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BINARY TRUNCATED MOMENT PROBLEMS AND THE HADAMARD PRODUCT

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Abstract. Up to the present day, the best solution we can get to the truncated moment problem (TMP) is probably the Flat Extension Theorem. It says that if the corresponding moment matrix of a moment sequence admits a rank-preserving positive extension, then the sequence has a representing measure. However, constructing a flat extension for most higher-order moment sequences cannot be executed easily because it requires to allow many parameters. Recently, the author has considered various decompositions of a moment matrix to find a solution to TMP instead of an extension. Using a new approach with the Hadamard product, the author would like to introduce more techniques related to moment matrix decompositions.

1. Introduction

A *truncated moment sequence* is a d-dimensional multisequence of the form, $\beta \equiv \beta^{(m)} = \left\{\beta_{\mathbf{i}} \in \mathbb{R} : \mathbf{i} \in \mathbb{Z}_+^d,\ |\mathbf{i}| \leq m\right\}$, with $\beta_{\mathbf{0}} \neq 0$. The number m is said to be the *degree* of β and d is the *dimension* of the sequence. For a closed set K in $\mathbb{R}^d$, the *truncated K-moment problem* (TKMP) aims to find necessary and sufficient conditions for the existence of a positive Borel measure μ supported on K such that

$$\beta_{\mathbf{i}} = \int_K \mathbf{x}^{\mathbf{i}}\, d\mu(\mathbf{x}) \ \ (\mathbf{i} \in \mathbb{Z}_+^d,\ |\mathbf{i}| \leq m), \tag{1.1}$$

where $\mathbf{x} \equiv (x_1, \ldots, x_d)$, $\mathbf{i} \equiv (i_1, \ldots, i_d) \in \mathbb{Z}_+^d$, and $\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_d^{i_d}$. In this case, μ is called a *K-representing measure* for β. For the case $K = \mathbb{R}^d$, the problem is just referred to as the *truncated real moment problem* (TRMP) and μ is said to be a *representing measure*.

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In a similar manner, the full moment problem for an infinite sequence $s = \left\{s_{\mathbf{i}} : \mathbf{i} \in \mathbb{Z}_+^d\right\}$ can be defined. As well known for $d = 1$, the sequence has a representing measure supported on $\mathbb{R}$ if and only if the Hankel matrix $H_0 := [s_{i+j}]_{0 \le i,\, j \le k}$ is positive semidefinite for all $k \ge 0$. In addition, if $H_1 := [s_{i+j+1}]_{0 \le i,\, j \le k}$ is also positive semidefinite for all $k \ge 0$, then the sequence admits a representing measure supported on $[0, \infty)$ [16]. When $K = \mathbb{R}$ (respectively, $K = [0, \infty)$, $K = [a, b]$), the sequence s is also called a *Hamburger*(respectively, *Stieltjes, Hausdorff*) *moment sequence.* A moment sequence is said to be *determinate*, if there is a unique representing measure satisfying (1.1); otherwise, it is said to be *indeterminate*.

When $m = 2n$, we define the *moment matrix* $M_d(n)$ of $\beta \equiv \beta^{(2n)}$ as

$$M_d(n) \equiv M_d(n)(\beta) := (\beta_{\ \mathbf{i}+\mathbf{j}})_{\ \mathbf{i},\,\mathbf{j} \in \mathbb{Z}_+^d;\ |\mathbf{i}|,\,|\mathbf{j}| \le n}.$$

Algebraic properties of $M_d(n)$ have played an important role for the existence of a representing measure for β; for example, $M_d(n)$ is necessarily positive semidefinite. However, different from the full moment problem, the positive semidefiniteness of $M_d(n)$ is not sufficient for $d \ge 2$.

