

A NOTE ON (p, q) -ANALOGUE TYPE OF FROBENIUS-GENOCCHI NUMBERS AND POLYNOMIALS

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ABSTRACT. The main purpose of this paper is to introduce Apostol type (p, q) -Frobenius-Genocchi numbers and polynomials of order α and investigate some basic identities and properties for these polynomials and numbers including addition theorems, difference equations, derivative properties, recurrence relations. We also obtain integral representations, implicit and explicit formulas and relations for these polynomials and numbers. Furthermore, we consider some relationships for Apostol type (p, q) -Frobenius-Genocchi polynomials of order α associated with (p, q) -Apostol Bernoulli polynomials, (p, q) -Apostol Euler polynomials and (p, q) -Apostol Genocchi polynomials.

1. Introduction

The (p, q) -numbers are defined as:

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}.$$

We can write easily that $[n]_{p,q} = p^{n-1}[n]_{q/p}$, where $[n]_{q/p}$ is the q -number in q -calculus given by $[n]_{q/p} = \frac{(q/p)^n - 1}{(q/p) - 1}$. Thereby this implies that (p, q) -numbers and q -numbers are different, that is, we cannot obtain (p, q) -numbers just by substituting q by q/p in the definition of q -numbers. In the case of $p = 1$, (p, q) -numbers reduce to q -numbers, (see [2], [3]).

The (p, q) -derivative of a function f with respect to x is defined by

$$D_{p,q}f(x) = D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, (x \neq 0) \quad (1.1)$$

and $(D_{p,q}f(0)) = f'(0)$, provided that f is differentiable at 0. The number (p, q) -derivative operator holds the following properties

$$D_{p,q}(f(x)g(x)) = g(p(x))D_{p,q}f(x) + f(qx)D_{p,q}g(x), \quad (1.2)$$

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and

$$D_{p,q} \left(\frac{f(x)}{g(x)} \right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}. \quad (1.3)$$

The (p, q) -analogue of $(x + a)^n$ is given by

$$\begin{aligned} (x + a)_{p,q}^n &= (x + a)(px + aq) \cdots (p^{n-2}x + aq^{n-2})(p^{n-1}x + aq^{n-1}), n \geq 1 \\ &= \sum_{k=0}^n \binom{n}{k}_{p,q} p^{\binom{n}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}, \end{aligned} \quad (1.4)$$

where the (p, q) -Gauss binomial coefficients $\binom{n}{k}_{p,q}$ and (p, q) -factorial $[n]_{p,q}!$ are defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} (n \geq k) \text{ and } [n]_{p,q}! = [n]_{p,q} \cdots [2]_{p,q}[1]_{p,q}, (n \in \mathbb{N}).$$

The (p, q) -exponential functions are defined by

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} x^n}{[n]_{p,q}!} \text{ and } E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{[n]_{p,q}!} \quad (1.5)$$

hold the identities

$$e_{p,q}(x)E_{p,q}(-x) = 1 \text{ and } e_{p-q}(x) = E_{p,q}(x), \quad (1.6)$$

and have the (p, q) -derivatives

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \text{ and } D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \quad (1.7)$$

The definition (p, q) -integral is defined by

$$\int_0^a f(x)d_{p,q}x = (p - q)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(a \frac{p^k}{q^{k+1}}\right),$$

in conjunction with

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x, \quad (\text{see [20]}). \quad (1.8)$$

The generalized (p, q) -Bernoulli, the generalized (p, q) -Euler, and the generalized (p, q) -Genocchi numbers and polynomials of order α are defined by means of the following generating function as follows (see [1], [2],[3], [4],[5], [6],[7], [8],[9],[10],[11],[12],[13],[14],[15],[16],[17],[18],[19],[20], [21]):

$$\left(\frac{t}{e_{p,q}(t) - 1} \right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x : p, q) \frac{t^n}{[n]_{p,q}!}, \quad |t| < 2\pi, \quad (1.9)$$

$$\left(\frac{2}{e_{p,q}(t) + 1} \right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x : p, q) \frac{t^n}{[n]_{p,q}!}, \quad |t| < \pi, \quad (1.10)$$

and

$$\left(\frac{2t}{e_{p,q}(t) + 1} \right)^\alpha e_{p,q}(xt) = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x : p, q) \frac{t^n}{[n]_{p,q}!}, \quad |t| < \pi. \quad (1.11)$$

