East Asian Math. J.
Vol. 36 (2020), No. 1, pp. 073–079
http://dx.doi.org/10.7858/eamj.2020.007

CHARACTERIZING ABELIAN GENERALIZED REGULAR RINGS THAT ARE NOETHERIAN

Juncheol Han and Hyo-Seob Sim*

Abstract. A ring R is called *generalized regular* if for every nonzero x in R there exists y in R such that xy is a nonzero idempotent. In this paper, we observe some equivalent conditions for the generalized regular rings that are abelian in terms of idempotents, and we also investigate the primitivity of an idempotent for such a ring. By using the investigation, we characterize such a kind of rings that are noetherian by showing that an abelian generalized regular ring R is noetherian if and only if R is isomorphic to a direct product of finitely many division rings. We also observe some interesting consequences of our results.

1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise specified. Most of notation and terminology not defined in this paper may be found in, for example, [6].

A ring R is called *generalized regular* if for every nonzero x in R there exists y in R such that xy is a nonzero idempotent. As a generalization of von Neumann regular rings, generalized regular rings are introduced in [2] with some basic properties. A ring R is called *strongly generalized regular* if for every nonzero x in R there exists y in R such that $0 \neq xy = x^2y^2$.

A ring R is called *abelian* if every idempotent of R is central. A ring R is called *abelian generalized regular* if R is abelian and generalized regular. It is known in [2, Theorem 2.5] that a ring R is abelian generalized regular if and only if R is strongly generalized regular.

Let R be a ring. Then R is called *reduced* if R has no nonzero nilpotent elements; R is called *reversible* if $ab = 0$ implies $ba = 0$ for a, b in R; and R is called *symmetric* if $abc = 0$ implies $acb = 0$ for a, b, c in R. In addition, R is called *e-reversible* for an idempotent e of R if $ab = e$ implies $ba = e$ for a, b in R.

Received October 28, 2019; Accepted January 9, 2020.
2010 *Mathematics Subject Classification.* 16E50, 16P40.
Key words and phrases. abelian ring, generalized regular ring, noetherian ring.
This work was supported by a Research Grant of Pukyong National University (2019 year).
*Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

It is easy to see that every reduced ring is reversible. In fact, it follows easily from the fact that $xy = 0$ implies $(yx)^2 = 0$ for x, y in R. As proved in [1, Theorem 1.3], every reduced ring is symmetric as well; a more general observation can be found in [4, Theorem 2.3].

It is obvious that every symmetric ring is reversible. Moreover, it is also easy to see that every reversible ring is abelian. In fact, if R is reversible, for every idempotent e of R, $ex(1-e) = 0$, $(1-e)xe = 0$ for all x in R, and so every idempotent is central.

We shall continue to characterize abelian generalized regular rings in the subsequent section. In the section 3, we study some properties about idempotents of generalized regular rings. In the section 4, we investigate several equivalent conditions to the primitivity of idempotents in abelian generalized regular rings. In the section 4, we shall observe the cases when an abelian generalized regular ring is noetherian. Our observation indeed characterizes such rings, by using the results obtained in the previous section. Finally, we also verify some interesting properties as consequences of the characterization.

2. Abelian generalized regular rings

We continue to investigate several equivalent conditions under which a generalized regular ring is abelian.

By observing for strongly generalized regular ring in [2], they show that if a generalized regular ring R is abelian then R is reduced in [2, Corollary 2.6]. In fact, for every nonzero x in R there exists y in R such that xy is a nonzero idempotent, since R is a generalized regular ring. If R is abelian, since the idempotent xy is central, it follows that $0 \neq xy = x^n y^n$, and so $x^n \neq 0$; thus every nonzero element is not nilpotent. This implies that every abelian generalized regular ring is reduced. Consequently, from our observation in the previous section, we see that the conditions to be reduced, symmetric, reversible and abelian are equivalent each other.

Moreover, it follows from Proposition 2.1 in [3] that a ring R is reversible if and only if R is e-reversible for every idempotent e. We notice that this result was also shown already in [5]. We now summarize our observations so far as follows:

Theorem 2.1. *Let R be a generalized regular ring. The following conditions are equivalent:*

(1) *R is abelian.*
(2) *R is reduced.*
(3) *R is symmetric.*
(4) *R is reversible.*
(5) *R is e-reversible for every idempotent e.*

Moreover, it follows from Proposition 2.3 in [2] that a ring R is abelian generalized regular if and only if every principal nonzero right(left) ideal of R contains a nonzero central idempotent.

