to prove (4.1).

Similarly, to prove (3.1), it is sufficient to prove

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} X_i > \epsilon n^{\alpha}\right) < \infty \tag{4.3}$$

when $0 < p < 1$.

Without loss of generality, we assume that $\hat{\mathbb{E}}[X_i] = 0$ when $p \geq 1$. Thus, to prove (3.1) and (3.2), we only need to prove (4.3) for any $p > 0$.

For any $r, \epsilon > 0$ and fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$Y_{ni} = -\frac{\epsilon n^{\alpha}}{4r} I\left(X_i < -\frac{\epsilon n^{\alpha}}{4r}\right) + X_i I\left(|X_i| \leq \frac{\epsilon n^{\alpha}}{4r}\right) + \frac{\epsilon n^{\alpha}}{4r} I\left(X_i > \frac{\epsilon n^{\alpha}}{4r}\right),$$

$$Y'_{ni} = X_i - Y_{ni} = \left(X_i + \frac{\epsilon n^{\alpha}}{4r}\right) I\left(X_i < -\frac{\epsilon n^{\alpha}}{4r}\right) + \left(X_i - \frac{\epsilon n^{\alpha}}{4r}\right) I\left(X_i > \frac{\epsilon n^{\alpha}}{4r}\right).$$

Let g_υ be a function such that $g_\upsilon \in C_{l,Lip}(\mathbb{R})$, $g_\upsilon = 1$ if $|x| \geq 1$, $g_\upsilon = 0$ if $|x| \leq 1 - \upsilon$, and $0 \leq g_\upsilon \leq 1$ for all x, where $0 < \upsilon < 1$. Thus

$$\begin{aligned} &I(|x| \geq 1) \leq g_\upsilon(x) \leq I(|x| > 1 - \upsilon), \\ &I(|x| \leq 1 - \upsilon) \leq 1 - g_\upsilon(x) \leq I(|x| < 1). \end{aligned} \tag{4.4}$$

Note that $\hat{\mathbb{E}}\left[|X_1|^p\right] \leq C_{\mathbb{V}}(|X_1|^p) < \infty$ by $\lim_{c\to\infty} \hat{\mathbb{E}}\left[(|X_1|^p - c)^+\right] = 0$.

If $p \geq 1$, by Lemma 2.2, $\hat{\mathbb{E}}[X_i]=0$, (4.4) and $\alpha p>1$, we have that

$$\begin{aligned}
n^{-\alpha}\left|\sum_{i=1}^{n}\hat{\mathbb{E}}[Y_{ni}]\right| &\leq n^{-\alpha}\sum_{i=1}^{n}\left(\left|\hat{\mathbb{E}}[X_i]\right|+\hat{\mathbb{E}}\left[\left|Y'_{ni}\right|\right]\right)\\
&= n^{-\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(|X_i|-\frac{\epsilon n^{\alpha}}{4r}\right)I\left(|X_i|>\frac{\epsilon n^{\alpha}}{4r}\right)\right]\\
&\leq n^{-\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(|X_i|-\frac{\epsilon n^{\alpha}}{4r}\right)g_{\frac{1}{2}}\left(\frac{4rX_i}{\epsilon n^{\alpha}}\right)\right]\\
&\leq n^{-\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X_i|g_{\frac{1}{2}}\left(\frac{4rX_i}{\epsilon n^{\alpha}}\right)\right]\\
&= n^{1-\alpha}\hat{\mathbb{E}}\left[|X_1|g_{\frac{1}{2}}\left(\frac{4rX_1}{\epsilon n^{\alpha}}\right)\right]\\
&\leq Cn^{1-\alpha p}\hat{\mathbb{E}}\left[|X_1|^p\right]\to 0 \text{ as } n\to\infty. \qquad (4.5)
\end{aligned}$$