It is clear that

$$B_n^{(\alpha)}(0 : p, q) = B_n^{(\alpha)}(p, q), \quad E_n^{(\alpha)}(0 : p, q) = E_n^{(\alpha)}(p, q),$$

and

$$G_n^{(\alpha)}(0 : p, q) = G_n^{(\alpha)}(p, q) \quad (n \in \mathbb{N}).$$

Recently, Duran et al. [6] defined Apostol type (p, q) -Bernoulli, Apostol type (p, q) -Euler and the Apostol type (p, q) -Genocchi numbers and polynomials $G_n^{(\alpha)}(x, y; \lambda : p, q)$ of order α are defined by means of the following generating functions:

$$\left(\frac{t}{\lambda e_{p,q}(t) - 1} \right)^\alpha e_{p,q}(xt) E_{p,q}(yt) = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x, y, \lambda : p, q) \frac{t^n}{[n]_{p,q}!}, \quad (|t + \log \lambda| < 2\pi), \quad (1.12)$$

$$\left(\frac{2}{\lambda e_{p,q}(t) + 1} \right)^\alpha e_{p,q}(xt) E_{p,q}(yt) = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x, y, \lambda : p, q) \frac{t^n}{[n]_{p,q}!}, \quad (|t + \log \lambda| < \pi), \quad (1.13)$$

and

$$\left(\frac{2t}{\lambda e_{p,q}(t) + 1} \right)^\alpha e_{p,q}(xt) E_{p,q}(yt) = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x, y, \lambda : p, q) \frac{t^n}{[n]_{p,q}!}, \quad (|t + \log \lambda| < \pi). \quad (1.14)$$

So that obviously

$$B_n^{(\alpha)}(0, 0, \lambda : p, q) = B_n^{(\alpha)}(\lambda : p, q), \quad E_n^{(\alpha)}(0, 0, \lambda : p, q) = E_n^{(\alpha)}(\lambda : p, q),$$

and

$$G_n^{(\alpha)}(0, 0, \lambda : p, q) = G_n^{(\alpha)}(\lambda : p, q) \quad (n \in \mathbb{N}).$$

Very recently, Yaşar and Özarslan [21] introduced Frobenius-Genocchi polynomials are defined by means of the following generating relation:

$$\frac{(1-\lambda)t}{e^t - \lambda} e^{xt} = \sum_{n=0}^{\infty} G_n^F(x; \lambda) \frac{t^n}{n!} \quad (1.15)$$

Taking $\lambda = -1$ in (1.15), we get the Genocchi polynomials (see [13])

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi. \quad (1.16)$$

The following section provides some identities and properties for Apostol type (p, q) -Frobenius-Genocchi numbers and polynomials of order α involving

addition property, difference equations, derivative properties, recurrence relationships. We also provide integral representations, implicit and explicit formulas and relations for mentioned polynomials and numbers. By using generating function of the polynomial stated in Definition (2.1), we derive some relationship for Apostol type (p, q) -Frobenius Genocchi polynomials of order α related to Apostol type (p, q) -Bernoulli polynomials, the Apostol type (p, q) -Euler polynomials and the Apostol type (p, q) -Genocchi polynomials.

2. (p, q) -analogue type of Apostol-type Frobenius-Genocchi polynomials $g_n^{(\alpha)}(x, y; u; \lambda : p, q)$

In this section, we introduce Apostol-type (p, q) -Frobenius-Genocchi polynomials of order α and provide some basic formulas and identities for Apostol type (p, q) -Frobenius-Genocchi polynomials of order α .

Definition 1. The Apostol type (p, q) -Frobenius-Genocchi polynomials $g_n^{(\alpha)}(x, y; u; \lambda : p, q)$ of order α are defined by means of the following generating function:

$$\left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^\alpha e_{p,q}(xt) E_{p,q}(yt) = \sum_{n=0}^{\infty} g_n^{(\alpha)}(x, y; u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!}, \quad (2.1)$$

where α and λ is suitable (real or complex) parameters, $p, q \in \mathbb{C}$ with $0 < |q| < |p| \leq 1$ and $u \in \mathbb{C}/\{1\}$.