We now observe some basic properties about the center of an abelian generalized regular ring as follows:

Proposition 2.2. *The center of an abelian generalized regular ring is abelian generalized regular.*

Proof. Let R be an abelian generalized regular ring, and let Z be the center of R. Let x be a nonzero element in Z. Then there exists y in R such that $e = xy$ is a nonzero idempotent, and so $e = xy = yx$. Let $z = yxy$. Then

$$rz = r(yx)y = y(xr)y = yr(xy) = (yxy)r = zr$$

for all r in R, and so $z \in Z$. Moreover, $xz = xyxy = e$ is a nonzero idempotent. So Z is generalized regular. It is obvious that Z is abelian. $\square$

A nonzero ring R is called *indecomposable* if R can not be a direct sum of two nonzero ideals. It is well known that R is indecomposable if and only if R has no nontrivial central idempotents.

Proposition 2.3. *A nonzero abelian generalized regular ring R is indecomposable if and only if the center of R is a field.*

Proof. Let R be a nonzero abelian generalized regular ring, and let Z be the center of R. Then from Proposition 2.2, Z is also abelian generalized regular. Suppose that R indecomposable. Let x be a nonzero element in Z. The there exists z in Z such that $e = xz$ is a nonzero idempotent of Z. Since R is indecomposable, $xz = 1$. It follows that Z is a field. The converse is clear. $\square$

3. Primitive idempotents of abelian generalized regular rings

In this section, we investigate primitive idempotents of generalized regular rings, especially for such abelian rings.

Lemma 3.1. *Let R be a generalized regular ring, and let e be an idempotent of R. If e is primitive, then e is right(left) irreducible.*

Proof. Suppose that e is primitive. Let I be a nonzero right ideal contained in eR. Since R is generalized regular, there exists a nonzero idempotent f in I. Then $f = ea$ for some a in R, and so $ef = f$. Thus fe and $e - fe$ are orthogonal idempotents such that $e = fe + (e - fe)$. Since e is primitive, either $fe = 0$ or $e = fe$. Assume that $fe = 0$. Then $f \in fI \subseteq feR = \{0\}$, which yields $f = 0$, a contradiction. Hence, $e = fe \in fR \subseteq I$. It follows that $eR \subseteq I$, and so $eR = I$. Therefore, eR is right irreducible module, and so e is a right irreducible. By the same argument, e is left irreducible. $\square$

As an immediate consequence of the above lemma, we have the following theorem.

Lemma 3.2. *Let R be a generalized regular ring, and let e be an idempotent of R. The following conditions are equivalent:*
(1) *e is primitive.*
(2) *e is local.*
(3) *e is right(left) irreducible.*

Proof. First of all, the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are true as known in general. It follows from Lemma 3.1 that (1) implies (3). $\square$

Theorem 3.3. *Let R be an abelian generalized regular ring, and let e be an idempotent of R. The following conditions are equivalent:*
(1) *e is primitive.*
(2) *e is local.*
(3) *e is irreducible.*
(4) $\operatorname{ann}(e)$ *is a maximal ideal of R.*
(5) $\operatorname{ann}(e)$ *is a prime ideal of R.*

Proof. First of all, (1), (2) and (3) are equivalent from Lemma 3.2. Therefore, it suffices to show that (3), (4) and (5) are equivalent. To show that (3) implies (4), assume that e is irreducible. Since e is central, $eR = eRe$, and so eR is a division ring by Schur's lemma. On the other hand, $R/\operatorname{ann}(e) \cong eR$ because $\operatorname{ann}(e)$ is the kernel of a homomorphism from R onto eR, and so $\operatorname{ann}(e)$ is a maximal ideal of R. It is well known that (4) implies (5) in general. To show that (5) implies (3), we finally assume that $\operatorname{ann}(e)$ is a prime ideal of R. Let $e = x + y$ for some orthogonal idempotents x, y in R. Then $xry = rxy = 0 \in \operatorname{ann}(e)$ for all $r \in R$. Since $\operatorname{ann}(e)$ is a prime ideal, either $x \in \operatorname{ann}(e)$ or $y \in \operatorname{ann}(e)$, and so either $x = xe = 0$ or $y = ye = 0$. Thus, e is primitive and hence e is irreducible. $\square$

4. Abelian generalized regular rings that are noetherian

In this section, we shall give a characterization of the abelian generalized regular rings that are noetherian.

Let $\mathrm{E}(R)$ be the set of all idempotents of a ring R. Define a relation $\leq$ on $\mathrm{E}(R)$ by $e \leq f$ if and only if $e = ef = fe$ for every e, f in $\mathrm{E}(R)$. Clearly, $\mathrm{E}(R)$ is a partially ordered set with respect to the relation $\leq$. By $e < f$ we here means both $e \leq f$ and $e \neq f$.

Lemma 4.1. *Let R be a ring, and let e, f be central idempotents in R.*
(1) *$e \leq f$ if and only if $1 - f \leq 1 - e$.*
(2) *$e \leq f$ if and only if* $\operatorname{ann}(f) \subseteq \operatorname{ann}(e)$.