If $0<p<1$, by (4.4), Markov's inequality and $\alpha p>1$, we have that

$$\begin{aligned}
n^{-\alpha}\left|\sum_{i=1}^{n}\hat{\mathbb{E}}[Y_{ni}]\right| &\leq n^{-\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X_i|I\left(|X_i|\leq\frac{\epsilon n^{\alpha}}{4r}\right)+\frac{\epsilon n^{\alpha}}{4r}I\left(|X_i|>\frac{\epsilon n^{\alpha}}{4r}\right)\right]\\
&\leq n^{-\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X_i|\left(1-g_{\frac{1}{2}}\left(\frac{2rX_i}{\epsilon n^{\alpha}}\right)\right)+\frac{\epsilon n^{\alpha}}{4r}g_{\frac{1}{2}}\left(\frac{4rX_i}{\epsilon n^{\alpha}}\right)\right]\\
&\leq n^{-\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X_i|\left(1-g_{\frac{1}{2}}\left(\frac{2rX_i}{\epsilon n^{\alpha}}\right)\right)\right]+C\sum_{i=1}^{n}\hat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{4rX_i}{\epsilon n^{\alpha}}\right)\right]\\
&= n^{1-\alpha}\hat{\mathbb{E}}\left[|X_1|\left(1-g_{\frac{1}{2}}\left(\frac{2rX_1}{\epsilon n^{\alpha}}\right)\right)\right]+Cn\hat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{4rX_1}{\epsilon n^{\alpha}}\right)\right]\\
&\leq Cn^{1-\alpha p}\hat{\mathbb{E}}\left[|X_1|^p\right]\to 0 \text{ as } n\to\infty. \qquad (4.6)
\end{aligned}$$

Hence for all n large enough, we have

$$n^{-\alpha}\left|\sum_{i=1}^{n}\hat{\mathbb{E}}[Y_{ni}]\right|<\frac{\epsilon}{2}.$$

It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n}X_i>\epsilon n^{\alpha}\right)\\
\leq\ &\sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left(\max_{1\leq i\leq n}|X_i|>\frac{\epsilon n^{\alpha}}{4r}\right)+\sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n}Y_{ni}>\epsilon n^{\alpha}\right)\\
\leq\ &\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}\left(|X_i|>\frac{\epsilon n^{\alpha}}{4r}\right)
\end{aligned}$$

$$+ C\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n}(Y_{ni}-\hat{\mathbb{E}}[Y_{ni}]) > \frac{\epsilon n^{\alpha}}{2}\right)$$

$$\doteq I + CJ. \tag{4.7}$$

It is easily seen that

$$\begin{aligned}
I &\le \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{4rX_i}{\epsilon n^{\alpha}}\right)\right] \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1}\hat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{4rX_1}{\epsilon n^{\alpha}}\right)\right] \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\left(|X_1| \ge \frac{\epsilon n^{\alpha}}{8r}\right) \\
&\le C\int_0^{\infty} x^{\alpha p-1}\mathbb{V}\left(|X_1|^p \ge \left(\frac{\epsilon}{8r}\right)^p \cdot x^{\alpha p}\right)dx \;\left(\text{taking } t = \left(\frac{\epsilon}{8r}\right)^p \cdot x^{\alpha p}\right) \\
&= C\int_0^{\infty} \mathbb{V}(|X_1|^p \ge t)dt \\
&= C \cdot C_{\mathbb{V}}(|X_1|^p) < \infty.
\end{aligned} \tag{4.8}$$

Next, we will show that $J < \infty$. It is easily seen that for fixed $n \ge 1$, $\left\{\frac{1}{n^{\alpha}}\left(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right), 1 \le i \le n\right\}$ are still EI random variables by Lemma 2.1. Noting that $\hat{\mathbb{E}}\left[\frac{1}{n^{\alpha}}\left(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right)\right] = 0$ and $\max_{1\le i\le n}\frac{1}{n^{\alpha}}\left|Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right| \le \frac{\epsilon}{2r}$ for fixed $n \ge 1$, we have by (2.1) in Lemma 2.3 (taking $x = \frac{\epsilon}{2}$ and $y = \frac{\epsilon}{2r}$) that

$$J \le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha r-2}\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right)^2\right]\right)^r. \tag{4.9}$$

If $p \ge 2$, then by taking $r > \frac{\alpha p-1}{2\alpha-1}$ and Hölder's inequality, we have that

$$\begin{aligned}
J &\le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha r-2}\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[Y_{ni}^2\right]\right)^r \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha r-2}\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[X_i^2\right]\right)^r \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha r-2}\left(n\left(\hat{\mathbb{E}}\left[|X_1|^p\right]\right)^{2/p}\right)^r \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha r-2+r} < \infty.
\end{aligned} \tag{4.10}$$