Remark 1. For $x = y = 0$ and $\alpha = 1$ in (2.1), the result reduces to

$$\left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right) = \sum_{n=0}^{\infty} g_n(u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!}, \quad (2.2)$$

where $g_n(u; \lambda : p, q)$ denotes the Apostol-type (p, q) -Frobenius-Genocchi numbers.

Remark 2. On setting $u = -1$, equation (2.1) reduces to

$$\left(\frac{2t}{\lambda e_{p,q}(t) + 1} \right)^\alpha e_{p,q}(xt) E_{p,q}(yt) = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x, y; \lambda : p, q) \frac{t^n}{[n]_{p,q}!}, \quad (2.3)$$

where $G_n^{(\alpha)}(x, y; \lambda : p, q)$ denotes the Apostol-type (p, q) -Genocchi polynomials of order α , (see [13]).

Remark 3. For $y = 0$ in (2.1), the result reduces to known result of Khan et al. [13] as follows

$$\left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^\alpha e_{p,q}(xt) = \sum_{n=0}^{\infty} g_n^{(\alpha)}(x; u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!}. \quad (2.4)$$

From (2.1), we have

$$\begin{aligned} g_n^{(1)}(x, y; u; \lambda : p, q) &:= g_n(x, y; u; \lambda : p, q), \\ g_n^{(\alpha)}(x, y; u; \lambda : p, q)|_{p=1} &:= g_{n,q}^{(\alpha)}(x, y; u; \lambda), \quad (\text{see [9]}) \\ \liminf_{q \rightarrow 1^-_{p=1}} g_n^{(\alpha)}(x, y; u; \lambda : p, q) &:= G_n^{(\alpha)}(x + y; u; \lambda), \quad (\text{see [14], [15]}). \end{aligned}$$

Note that

$$g_n^{(0)}(x, y; u; \lambda : p, q) = \sum_{k=0}^n \binom{n}{k}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k y^{n-k} = (x + y)_{p,q}^n.$$

Theorem 2.1. (*Addition Theorems*) *The following relationship holds true:*

$$g_n^{(\alpha)}(x, y; u; \lambda : p, q) = \sum_{k=0}^n \binom{n}{k}_{p,q} g_k^{(\alpha)}(u : p, q) (x + y)_{p,q}^{n-k}, \quad (2.5)$$

$$g_n^{(\alpha)}(x, y; u; \lambda : p, q) = \sum_{k=0}^n \binom{n}{k}_{p,q} q^{\binom{n-k}{2}} y^{n-k} g_k^{(\alpha)}(x, 0; u : p, q), \quad (2.6)$$

$$g_n^{(\alpha)}(x, y; u; \lambda : p, q) = \sum_{k=0}^n \binom{n}{k}_{p,q} p^{\binom{n-k}{2}} x^{n-k} g_k^{(\alpha)}(0, y; u : p, q). \quad (2.7)$$

Proof. Suitably using equations (1.5) and (1.6) in generating function (2.1) to get three different forms. Further, making use of the Cauchy product rule in the resultant expressions and then comparing the like powers of t in both sides of resultant equation, we find formulas (2.5)-(2.7). \square

Theorem 2.2. (*Difference equations*) *The following recursive formula holds true:*

$$\lambda g_n^{(\alpha)}(1, y; u; \lambda : p, q) - u g_n^{(\alpha)}(0, y; u; \lambda : p, q) = (1 - u) g_{n-1}^{(\alpha-1)}(0, y; u; \lambda : p, q), \quad (2.8)$$

$$\lambda g_n^{(\alpha)}(x, 0; u; \lambda : p, q) - u g_n^{(\alpha)}(x, -1; u; \lambda : p, q) = (1 - u) g_{n-1}^{(\alpha-1)}(x, -1; u; \lambda : p, q). \quad (2.9)$$

Proof. By using generating function (2.1) and use of the Cauchy product rule in the resultant expressions and then comparing the coefficients of t in both sides, we get (2.8)-(2.9). \square