R. Curto and L. Fialkow have established many elegant results for various moment problems based on a positive extension of $M_d(n)$. They also have used the functional calculus in the column space of $M_d(n)$; to introduce the functional calculus, we label the columns and rows of $M_d(n)$ with monomials $X^{\mathbf{i}} := X_1^{i_1} \cdots X_d^{i_d}$ in the degree-lexicographic order. Note that each block with the moments of the same order in $M_d(n)$ is Hankel and that $M_d(n)$ is symmetric. In addition, one can define a sesquilinear form: For $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^d$,

$$\langle X^{\mathbf{i}}, X^{\mathbf{j}} \rangle_{M_d(n)} := \langle M_d(n) \widehat{X^{\mathbf{i}}}, \widehat{X^{\mathbf{j}}} \rangle = \beta_{\mathbf{i}+\mathbf{j}},$$

where $\widehat{X^{\mathbf{i}}}$ denotes the column vector associated to the monomial $X^{\mathbf{i}}$.

The key ideas in the main results of this note can be applied to TRMP of any dimension, but we only focus on the bivariate truncated moment problem, which is the case when $d = 2$. From now on we simply denote $M_2(n)$ as $M(n)$ and its columns are labeled as $\mathit{1}, X, Y, X^2, XY, Y^2, \ldots, X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n$.

1.1. Necessary Conditions

First, when μ is a representing measure for β, we compute for $p(x, y) \in \mathbb{R}[x, y]$

$$0 \le \int p(x, y)^2 \, d\mu = \sum_{i,j,k,l} a_{ij} a_{kl} \int x^{i+k} y^{j+l} \, d\mu = \sum_{i,j,k,l} a_{ij} a_{kl} \beta_{i+k, j+l},$$

which is equivalent to the positive semidefiniteness of $M(n)$; that is, the most basic necessary condition for the existence of a representing measure for β is that $M(n)$ is positive semidefinite.

Second, we define an assignment from $\mathcal{P}_n \equiv \mathcal{P}_n[x, y]$, the set of all polynomials of degree at most n, to $\mathcal{C}_{M(n)}$, the column space of $M(n)$; a polynomial $p(x, y) \equiv \sum_{ij} a_{ij} x^i y^j$ is mapped to $p(X, Y) := \sum_{ij} a_{ij} X^i Y^j$. This mapping is the above-mentioned functional calculus. We then let $\mathcal{Z}(p)$ be the zero set of p and define an algebraic set:

$$\mathcal{V} \equiv \mathcal{V}(\beta) \equiv \mathcal{V}(M(n)) := \bigcap_{p(X,Y)=\mathbf{0},\, \deg\, p \leq n} \mathcal{Z}(p), \tag{1.2}$$

which is called the *algebraic variety* of β or $M(n)$. If $\widehat{p}$ denotes the column vector of coefficients of a polynomial p, then we can easily check that $p(X,Y) = M(n)\widehat{p}$, that is, $p(X,Y) = \mathbf{0}$ if and only if $\widehat{p} \in \ker M(n)$. It is also known that the existence of a representing measure for β requires the conditions, supp $\mu \subseteq \mathcal{V}(\beta)$ and $r :=$ rank $M(n) \leq$ card supp $\mu \leq v :=$ card $\mathcal{V}$; the later is referred to as the *variety condition* [3].

Finally, the *Riesz functional* is a real-valued map on $\mathcal{P} \equiv \mathcal{P}[x,y]$, the set of all polynomials, defined by $\Lambda\left(\sum_{ij} a_{ij}x^i y^j\right) = \sum_{ij} a_{ij}\beta_{ij}$. If p is any polynomial of degree at most $2n$ such that $p|_{\mathcal{V}} \equiv 0$ and if μ is a representing measure for β, then the *Riesz functional* Λ must satisfy $\Lambda(p) = \int p\, d\mu = 0$. This property is referred to as *consistency* of the moment sequence. When $r = v$, β or $M(n)$ is said to be *extremal*. The consistency is also sufficient for the extremal problems [7]. In addition, when $M(n)$ satisfies that

$$p(X,Y) = \mathbf{0} \Longrightarrow (p\,q)(X,Y) = \mathbf{0} \quad \text{for each } q \in \mathcal{P}_n \text{ with } \deg(p\,q) \leq n,$$

β or $M(n)$ is said to be *recursively generated*. Note that the recursively generatedness is a weaker condition than the consistency.