If $0 < p < 2$, then by taking $r = 1$ and the sub-additivity of $\hat{\mathbb{E}}$, we have that

$$\begin{aligned} J &\le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[X_i^2 I\left(|X_i|\le \frac{\epsilon n^{\alpha}}{4}\right)+\frac{\epsilon^2 n^{2\alpha}}{16} I\left(|X_i|>\frac{\epsilon n^{\alpha}}{4}\right)\right] \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[X_i^2\left(1-g_{\frac{1}{2}}\left(\frac{2X_i}{\epsilon n^{\alpha}}\right)\right)+\frac{\epsilon^2 n^{2\alpha}}{16} g_{\frac{1}{2}}\left(\frac{4X_i}{\epsilon n^{\alpha}}\right)\right] \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[X_i^2\left(1-g_{\frac{1}{2}}\left(\frac{2X_i}{\epsilon n^{\alpha}}\right)\right)\right] \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[g_{\frac{1}{2}}\left(\frac{4X_i}{\epsilon n^{\alpha}}\right)\right] \\ &\doteq CJ_1 + CJ_2. \end{aligned} \tag{4.11}$$

In view of (4.8), we have $J_2 < \infty$. In order to prove $J_1 < \infty$, we construct functions $g_k(\cdot)$, $k = 1, 2, \ldots$. Let $g_k(x) \in C_{l,Lip}(\mathbb{R})$ such that $0 \le g_k(x) \le 1$ and

$$g_k\left(\frac{x}{\epsilon\cdot 2^{k\alpha}}\right) = \begin{cases} 0 & \text{if } \epsilon\cdot 2^{(k-1)\alpha} < |x| \le \epsilon\cdot 2^{k\alpha}, \\ 1 & \text{if } |x| \le \frac{\epsilon}{2}\cdot 2^{(k-1)\alpha} \text{ or } |x| > \frac{3\epsilon}{2}\cdot 2^{k\alpha}. \end{cases}$$

It can be checked that

$$1 - g_k\left(\frac{X_1}{\epsilon\cdot 2^{k\alpha}}\right) \le I\left(\frac{\epsilon}{2}\cdot 2^{(k-1)\alpha} < |X_1| \le \frac{3\epsilon}{2}\cdot 2^{k\alpha}\right) \tag{4.12}$$

and

$$X_1^2\left(1-g_{\frac{1}{2}}\left(\frac{X_1}{\epsilon\cdot 2^{j\alpha}}\right)\right) \le \epsilon^2 + \sum_{k=1}^{j} X_1^2\left(1-g_k\left(\frac{X_1}{\epsilon\cdot 2^{k\alpha}}\right)\right). \tag{4.13}$$

Since $p < 2$, it follows by (4.4) and (4.13) that

$$\begin{aligned} J_1 &= \sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1}\hat{\mathbb{E}}\left[X_1^2\left(1-g_{\frac{1}{2}}\left(\frac{2X_1}{\epsilon n^{\alpha}}\right)\right)\right] \\ &= \sum_{j=1}^{\infty}\sum_{n=2^{j-1}}^{2^j-1} n^{\alpha p-2\alpha-1}\hat{\mathbb{E}}\left[X_1^2\left(1-g_{\frac{1}{2}}\left(\frac{2X_1}{\epsilon n^{\alpha}}\right)\right)\right] \\ &\le \sum_{j=1}^{\infty}\sum_{n=2^{j-1}}^{2^j-1} n^{\alpha p-2\alpha-1}\hat{\mathbb{E}}\left[X_1^2\left(1-g_{\frac{1}{2}}\left(\frac{X_1}{\epsilon\cdot 2^{j\alpha}}\right)\right)\right] \\ &\le C\sum_{j=1}^{\infty} 2^{\alpha(p-2)j}\hat{\mathbb{E}}\left[X_1^2\left(1-g_{\frac{1}{2}}\left(\frac{X_1}{\epsilon\cdot 2^{j\alpha}}\right)\right)\right] \\ &\le C\sum_{j=1}^{\infty} 2^{\alpha(p-2)j}\hat{\mathbb{E}}\left[\epsilon^2+\sum_{k=1}^{j} X_1^2\left(1-g_k\left(\frac{X_1}{\epsilon\cdot 2^{k\alpha}}\right)\right)\right] \end{aligned}$$

$$\leq C\sum_{j=1}^{\infty}2^{\alpha(p-2)j}+C\sum_{j=1}^{\infty}2^{\alpha(p-2)j}\sum_{k=1}^{j}\hat{\mathbb{E}}\left[X_1^2\left(1-g_k\left(\frac{X_1}{\epsilon\cdot 2^{k\alpha}}\right)\right)\right]$$