Theorem 2.3. (*Derivative properties*) *The following relationship holds true:*

$$D_{p,q;x} g_n^{(\alpha)}(x, y; u; \lambda : p, q) = [n]_{p,q} g_{n-1}^{(\alpha)}(px, y; u; \lambda : p, q), \quad (2.10)$$

$$D_{p,q;y} g_n^{(\alpha)}(x, y; u; \lambda : p, q) = [n]_{p,q} g_{n-1}^{(\alpha)}(x, qy; u; \lambda : p, q). \quad (2.11)$$

Proof. Differentiating generating function (2.1) with respect to x and y with the help of equation (1.2) and then simplifying with the help of Cauchy product rule, formulas (2.10) and (2.11) are obtained. \square

Theorem 2.4. (*Recurrence relationship*) *The following formula holds true:*

$$\begin{aligned} & \alpha \sum_{k=0}^n \binom{n}{k}_{p,q} p^{\binom{n-k}{2}} m^k g_k^{(\alpha)}(x, 0; u; \lambda : p, q) \\ & - u \sum_{k=0}^n \binom{n}{k}_{p,q} p^{\binom{n-k}{2}} m^k g_k^{(\alpha)}(x, -1; u; \lambda : p, q) \\ & = (1-u) \sum_{k=0}^n \binom{n-1}{k}_{p,q} p^{\binom{n-k-1}{2}} m^k g_k^{(\alpha-1)}(x, -1; u; \lambda : p, q). \end{aligned} \quad (2.12)$$

Proof. By using definition (2.1) and use Cauchy product rule, we obtain the result (2.12). \square

Theorem 2.5. *For $\alpha, \beta \in \mathbb{N}$, (p, q) -Apostol type Frobenius-Genocchi polynomials satisfy the following relations:*

$$g_n^{(\alpha+\beta)}(x, y; u; \lambda : p, q) = \sum_{k=0}^n \binom{n}{k}_{p,q} g_{n-k}^{(\alpha)}(x, y; u; \lambda : p, q) g_k^{(\beta)}(u; \lambda : p, q), \quad (2.13)$$

$$g_k^{(\alpha-\beta)}(x, y; u; \lambda : p, q) = \sum_{k=0}^n \binom{n}{k}_{p,q} g_{n-k}^{(\alpha)}(x, y; u; \lambda : p, q) g_k^{(-\beta)}(u; \lambda : p, q). \quad (2.14)$$

3. Main Results

In this section, we includes implicit and explicit formulas, integral representations, some identities for $g_k^{(\alpha)}(x, y; u; \lambda : p, q)$. Also we present new theorems and some (p, q) -extensions of known results in Carlitz [1], Kurt [9], Simsek [17], [18] and so on. We start with the following implicit formula for Apostol type (p, q) -Frobenius Genocchi polynomials of order α by the following theorem.

Theorem 3.1. *The following implicit summation formula holds true:*

$$\begin{aligned} & g_{k+l}^{(\alpha)}(z, y; u; \lambda : p, q) \\ & = \sum_{n,m=0}^{k,l} \binom{l}{m}_{p,q} \binom{k}{n}_{p,q} (z-x)^{m+n} g_{k-n, l-m}^{(\alpha)}(x, y; u; \lambda : p, q). \end{aligned} \quad (3.1)$$

Proof. Replacing t with $(t+w)$ in (2.1) and using result [19], we get

$$\begin{aligned} & \left(\frac{(1-u)(t+w)}{\lambda e_{p,q}(t+w) - u} \right)^\alpha E_{p,q}(yt) \\ & = e_{p,q}(-x(t+w)) \sum_{l,k=0}^{\infty} g_{k+l}^{(\alpha)}(x, y; u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \frac{w^l}{[l]_{p,q}!}. \end{aligned} \quad (3.2)$$

Replacing x by z and equating the obtained equation with the above equation, we arrive at

$$\begin{aligned} e_{p,q}((z-x)(t+w)) \sum_{k,l=0}^{\infty} g_{k+l}^{(\alpha)}(x, y; u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \frac{w^l}{[l]_{p,q}!} \\ = \sum_{k,l=0}^{\infty} g_{k+l}^{(\alpha)}(z, y; u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \frac{w^l}{[l]_{p,q}!}. \end{aligned} \quad (3.3)$$