For solutions of the quadratic ($n = 1$) and quartic ($n = 2$) moment problems, the positive semidefiniteness, the recursively generatedness, and the variety condition were sufficient (see [3], [6], [11], [13]); that is, complete solutions had been established. For $M(n)$ with $n \geq 3$, the moment problems get more sophisticated; many instances require an additional condition such as numerical conditions involving moments as seen in [9], [10], and [12].

1.2. Flat Extension

The Flat Extension Theorem states that if $M(n)$ admits a rank-preserving positive semidefinite extension $M(n+1)$, then β has a rank $M(n)$-atomic measure [3]. In this case, an extension $M(n+1)$ is called a *flat extension*. This result seems to be the most general solution to truncated moment problems up to date, but the construction of an extension is not handy for many cases when $n \geq 3$.

We briefly summarize the way to build a flat extension. Observe that each rectangular block with the same order moments of $M(n)$ is Hankel, and that an extension $M(n+1)$ can be written as $M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$, for some matrices B and C. To make sure a prospective moment matrix $M(n+1)$ is positive semidefinite, we use the following classical result:

Theorem 1.1. *(Smul'jan's Theorem* [17]*) Let A, B, C be matrices of complex numbers, with A and C square matrices. Then*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff \begin{cases} A \geq 0 \\ B = AW \text{ (for some } W) \\ C \geq W^*AW \end{cases}.$$

Moreover, $\operatorname{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A$ *if and only if* $C = W^*AW$.

Remark 1. If the equality about the rank holds in Theorem 1.1, we may write $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \equiv \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix}$, which is a flat extension of A. It looks easy to construct a flat extension but to keep C-block being Hankel is an extremely nontrivial process. In other words, it is quite difficult to maintain the positive semidefiniteness and the moment matrix structure of $M(n+1)$ at the same time.

An important contribution of the Flat Extension Theorem is that it enables us to find an explicit formula of a representing measure. An extended version of the Flat Extension Theorem tells us that if $M(n)$ admits a positive extension $M(n+k)$ for some $k \in \mathbb{Z}_+$ that has a flat extension $M(n+k+1)$, then β has a rank $M(n+k)$-atomic measure, namely μ [5]. According to this result, we know the algebraic variety $\mathcal{V}(M(n+k))$ consists of exactly $\tau := \operatorname{rank} M(n+k)$ points. If we write $\mathcal{V}(M(n+k)) = \{(x_1, y_1), \ldots, (x_\tau, y_\tau)\}$, then the Vandermonde matrix V is written as

$$V = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & \cdots & x_1^n & \cdots & y_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & x_\tau & y_\tau & x_\tau^2 & x_\tau y_\tau & y_\tau^2 & \cdots & x_\tau^n & \cdots & y_\tau^n \end{pmatrix}. \tag{1.3}$$

Suppose that $\mathcal{B} := \{\mathbf{t}_1, \ldots, \mathbf{t}_\tau\}$ is the basis for the column space of $M(n+k)$ and that $V_{\mathcal{B}}$ is the submatrix of V with columns selected from $\mathcal{B}$. Then we can compute the densities of the representing measure by solving the matrix equation:

$$V_{\mathcal{B}}^T \left(\rho_1, \rho_2, \ldots, \rho_\tau\right)^T = \left(\Lambda(\mathbf{t}_1), \Lambda(\mathbf{t}_2), \ldots, \Lambda(\mathbf{t}_\tau)\right)^T \tag{1.4}$$

Thus, we can write $\mu = \sum_{k=1}^{\tau} \rho_k \delta_{(x_k, y_k)}$, where δ denotes the point mass.

2. The Hadamard Product

The Hadamard product is an entrywise product of two matrices of the same dimension. We begin with setting up some notations which will be used in the sequel:

- When a Hermitian matrix A is positive semidefinite (respectively, positive definite), we denote $A \geq 0$ (respectively, $A > 0$).