$$\doteq CJ_{11}+CJ_{12}. \tag{4.14}$$

It is obvious that $J_{11}<\infty$. According to (4.12) and the proof of (4.8), we can obtain

$$\begin{aligned} J_{12} &\leq C\sum_{k=1}^{\infty}2^{\alpha(p-2)k}\hat{\mathbb{E}}\left[X_1^2\left(1-g_k\left(\frac{X_1}{\epsilon\cdot 2^{k\alpha}}\right)\right)\right] \\ &\leq C\sum_{k=1}^{\infty}2^{\alpha pk}\mathbb{V}\left(|X_1|>\frac{\epsilon}{2}\cdot 2^{(k-1)\alpha}\right) \\ &\leq C\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\left(2^k\right)^{\alpha p-1}\mathbb{V}\left(|X_1|>\frac{\epsilon}{2^{1+2\alpha}}\cdot 2^{(k+1)\alpha}\right) \\ &\leq C\sum_{n=1}^{\infty}n^{\alpha p-1}\mathbb{V}\left(|X_1|>\frac{\epsilon}{2^{1+2\alpha}}\cdot n^{\alpha}\right) \\ &\leq C\cdot C_{\mathbb{V}}(|X_1|^p)<\infty. \end{aligned} \tag{4.15}$$

Hence, $J<\infty$ from (4.11), $J_2<\infty$, (4.14), $J_{11}<\infty$ and (4.15) when $0<p<2$.

This completes the proof of the theorem. □

Proof of Corollary 3.1. Taking $\alpha=1$ and $p=2$ in Theorem 3.1(ii), we have by (3.2) that

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\hat{\mathbb{E}}[X_1]>\epsilon\right)<\infty \tag{4.16}$$

and

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\hat{\varepsilon}[X_1]<-\epsilon\right)<\infty.$$

By (4.16) and Lemma 2.4, we can obtain that for any $\epsilon>0$,

$$\mathbb{V}\left(\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\hat{\mathbb{E}}[X_1]>\epsilon\right),i.o.\right)=0,$$

and thus,

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i-\hat{\mathbb{E}}[X_1]>\epsilon\right)=0. \tag{4.17}$$

It follows by the countable sub-additivity of $\mathbb{V}$ and (4.17) that

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i>\hat{\mathbb{E}}[X_1]\right)=\mathbb{V}\left(\bigcup_{k=1}^{\infty}\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i>\hat{\mathbb{E}}[X_1]+\frac{1}{k}\right)\right)$$

$$\leq \sum_{k=1}^{\infty} \mathbb{V}\left(\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i > \hat{\mathbb{E}}[X_1] + \frac{1}{k}\right) = 0,$$

and thus,

$$\mathbb{V}\left(\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i < \hat{\varepsilon}[X_1]\right) = \mathbb{V}\left(\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(-X_i) > \hat{\mathbb{E}}[-X_1]\right) = 0.$$

Therefore, the desired result (3.3) is obtained, and (3.4) follows from (3.3) immediately. This completes the proof of the corollary. □

Proof of Theorem 3.2. Without loss of generality, we assume that $\hat{\mathbb{E}}[X_i] = 0$ when $p \geq 1$. Similar to the proof of Theorem 3.1, it is sufficient to show that for any $p > 0$,

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{V}\left(\sum_{i=1}^{n} X_i > \epsilon n^{1/p}\right) < \infty. \tag{4.18}$$

Here, we use the same notations as those in Theorem 3.1.

If $p \geq 1$, since

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_1|^p > Cn) \leq C \cdot C_{\mathbb{V}}(|X_1|^p) < \infty \tag{4.19}$$

and $\mathbb{V}(|X_1|^p > Cn)$ is decreasing with respect to n, we have $n\mathbb{V}(|X_1|^p > Cn) \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}\hat{\mathbb{E}}\left[|X_1|^p g_{\frac{1}{2}}\left(\frac{X_1^p}{Cn}\right)\right] &\leq \hat{\mathbb{E}}\left[(|X_1|^p - n) g_{\frac{1}{2}}\left(\frac{X_1^p}{Cn}\right)\right] + \hat{\mathbb{E}}\left[n g_{\frac{1}{2}}\left(\frac{X_1^p}{Cn}\right)\right] \\ &\leq \hat{\mathbb{E}}\left[(|X_1|^p - n)^+\right] + n\mathbb{V}(|X_1|^p > Cn) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.20}$$