Expanding the exponent part in the above equation, we have

$$\begin{aligned} \sum_{N=0}^{\infty} \frac{[(z-x)(t+w)]^N}{[N]_{p,q}!} \sum_{k,l=0}^{\infty} g_{k+l}^{(\alpha)}(x, y; u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \frac{w^l}{[l]_{p,q}!} \\ = \sum_{k,l=0}^{\infty} g_{k+l}^{(\alpha)}(z, y; u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \frac{w^l}{[l]_{p,q}!}. \end{aligned} \quad (3.4)$$

$$\begin{aligned} \sum_{n,m=0}^{\infty} \frac{(z-x)^{n+m} t^n w^m}{[n]_{p,q}! [m]_{p,q}!} \sum_{k,l=0}^{\infty} g_{k+l}^{(\alpha)}(x, y; u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \frac{w^l}{[l]_{p,q}!} \\ = \sum_{k,l=0}^{\infty} g_{k+l}^{(\alpha)}(z, y; u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \frac{w^l}{[l]_{p,q}!}. \end{aligned} \quad (3.4)$$

Using Lemma [19] and then on comparing the coefficients of t^k and w^l , we get the required result. \square

Corollary 3.2. *For $l = 0$ in Theorem 3.1, we get*

$$g_k^{(\alpha)}(z, y; u; \lambda : p, q) = \sum_{n=0}^k \binom{k}{n}_{p,q} (z-x)^n g_{k-n}^{(\alpha)}(x, y; u; \lambda : p, q). \quad (3.6)$$

Theorem 3.3. *The following (p, q) -integral is valid*

$$\int_a^b g_n^{(\alpha)}(x, y; u; \lambda : p, q) d_{p,q} x = p \frac{g_{n+1}^{(\alpha)}\left(\frac{b}{p}, y; u; \lambda : p, q\right) - g_{n+1}^{(\alpha)}\left(\frac{a}{p}, y; u; \lambda : p, q\right)}{[n+1]_{p,q}}, \quad (3.7)$$

$$\int_a^b g_n^{(\alpha)}(x; u; \lambda : p, q) d_{p,q} y = p \frac{g_{n+1}^{(\alpha)}(x, \frac{b}{q}; u; \lambda : p, q) - g_{n+1}^{(\alpha)}(x, \frac{a}{q}; u; \lambda : p, q)}{[n+1]_{p,q}}. \quad (3.8)$$

Proof. Since

$$\int_a^b \frac{\delta}{\delta_{p,q} x} g_n^{(\alpha)}(x, y; u; \lambda : p, q) d_{p,q} x = f(b) - f(a), \text{ (see [20])}$$

in terms of equation (2.10) and equations (1.7) and (1.8), we arrive at the asserted result

$$\int_a^b \frac{\delta}{\delta_{p,q} x} g_n^{(\alpha)}(x, y; u; \lambda : p, q) d_{p,q} x = \frac{p}{[n+1]_{p,q}} \int_a^b g_n^{(\alpha)}\left(\frac{x}{p}, y; u; \lambda : p, q\right) d_{p,q} x$$

$$p \frac{g_{n+1}^{(\alpha)}\left(\frac{b}{p}, y; u; \lambda : p, q\right) - g_{n+1}^{(\alpha)}\left(\frac{b}{p}, y; u; \lambda : p, q\right)}{[n+1]_{p,q}}.$$

The other can be shown using similar method. Therefore, the complete proof of this theorem. \square

Theorem 3.4. *The following result holds true:*

$$\begin{aligned} & (2u-1) \sum_{k=0}^n \binom{n}{k}_{p,q} g_k(u; \lambda : p, q) g_{n-k}(x, y; 1-u; \lambda : p, q) \\ & = ug_n(x, y; u; \lambda : p, q) - (1-u)g_k(x, y; 1-u; \lambda : p, q). \end{aligned} \quad (3.9)$$