- $P(a,b) := \mathbf{v}(a,b)^T\mathbf{v}(a,b)$, where $\mathbf{v}(a,b) := \big(1, a, b, \cdots, a^n, a^{k-1}b^k, \cdots, ab^{n-1}, b^n$ is a row vector. For example, when $n = 2$,

$$P(a,b) = \begin{pmatrix} 1 & a & b & a^2 & ab & b^2 \\ a & a^2 & ab & a^3 & a^2b & ab^2 \\ b & ab & b^2 & a^2b & ab^2 & b^3 \\ a^2 & a^3 & a^2b & a^4 & a^3b & a^2b^2 \\ ab & a^2b & ab^2 & a^3b & a^2b^2 & ab^3 \\ b^2 & ab^2 & b^3 & a^2b^2 & ab^3 & b^4 \end{pmatrix}.$$

  Note that $P(a,b)$ is a rank-one moment matrix corresponding to the point mass $\delta_{(a,b)}$. Since it has a representing measure, it is clearly positive semidefinite.

One can immediately observe the following: For square matrices A and B, $A\circ B = AB$ if and only if both A and B are diagonal. In general, $P(a,b)P(c,d) \neq P(c,d)P(a,b)$ but obviously, $P(a,b) \circ P(c,d) = P(c,d) \circ P(a,b) = P(ac, bd)$.

By the result of C. Bayer and J. Teichmann, once $M(n)$ admits at lease one representing measure, then one of them must be finitely atomic. We naturally should be able to write

$$M(n) = \sum_{k=1}^{\ell} \rho_k P(x_k, y_k),$$

where $\rho_k > 0$ and $(x_k, y_k) \in \mathbb{R}^2$ for $k = 1, \ldots, \ell \leq \dim \mathcal{P}_{2n}$; that is, $\beta_{i,j} = \sum_{k=1}^{\ell} \rho_k x_k^i y_k^j$. Since the rank-one moment matrix $P(a,b)$ is generated by $\delta_{(a,b)}$, in the presence of a representing measure for $M(n)$, we can write

$$M(n) \circ P(a,b) = \sum_{k=1}^{\ell} \rho_k P(ax_k, by_k).$$

In particular, $M(n) \circ P(1,0)$ has nonzero entries only at the places corresponding to monomials $1, x, x^2, \ldots, x^{2n}$; similarly, all nonzero moments of $M(n) \circ P(0,1)$ correspond to monomials $1, y, y^2, \ldots, y^{2n}$. Indeed $M(n) \circ P(1,0)$ and $M(n) \circ P(0,1)$ can be considered as a projection of bivariate moment problems to univariate moment problems.

If $M(n)$ admits a finitely atomic representing measure $\mu = \sum_{k=1}^{\ell} \rho_k \delta_{(x_k, y_k)}$, then $\beta_{i,0} = \sum_{k=1}^{\ell} \rho_k x_k^i$ for $i = 0, 1, \ldots, 2n$ and $\beta_{0,j} = \sum_{k=1}^{\ell} \rho_k y_k^j$ for $j = 0, 1, \ldots, 2n$. We may consider using the disintegration of measures to solve TRMP [15]. It is much easier to solve a univariate moment problem, so we find proper representing measures for $M(n) \circ P(1,0)$ and $M(n) \circ P(0,1)$, respectively and try to see if a disintegration of two measures would represent $M(n)$. Also, for some $(a,b) \neq (0,0)$, it might be easier to find a representing measure for $M(n) \circ P(a,b)$ since it may have a simpler structure than that of $M(n)$.

To investigate a bound of rank of the Hadamard product of moment matrices, we review some auxiliary results.

Proposition 2.1. [14]*A is positive semidefinite if and only if $A \circ B \geq 0$ for all $B \geq 0$.*

Thus, we know that $M(n) \circ P(a,b) \geq 0$ for any $(a,b) \in \mathbb{R}^2$.

Proposition 2.2. [19]*Suppose $A \geq 0$ and $B \geq 0$. Then*

$$\text{rank } (A \circ B) \leq (\text{rank } A)(\text{rank } B).$$

Moreover, if $A > 0$, then rank $(A \circ B)$ *is equal to the number of nonzero diagonal entries of B.*

This proposition leads us to the fact that rank $(M(n) \circ P(a,b)) \leq$ rank $M(n)$ for any $(a,b) \in \mathbb{R}^2$; that is, a Hadamard product of a moment matrix by a rank-one moment matrix will not increase the rank.