Proof. (1) Clear.

(2) Suppose that $e \leq f$. Then $e = ef = fe$, and so $\operatorname{ann}(f) \subseteq \operatorname{ann}(ef) = \operatorname{ann}(e)$. Conversely, suppose that $\operatorname{ann}(f) \subseteq \operatorname{ann}(e)$. Then $\operatorname{ann}(f) \subseteq \operatorname{ann}(e) \subseteq \operatorname{ann}(ef)$. If $x \in \operatorname{ann}(ef)$ then $xe \in \operatorname{ann}(f) \subseteq \operatorname{ann}(e)$ and so $xe = (xe)e = 0$, which implies that $x \in \operatorname{ann}(e)$. Therefore, $\operatorname{ann}(ef) \subseteq \operatorname{ann}(e)$, and so $\operatorname{ann}(e) =$

ann(ef). Since $1 - ef = 1 - fe \in \text{ann}(e)$, it follows that $e = ef = fe$ and so $e \leq f$. $\square$

Lemma 4.2. *Let R be a ring, and let e be a nonzero idempotent in R. Then e is primitive if and only if there exists no nonzero idempotent f such that $f < e$.*

Proof. Suppose that e is not primitive. Then there exist nonzero orthogonal idempotents f, g such that $e = f + g$. Then it is easy to see that $f < e$. Conversely, if there exists a nonzero idempotent f such that $f < e$, then $e - f$ is a nonzero idempotent such that f and $e - f$ are orthogonal and $e = f + (e - f)$, which implies that e is not primitive. $\square$

Lemma 4.3. *Every distinct primitive central idempotents of a ring are orthogonal.*

Proof. Let e and f be primitive central idempotents in a ring R. Since both e and f are central, $ef, e(1 - f), (1 - e)f$ are idempotents. Thus, both $e = ef + e(1 - f)$ and $f = ef + (1 - e)f$ are sums of orthogonal idempotents. Therefore, either $ef = 0$ or $e = ef = f$. Consequently, either $ef = 0$ or $e = f$. $\square$

A *complete set of primitive idempotents* of a ring R is a finite set of pairwise orthogonal primitive idempotents whose sum is the identity 1.

Theorem 4.4. *Let R be an abelian generalized regular ring. The following conditions are equivalent:*
(1) *R is noetherian.*
(2) *R satisfies the ascending chain condition on annihilators of idempotents.*
(3) *R has a complete set of primitive idempotents.*
(4) *R is isomorphic to a direct product of finitely many division rings.*
(5) *R is semisimple.*
(6) *R is artinian.*

Proof. First of all, (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1) $\Rightarrow$ (2) are true as known in general.

To show (2) $\Rightarrow$ (3), let $e_1 \leq e_2 \leq \cdots \leq e_i \leq \cdots$ be a maximal chain in $\mathrm{E}(R)$. Since 0 is the least element and 1 is the greatest element of $\mathrm{E}(R)$, the maximal chain should contain 0 and 1, and so

$$0 = e_1 \leq e_2 \leq \cdots \leq e_i \leq \cdots \leq 1.$$

Therefore, from Lemma 4.1, we have an ascending chain of annihilators:

$$\langle 0 \rangle = \text{ann}\,(1 - e_1) \subseteq \text{ann}\,(1 - e_2) \subseteq \cdots \subseteq \text{ann}\,(1 - e_i) \subseteq \cdots \subseteq \text{ann}\,(0) = R.$$

It follow from the assumption (2) that there exists a positive integer n such that

$$\text{ann}\,(1 - e_{n+1}) = \text{ann}\,(1 - e_i) = R,$$

and so $e_{n+1} = e_i = 1$ for all $i > n$. Now, without loss of generality, we may assume that $0 = e_1 < e_2 < \cdots < e_{n+1} = 1$. Denote $a_i := e_{i+1} - e_i$ for each $i = 1, 2, ..., n$. We then wan to show that $a_1, ..., a_n$ are primitive idempotents such that $a_1+\cdots+a_n = 1$. In fact, for every $i = 1, 2, ..., n$, each is an idempotent, since $e_i < e_j$, that is, $e_i = e_ie_j = e_je_i$ for all $j > i$. Suppose that a_k is not primitive for some $k = 1, 2, ..., n$. Then, from Lemma 4.2, there exists a nonzero idempotent b_k such that $b_k < a_k$. Therefore, it follows from $b_k < e_{k+1}$ and $b_ke_k = 0$ that $b_k + e_k$ is an idempotent such that $e_k < e_k + b_k < e_{k+1}$; this yields a contradiction, since $0 = e_1 < e_2 < \cdots < e_{n+1} = 1$ is a maximal chain in $\mathrm{E}(R)$. Consequently, $a_1, ..., a_n$ are primitive such that $a_1 + \cdots + a_n = 1$. Moreover, from Lemma 4.3, $a_1, ..., a_n$ are pairwise orthogonal, and so R has a complete set of primitive idempotents.