Proof. By utilizing the same method of Duran et al. [4] and Kurt [9], we first consider the identity

$$\frac{2u-1}{(\lambda e_{p,q}(t)-u)(\lambda e_{p,q}(t)-(1-u))} = \frac{1}{\lambda e_{p,q}(t)-u} - \frac{1}{\lambda e_{p,q}(t)-(1-u)},$$

then

$$\begin{aligned} & (2u-1) \frac{(1-u)te_{p,q}(t)E_{p,q}(yt)(1-(1-u)t)}{(\lambda e_{p,q}(t)-u)(\lambda e_{p,q}(t)-(1-u))} \\ & u \frac{(1-u)te_{p,q}(t)E_{p,q}(yt)}{\lambda e_{p,q}(t)-u} - \frac{(1-u)te_{p,q}(xt)(1-(1-u)t)E_{p,q}(yt)}{\lambda e_{p,q}(t)-(1-u)} \\ & (2u-1) \sum_{k=0}^{\infty} g_k(u; \lambda : p, q) \frac{t^k}{[k]_{p,q}!} \sum_{n=0}^{\infty} g_n(x, y; 1-u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!} \\ & = u \sum_{n=0}^{\infty} g_n(x, y; u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!} - (1-u) \sum_{n=0}^{\infty} g_n(x, y; 1-u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

On comparing the coefficients of t^n , we arrive at the required result (3.9). \square

Theorem 3.5. *The following result holds true:*

$$ug_n(x, y; u; \lambda : p, q) = \lambda \sum_{k=0}^n g_{n-k}(x, y; u; \lambda : p, q) p\binom{n}{k} - (1-u)(x+y)_{p,q}^{n-1}. \quad (3.10)$$

Proof. By using $e_{p,q}(x)E_{p,q}(-x) = 1$, consider the identity

$$\frac{u}{(\lambda e_{p,q}(t)-u)\lambda e_{p,q}(t)} = \frac{1}{\lambda e_{p,q}(t)-u} - \frac{1}{\lambda e_{p,q}(t)},$$

then

$$\begin{aligned} & \frac{u(1-u)te_{p,q}(t)E_{p,q}(yt)}{(\lambda e_{p,q}(t)-u)\lambda e_{p,q}(t)} \\ & = \frac{(1-u)te_{p,q}(xt)E_{p,q}(yt)}{\lambda e_{p,q}(t)-u} - \frac{(1-u)te_{p,q}(xt)(1-(1-u)t)E_{p,q}(yt)}{\lambda e_{p,q}(t)} \\ & \quad \frac{u}{\lambda} \sum_{n=0}^{\infty} g_n(x, y; u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} g_n(x, y; u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!} \sum_{k=0}^{\infty} p\binom{n}{k} \frac{t^k}{[k]_{p,q}!} - \frac{1-u}{\lambda} \sum_{n=0}^{\infty} (x+y)_{p,q}^n \frac{t^{n+1}}{[n]_{p,q}!}.$$

On comparing the coefficients of t^n , we arrive at the required result (3.10). \square

4. Relationship between Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials

In this section, we prove some relationships for Apostol type (p, q) -Frobenius-Euler polynomials of order α related to Apostol type (p, q) -Bernoulli polynomials, Apostol type (p, q) -Euler polynomials and Apostol type (p, q) -Genocchi polynomials in Theorems 4.1, 4.2, 4.3.

Theorem 4.1. *The following recurrence relations are valid:*

$$\begin{aligned} g_n^{(\alpha)}(x, y; u; \lambda : p, q) &= \sum_{s=0}^{n+1} \binom{n+1}{s}_{p,q} \\ &\times \left[\sum_{k=0}^s \binom{s}{k}_{p,q} B_{s-k}(x, 0; \lambda : p, q) p\binom{k}{2} - B_s(x, 0; \lambda : p, q) \right] \frac{g_{n+1-s}^{(\alpha)}(0, y; u; \lambda : p, q)}{[n+1]_{p,q}!}. \end{aligned} \quad (4.1)$$