Proposition 2.3. [14]*Suppose $A > 0$ and $B \geq 0$. Let $\nu(B)$ be the number of nonzero main diagonal entries of B. Then* rank $B \leq \nu(B) \leq$ rank $(A \circ B)$.

Remark 2. If $M(n) > 0$ and $ab \neq 0$, then by Proposition 2.3,

$$\frac{(n+1)(n+2)}{2} = \nu(P(a,b)) \leq \text{rank } (M(n) \circ P(a,b)) \leq \text{rank } M(n) = \frac{(n+1)(n+2)}{2},$$

that is, rank $(M(n) \circ P(a,b)) = \frac{(n+1)(n+2)}{2}$. Thus the Hadamard product by $P(a,b)$ to $M(n)$ cannot reduce the rank of $M(n)$.

Suppose the eigenvalues of an $n \times n$ matrix A are arranged as $\lambda_n(A) \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_1(A)$. Then A is positive semidefinite if and only if $\lambda_n(A) \geq 0$. The rank-one decomposition method has been useful to handle TMP; we have to maintain the positivity of $M(n)$ after a Hadamard product or a perturbation by a rank-one matrix. Through the following results, we know how to control the minimum eigenvalue.

Proposition 2.4. [14]*Suppose A and $B = [b_{ij}]$ are positive semidefinite. Then the following hold:*

(i) $\lambda_{\min}(A \circ B) \geq \lambda_{\min}(A) \min\{b_{ii}\}$
(ii) $\lambda_{\max}(A \circ B) \leq \lambda_{\max}(A) \max\{b_{ii}\}$

If $A = M(n) > 0$ and $B = P(a,b)$ with $ab \neq 0$, then it follows from Proposition 2.4 (i) that

$$\lambda_{\min}(A \circ B) \geq \lambda_{\min}(A) \min\{b_{ii}\} > 0.$$

This observation is exactly relevant to Remark 2 and implies that the rank-reduction method introduced in [10] via the Hadmard product is not applicable for a positive definite $M(n)$.

3. Truncated Moment Problems via the Hadamard Product

For the two real moment sequences $\sigma \equiv \sigma^{(2n)} = \{s_{ij}\}$ and $\tau \equiv \tau^{(2n)} = \{t_{ij}\}$, we denote $\sigma \circ \tau$ whose moments are given by $\{s_{ij}t_{ij}\}$, that is, the moment metrix of $\sigma \circ \tau$ is just as $M(\sigma) \circ M(\tau)$. It is reasonable enough to call $\sigma \circ \tau$ the *Hadamard product* of the two sequences σ and τ.

Proposition 3.1. *If moment sequences σ and τ admit representing measures, then so does $\sigma \circ \tau$.*

Proof. Since σ and τ admit representing measures, we may write

$$M(\sigma) = \sum_{k=1}^{m} \rho_k P(a_k, b_k), \qquad M(\tau) = \sum_{\ell=1}^{r} \kappa_\ell P(c_\ell, d_\ell),$$

where $\rho_1, \ldots, \rho_m > 0$ and $\kappa_1, \ldots, \kappa_r > 0$. Thus, we get

$$M(\sigma) \circ M(\tau) = \sum_{k,\ell} \rho_k \kappa_\ell P(a_k c_\ell, b_k d_\ell),$$

which implies that a representing measure for $\sigma \circ \tau$ is $\sum_{k,\ell} \rho_k \kappa_\ell \delta_{(a_k c_\ell, b_k d_\ell)}$. □

Remark 3. Even though two moment sequences do not have a representing measure, the Hadamard product of them may have a representing measure. For example, consider

$$M(\sigma^{(2)}) = M(\tau^{(2)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the converse of Proposition 3.1 is not true.