Hence, similar to the proof of (4.5), we have by (4.20) that

$$n^{-1/p}\left|\sum_{i=1}^{n} \hat{\mathbb{E}}[Y_{ni}]\right| \leq \hat{\mathbb{E}}\left[|X_1|^p g_{\frac{1}{2}}\left(\frac{X_1^p}{Cn}\right)\right] \to 0 \text{ as } n \to \infty.$$

If $0 < p < 1$, similar to the proof of (4.6), we have that

$$n^{-1/p}\left|\sum_{i=1}^{n} \hat{\mathbb{E}}[Y_{ni}]\right| \leq n^{1-1/p}\hat{\mathbb{E}}\left[|X_1|\right] \to 0 \text{ as } n \to \infty.$$

Therefore, for all n large enough, we have

$$n^{-1/p}\left|\sum_{i=1}^{n} \hat{\mathbb{E}}[Y_{ni}]\right| < \frac{\epsilon}{2}.$$

In view of (4.7), (4.19) and the proofs of $I < \infty$ and $J < \infty$ in Theorem 3.1, we can get the desired result (4.18) immediately. This completes the proof of the theorem. □

Proof of Theorem 3.3. In order to prove (3.7), it is equivalent to prove

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n}(X_{ni} - \hat{\mathbb{E}}[X_{ni}]) > a_n\epsilon\right) < \infty \tag{4.21}$$

and

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n}(X_{ni} - \hat{\varepsilon}[X_{ni}]) < -a_n\epsilon\right) < \infty. \tag{4.22}$$

It is obvious that (4.22) follows from (4.21) by replacing X_{ni} by $-X_{ni}$. So we only need to prove (4.21). Without loss of generality, we assume that $\hat{\mathbb{E}}[X_{ni}] = 0$. Thus it is sufficient to show

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} X_{ni} > a_n\epsilon\right) < \infty. \tag{4.23}$$

For fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$Y_{ni} = -a_n I(X_{ni} < -a_n) + X_{ni} I(|X_{ni}| \leq a_n) + a_n I(X_{ni} > a_n),$$
$$Y_{ni}^{'} = X_{ni} - Y_{ni} = (X_{ni} - a_n) I(X_{ni} > a_n) + (X_{ni} + a_n) I(X_{ni} < -a_n).$$

Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} X_{ni} > a_n\epsilon\right) \\
\leq\ &\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} Y_{ni} > \frac{1}{2}a_n\epsilon\right) + \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} Y_{ni}^{'} > \frac{1}{2}a_n\epsilon\right) \\
\leq\ &\sum_{n=1}^{\infty} \mathbb{V}\left(\frac{1}{a_n}\sum_{i=1}^{n}\Big(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\Big) > \frac{1}{2}\epsilon - \frac{1}{a_n}\sum_{i=1}^{n}\Big|\hat{\mathbb{E}}[Y_{ni}]\Big|\right) \\
&+ \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} Y_{ni}^{'} > \frac{1}{2}a_n\epsilon\right) \\
\doteq\ &I_1 + I_2.
\end{aligned} \tag{4.24}$$

Let f_u be a function such that $f_u \in C_{l,Lip}(\mathbb{R})$, $f_u = 1$ if $|x| \geq 1$, $f_u = 0$ if $|x| \leq u$, and $0 \leq f_u \leq 1$ for all x. Thus

$$I(|x| \geq 1) \leq f_u(x) \leq I(|x| > u),\ I(|x| \leq u) \leq 1 - f_u(x) \leq I(|x| < 1). \tag{4.25}$$

Noting that $\hat{\mathbb{E}}[X_{ni}] = 0$ and $q \geq 1$, we have by Lemma 2.2, (3.5) and (3.6) that

$$\begin{aligned}
\frac{1}{a_n}\sum_{i=1}^{n}\Big|\hat{\mathbb{E}}[Y_{ni}]\Big| &= \frac{1}{a_n}\sum_{i=1}^{n}\Big|\hat{\mathbb{E}}[X_{ni} - Y_{ni}^{'}]\Big| \\
&\leq \frac{1}{a_n}\sum_{i=1}^{n}\Big\{\Big|\hat{\mathbb{E}}[X_{ni}]\Big| + \hat{\mathbb{E}}[|Y_{ni}^{'}|]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a_n}\sum_{i=1}^{n}\hat{\mathbb{E}}[(|X_{ni}|-a_n)I(|X_{ni}|>a_n)]\\
&\le \frac{1}{a_n}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[(|X_{ni}|-a_n)f_u\left(\frac{X_{ni}}{a_n}\right)\right]\\
&\le \frac{1}{a_n}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X_{ni}|f_u\left(\frac{X_{ni}}{a_n}\right)\right]\\
&\le C\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\Psi_i(X_{ni})}{\Psi_i(ua_n)}\right]\to 0 \text{ as } n\to\infty,
\end{aligned}
$$