Proof. By using Definition (2.1), we have

$$\begin{aligned} &\left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^{\alpha} e_{p,q}(xt) E_{p,q}(yt) \\ &= \left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^{\alpha} E_{p,q}(yt) \frac{t}{\lambda e_{p,q}(t) - 1} \frac{\lambda e_{p,q}(t) - 1}{t} e_{p,q}(xt) \\ &= \frac{1}{t} \left[\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}_{p,q} B_{n-k}(x, 0; \lambda : p, q) p\binom{k}{2} \right) \frac{t^n}{[n]_{p,q}!} - \sum_{s=0}^n \binom{n}{s}_{p,q} B_s(x, 0; \lambda : p, q) \right] \\ &\quad \times \sum_{n=0}^{\infty} G_n^{(\alpha)}(0, y; u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left[\lambda \sum_{s=0}^n \binom{n}{s}_{p,q} \sum_{k=0}^s B_{s-k}(x, 0; \lambda : p, q) p\binom{k}{2} - \sum_{s=0}^n \binom{n}{s}_{p,q} B_s(x, 0; \lambda : p, q) \right] \\ &\quad \times g_{n-s}^{(\alpha)}(0, y; u; \lambda : p, q) \frac{t^{n-1}}{[n]_{p,q}!}. \end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the required result (4.1). \square

Theorem 4.2. *Each of the following relationship holds true:*

$$\begin{aligned} g_n^{(\alpha)}(x, y; u; \lambda : p, q) &= \sum_{s=0}^n \binom{n}{s}_{p,q} \\ &\times \left[\sum_{k=0}^s \binom{s}{k}_{p,q} E_k(x, 0; \lambda : p, q) p^{\binom{s-k}{2}} + E_s(x, 0; \lambda : p, q) \right] \frac{g_{n-s}^{(\alpha)}(0, y; u; \lambda : p, q)}{[2]_{p,q}!}. \end{aligned} \quad (4.2)$$

Proof. The proof of this theorem is based on the following equalities

$$\begin{aligned} &\left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^\alpha e_{p,q}(xt) E_{p,q}(yt) \\ &= \left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^\alpha E_{p,q}(yt) \frac{[2]_{p,q}}{\lambda e_{p,q}(t) + 1} \frac{\lambda e_{p,q}(t) + 1}{[2]_{p,q}} e_{p,q}(xt) \\ &= \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left[\left(\sum_{s=0}^n \binom{n}{s}_{p,q} \sum_{k=0}^s \binom{s}{k}_{p,q} E_k(x, 0; \lambda : p, q) p^{\binom{s-k}{2}} \right) \right. \\ &\quad \left. + \sum_{s=0}^n \binom{n}{s}_{p,q} E_s(x, 0; \lambda : p, q) \right] g_{n-s}^{(\alpha)}(0, y; u; \lambda : p, q) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

On comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the required result (4.2). \square

Theorem 4.3. *Each of the following relationship holds true:*

$$\begin{aligned} g_n^{(\alpha)}(x, y; u; \lambda : p, q) &= \sum_{s=0}^{n+1} \binom{n+1}{s}_{p,q} \left[\sum_{k=0}^s \binom{s}{k}_{p,q} G_{s-k}(x, 0; \lambda : p, q) p^{\binom{k}{2}} \right. \\ &\quad \left. + G_s(x, 0; \lambda : p, q) \right] \frac{g_{n+1-s}^{(\alpha)}(0, y; u; \lambda : p, q)}{[2]_{p,q}! [n+1]_{p,q}!}. \end{aligned} \quad (4.3)$$

Proof. By making use of the following equalities

$$\begin{aligned} &\left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^\alpha e_{p,q}(xt) E_{p,q}(yt) \\ &= \left(\frac{(1-u)t}{\lambda e_{p,q}(t) - u} \right)^\alpha E_{p,q}(yt) \frac{[2]_{p,q} t}{\lambda e_{p,q}(t) + 1} \frac{\lambda e_{p,q}(t) + 1}{[2]_{p,q} t} e_{p,q}(xt) \\ &= \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left[\sum_{s=0}^n \binom{n}{s}_{p,q} \sum_{k=0}^s \binom{s}{k}_{p,q} G_{s-k}(x, 0; \lambda : p, q) p^{\binom{k}{2}} \right. \\ &\quad \left. + \sum_{s=0}^n \binom{n}{s}_{p,q} G_s(x, 0; \lambda : p, q) \right] g_{n+1-s}^{(\alpha)}(0, y; u; \lambda : p, q) \frac{t^n}{[n+1]_{p,q}!}. \end{aligned}$$

On comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the required result (4.3). \square

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