To show (3) $\Rightarrow$ (4), let $a_1 + \cdots + a_n = 1$ be a decomposition of pairwise orthogonal primitive idempotents in R. By Theorem 3.3, each e_i is irreducible central idempotent, and so $e_iR = e_iRe_i$ is a division ring by Schur's lemma. Therefore, R is isomorphic to the direct product of finitely many division rings $e_1R, ..., e_nR$. $\square$

We have an immediate consequence of Theorem 4.4, as follows:

Corollary 4.5. *Let R be an abelian ring. The followings are equivalent:*

(1) *R is von Neumann regular and noetherian.*

(2) *R is generalized regular and noetherian.*

(3) *R is semisimple.*

Finally, we observe some interesting consequences of Theorem 4.4.

Corollary 4.6. *Let R be an abelian generalized regular ring.*

(1) $\mathrm{E}(R)$ *is finite if and only if R is noetherian.*

(2) *R is finite if and only if R is isomorphic to a direct product of finitely many finite fields.*

Proof. (1) If $\mathrm{E}(R)$ is finite, then R satisfies the ascending chain condition on annihilators of idempotents. It follows from Theorem 4.4 that R is isomorphic to a direct product of finitely many division rings, and equivalently, R is noetherian. The converse is clear, since R is isomorphic to a direct product of finitely many division rings.

(2) It follows immediately from (1). $\square$

We recall that a prime ideal P of a ring R is called an *associated prime* if $P = \mathrm{ann}(x)$ for some x in R. It is known that if R is noetherian, there exist only finitely many associated primes.

Lemma 4.7. *Let R be a symmetric ring, and $\Lambda := \{\mathrm{ann}(x) : 0 \neq x \in R\}$. If $P = \mathrm{ann}(x)$ is maximal as subsets in Λ, then P is a prime ideal of R.*

Proof. Suppose that $P = \mathrm{ann}(x)$ is maximal in Λ. Then since R is reversible, P is an ideal of R. Since $x \neq 0$, it follows that $P \neq R$. Let a, b be elements in R such that $bc \in P$. Assume that $c \notin P$. Then $cx \neq 0$ and $b \in \mathrm{ann}(cx)$. Thus

$P \subseteq \operatorname{ann}(cx)$ because R is symmetric. It follows that $P = \operatorname{ann}(cx)$ since P is maximal in Λ. Therefore, $b \in P$. Consequently, P is a prime ideal of R. $\square$

We then have a consequence of Lemma 4.7 as follows:

Corollary 4.8. *Let R be an abelian generalized regular ring. If R is noetherian, then*

(1) *there exists at least one associated prime in R;*

(2) *a is a zero-divisor of R if and only if a is contained in some associated prime in R.*

Proof. (1) It follows from Theorem 4.4 that R has at least one primitive idempotent e if R is noetherian, and so $\operatorname{ann}(e)$ is a prime ideal of R by Theorem 3.3.

(2) Suppose that R is noetherian, and let a be a nonzero zero-divisor. Note that $\operatorname{ann}(x)$ is an ideal of R for every x in R since R is reversible. Then R satisfies the ascending chain condition on annihilators of nonzero elements. Therefore, there exists a maximal element $P = \operatorname{ann}(x)$ of Λ such that $a \in P$. From Lemma 4.7, P is an associated prime containing a. The converse is clear. $\square$

References

[1] D. D. Anderson and V. Camillo, *Semigroups and rings whose zero products commute*, Comm. in Algebra **27** (1999), 2847–2852.

[2] J. Han, Y. Lee and S. Park, *On idempotents in relation with regularity*, J. Korean Math. Soc. **53** (2016), 217–232.

[3] J. Han and H. Sim, *Rings with the symmetric property for idempotent-products*, East Asian Math. J. **51** (2018), 615–621.

[4] C. Huh, H. Kim, N. Kim and Y. Lee, *Basic examples and extensions of symmetric rings*, J. Pure and Applied Algebra **202** (2005), 154–167.

[5] D. Jung, N. Kim, Y. Lee and S. Ryu, *On properties related to reversible rings*, Bull. Korean Math. Soc. **52** (2015), 247–261.

[6] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, 1991.

Juncheol Han
Department of Mathematics Education,
Pusan National University, Pusan 46277, Korea
E-mail address: `jchan@pusan.ac.kr`

Hyo-Seob Sim
Department of Applied Mathematics,
Pukyong National University, Pusan 48547, Korea
E-mail address: `hsim@pknu.ac.kr`