Proposition 3.2. *A moment matrix σ admits a representing measure if and only if there are some moment matrices σ_1 and σ_2 having a representing measure respectively such that $M(\sigma) = M(\sigma_1) \circ M(\sigma_2)$.*

Proof. Assume that σ has a representing measure. Note that $P(1,1)$ is a rank-one moment matrix whose all the entries are equal to 1, so it is obvious that $P(1,1)$ is the identity element of the Hadamard product. Now, we may take σ_1 as σ and σ_2 as the one corresponding to $P(1,1)$, that is, $M(\sigma) = M(\sigma_1) \circ P(1,1)$. The converse is immediate from Proposition 3.1. □

Proposition 3.2 states that a moment matrix σ with a representing measure has a trivial Hadamard product. Thus, it is natural to ask that when $M(\sigma)$ may have a nontrivial Hadamard product, that is,

> If $M(\sigma)$ admit a representing measure, then are there moment matrices $M(\sigma_1)$ and $M(\sigma_2)$ (both different from $P(1,1)$) having a representing measure respectively such that $M(\sigma) = M(\sigma_1) \circ M(\sigma_2)$?

Note that the answer to the question seems to be negative for the case when a moment sequence σ has a unique r-atomic representing measure for a prime number r. For, if $\sigma = \sigma_1 \circ \sigma_2$, then by an observation about the cardinality of the corresponding measures, we can conclude that $\sigma = \sigma_1$ or $\sigma = \sigma_2$.

For a Stieltjes moment sequence $\{s_i\}$, it is well known that $\{s_i^m\}$ is also a Stieltjes moment sequence for $m \in \mathbb{N}$ (Proposition 3.2 is an analogue of this result for bivariate moment problems) but not necessary for a non-integer m. In particular, $\left\{s_i^{-1}\right\}$ is not a Stieltjes moment sequence; for a concrete example, we may consider $\{s_i\} = \left\{1^i + 2^i\right\}$. Through the next proposition, we can see that a similar phenomenon happens for bivariate moment problems.

Proposition 3.3. [14]*Suppose $A = [a_{ij}]$ is positive semidefinite and $a_{ij} \neq 0$ for all i, j. Then $A^{(-1)} := \left[a_{ij}^{-1}\right]$ is positive semidefinite if and only if* rank $A = 1$.

Suppose that rank $M(n) \geq 2$ and that $M(n)$ with no zero moments admits a representing measure. Proposition 3.3 says that $M(n)^{(-1)}$ cannot have a representing measure. The reason is that unless rank $M(n) = 1$, $M(n)^{(-1)}$ cannot be positive semidefinite. The only moment matrix $M(n)$ whose $M(n)^{(-1)}$ can have a representing is the case when $M(n) = P(a, b)$ for some $ab \neq 0$.

If all the odd-degree moments are zero, then the moment sequence and its moment matrix are said to be *symmetric*.

Theorem 3.4. *The moment matrix $S(n) := M(n) \circ \frac{1}{2}[P(1,1) + P(-1,-1)]$ is symmetric. In addition, if $M(n)$ admits a representing measure, then so does $S(n)$.*

Proof. Just observe that the matrix $\frac{1}{2}[P(1,1) + P(-1,-1)]$ captures only even-degree moments of $M(n)$ and leaves all the odd-degree moments as zero. $\square$

Thus, an approach to solving symmetric moment problems would be changing the odd-degree moments properly and making sure the new moment sequence has a representing measure.

Remark 4. The contrapositive of the above theorem seems to be more important; for $M(n)$ to have a representing measure, it is essential that $S(n)$ have a representing measure.

Remark 5. The converse of Theorem 3.4 is not true; consider

$$M(1) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that the second Hankel determinant is negative, and so it does not admit a representing measure. However, $S(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is positive definite and clearly it has infinitely many representing measures.