which, together with (2.2) in Lemma 2.3, (3.5) and (3.6), yields that

$$
\begin{aligned}
I_1 &\le C\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{a_n}\sum_{i=1}^{n}\left(Y_{ni}-\hat{\mathbb{E}}[Y_{ni}]\right)>\frac{1}{4}\epsilon\right)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(Y_{ni}-\hat{\mathbb{E}}[Y_{ni}]\right)^2\right]\\
&\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}\hat{\mathbb{E}}[Y_{ni}^2]\\
&\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}\hat{\mathbb{E}}[X_{ni}^2I(|X_{ni}|\le a_n)+a_n^2I(|X_{ni}|>a_n)]\\
&\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[X_{ni}^2\left(1-f_u\left(\frac{uX_{ni}}{a_n}\right)\right)\right]+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[f_u\left(\frac{X_{ni}}{a_n}\right)\right]\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\Psi_i(uX_{ni})}{\Psi_i(a_n)}\right]+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\Psi_i(X_{ni})}{\Psi_i(ua_n)}\right]<\infty. \quad (4.26)
\end{aligned}
$$

According to the definition of Y'_{ni}, we have by (3.5) and (3.6) again that

$$
\begin{aligned}
I_2 &\le \sum_{n=1}^{\infty}\mathbb{V}\left(\bigcup_{i=1}^{n}(|X_{ni}|>a_n)\right)\\
&\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathbb{V}(|X_{ni}|>a_n)\\
&\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\Psi_i(X_{ni})}{\Psi_i(ua_n)}\right]<\infty. \quad (4.27)
\end{aligned}
$$

Hence, (4.23) follows from (4.24), (4.26) and (4.27) immediately. The proof is completed. □

Proof of Corollary 3.2. The proof is similar to that of Corollary 3.1, so we omit the details. □

Proof of Theorem 3.4. Similar to the proof of Theorem 3.3, we only need to prove (4.26). It follows by Markov's inequality, Lemma 2.5 (taking $s \geq p$), Jensen's inequality, the sub-additivity of $\hat{\mathbb{E}}$, (3.5), (3.6) and (3.10) that

$$\begin{aligned}
I_1 &\leq C\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{a_n}\left(\sum_{i=1}^{n}\left(Y_{ni}-\hat{\mathbb{E}}[Y_{ni}]\right)\right)^{+} > \frac{1}{4}\epsilon\right)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^s}\hat{\mathbb{E}}\left[\left(\left(\sum_{i=1}^{n}(Y_{ni}-\hat{\mathbb{E}}[Y_{ni}])\right)^{+}\right)^{s}\right]\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^s}\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left|Y_{ni}-\hat{\mathbb{E}}[Y_{ni}]\right|^{s}\right]+\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(Y_{ni}-\hat{\mathbb{E}}[Y_{ni}]\right)^{2}\right]\right)^{s/2}\right)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^s}\sum_{i=1}^{n}\hat{\mathbb{E}}[|Y_{ni}|^s]+C\sum_{n=1}^{\infty}\frac{1}{a_n^s}\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[Y_{ni}^2\right]\right)^{s/2}\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^s}\sum_{i=1}^{n}\hat{\mathbb{E}}[|X_{ni}|^s I(|X_{ni}|\leq a_n)+a_n^s I(|X_{ni}|>a_n)]\\
&\quad+C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(\frac{X_{ni}}{a_n}\right)^{2}\right]\right)^{s/2}\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^s}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X_{ni}|^s\left(1-f_u\left(\frac{uX_{ni}}{a_n}\right)\right)\right]\\
&\quad+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[f_u\left(\frac{X_{ni}}{a_n}\right)\right]+C\\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\Psi_i(uX_{ni})}{\Psi_i(a_n)}\right]+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\Psi_i(X_{ni})}{\Psi_i(ua_n)}\right]+C<\infty.
\end{aligned}$$

This completes the proof of the theorem. □

Proof of Corollary 3.3. The proof is similar to that of Corollary 3.1, so we omit the details. □

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