Lemma 3.5. *Let* $p(x,y) = \sum_{i,\,j\in\mathbb{Z}^+;\,0\le i+j\le n} c_{ij}x^iy^j$*. If* $M(n)$ *has a column (dependent) relation* $p(X,Y) = \mathbf{0}$*, then* $\widetilde{M} := M(n)\circ P(a,b)$ *with* $ab \ne 0$ *has the column relation* $\sum_{i,\,j\in\mathbb{Z}^+;\,0\le i+j\le n} c_{ij}a^{-i}b^{-j}\widetilde{X}^i\widetilde{Y}^j = \mathbf{0}$*, where* $\widetilde{X}^i\widetilde{Y}^j$ *denotes a column in* $\widetilde{M}$*.*

Proof. Observe that the identity $\widetilde{X}^i\widetilde{Y}^j = a^ib^jX^iY^j \circ (1,a,b,a^n,\dots,b^n)^T$ holds for $i,\,j\in\mathbb{Z}^+;\ 0\le i+j\le n$. Thus, we get

$$p(X,Y) = \sum_{i,\,j\in\mathbb{Z}^+;\,0\le i+j\le n} c_{ij}a^{-i}b^{-j}\widetilde{X}^i\widetilde{Y}^j \circ \left(1,a^{-1},b^{-1},a^{-n},\dots,b^{-n}\right)^T = \mathbf{0},$$

which is equivalent to the desired identity. □

Example 3.6. *If* $M(n)$ *has a column relation* $Y+X^2 = Y^2$*, then* $M(n)\circ P(a,b)$ *with* $ab\ne 0$ *has the column relation* $a^2b\widetilde{Y} + b^2\widetilde{X}^2 = a^2\widetilde{Y}^2$ *and* rank $M(n) =$ rank $M(n)\circ P(a,b)$*.*

A consequence of Lemma 3.5 is that rank $M(n) =$ rank $M(n)\circ P(a,b)$ unless $ab = 0$. As for the case of a positive definite $M(n)$, the Hadamard multiplication by $P(a,b)$ to a singular $M(n)$ will not reduce the rank of $M(n)$ unless $ab=0$. Thus, we should find a suitable way to make $M(n)\circ P(a,b)$ feasible rather than reducing the rank of $M(n)$.

We conclude this note with a final observation. Column relations in $M(n)$ have important information to solve truncated moment problems; for example, they have something to do with the support of a representing measure and they need to satisfy the variety condition for the existence of a representing measure. Thus a singular moment problem is much easier to handle. However, when $M(n)$ is positive definite, $M(n)$ has no column relation and so it is difficult to collection information about its solution. When $n = 1$ and $n = 2$, it is known that $M(n)$ has infinitely many representing measure. However, there is an example of a positive definite $M(3)$ with no representing measure (See [4]). Here, we can consider an algorithm to determine if a positive definite $M(n)$ has a representing measure as follows: When a positive definite $M(n)$ admits a representing measure, its measure has at least one atom which is neither on the x-axis nor y-axis (otherwise, $M(n)$ has a column relation $XY = \mathbf{0}$.) Hence, we may write with such an atom $(a_0,b_0)\ne(0,0)$,

$$M(n) = \sum_{k=1}^{\ell} \rho_k P(a_k,b_k) + \rho_0 P(a_0,b_0),$$

where $\ell<\infty$. The Hadamard product by $P(1/a_0,1/b_0)$ will give:

$$M(n)\circ P(1/a_0,1/b_0) = \sum_{k=1}^{\ell} \rho_k P(a_k/a_0,b_k/b_0) + \rho_0 P(1,1). \tag{3.1}$$

Rearranging the terms in (3.1), we get

$$\widetilde{M} := \sum_{k=1}^{\ell} \rho_k P(a_k/a_0, b_k/b_0) = M(n) \circ P(1/a_0, 1/b_0) - \rho_0 P(1,1).$$

Since $\widetilde{M}$ has a representing measure, it must be positive semidefinite. Our test will be seeing if there are parameters ρ_0, a_0, and b_0 such that $\widetilde{M}$ remains to be positive semidefinite. We actually have found a necessary condition for the existence of a representing measure for a moment sequence and may summarize it as follows:

Theorem 3.7. *Let $M(n)(\beta)$ be a positive definite moment matrix for β. If β admits a representing measure, then there are real numbers ρ_0, a_0, and b_0 such that $M(n) \circ P(1/a_0, 1/b_0) - \rho_0 P(1,1)$ is positive semidefinite.*

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