SPECTRAL ANALYSIS OF THE MGSS PRECONDITIONER FOR SINGULAR SADDLE POINT PROBLEMS†

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ABSTRACT. Recently Salkuyeh and Rahimian in (COMPUT. MATH. APPL. 74 (2017) 2940–2949) proposed a modification of the generalized shift-splitting (MGSS) method for solving singular saddle point problems. In this paper, we present the spectral analysis of the MGSS preconditioner when it is applied to precondition the singular saddle point problems with the $(1,1)$ block being symmetric. Some eigenvalue bounds for the spectrum of the preconditioned matrix are given. We show that all the real eigenvalues of the preconditioned matrix are in a positive interval and all nonzero eigenvalues having nonzero imaginary part are contained in an intersection of two circles.

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1. Introduction

We consider the following saddle point problem

$$\mathcal{A}u \equiv \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite (SPD) matrix ($A = A^T$ and $x^T A x > 0$ for all $0 \neq x \in \mathbb{R}^n$), $B \in \mathbb{R}^{m \times n}$ is rank deficient ($rank(B) = r < m \leqslant n$), and $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are two given vectors. Furthermore, we presume that the matrices A and B are large and sparse. It is not difficult to see that under the above conditions the matrix $\mathcal{A}$ is singular. We also assume that the singular saddle point problem (1) is consistent. Saddle point problems of the form (1) appear in a variety of scientific and engineering problems; e.g., computational fluid dynamics, constrained optimization [8].

There are several iterative methods for solving the saddle point problems of the form (1). Bai et al. in [1], presented the shift-splitting iteration method for solving systems of linear equations with positive definite (PD) coefficient matrices. It is noted that the matrix $A \in \mathbb{R}^{n \times n}$ is called PD if the matrix $A + A^T$ is SPD. Indeed, for solving the system of linear equations $\mathbf{A}\mathrm{x} = \mathbf{b}$ with $\mathbf{A}$ being PD, the shift-splitting method can be written as

$$\frac{1}{2}(\alpha I + \mathbf{A})\mathrm{x}^{(k+1)} = \frac{1}{2}(\alpha I + \mathbf{A})\mathrm{x}^{(k)} + \mathbf{b},$$

where α is a positive number and I is the identity matrix. It was proved that this method is unconditionally convergent [1]. Naturally, the shift-splitting method serves the preconditioner $\mathbf{P} = (\alpha I + \mathbf{A})/2$ for the system $\mathbf{A}\mathrm{x} = \mathbf{b}$. In fact, a Krylov subspace method like GMRES [11] can be employed for solving the preconditioned system $\mathbf{P}^{-1}\mathbf{A}\mathrm{x} = \mathbf{P}^{-1}\mathbf{b}$

Cao et al. in [5] applied the shift-splitting iteration method for solving the saddle point problems. Next, Chen and Ma in [7] generalized the shift-splitting (GSS) method with the shift matrix

$$\Omega_{\alpha,\beta} = \begin{pmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{pmatrix},$$

for solving the saddle point problems where I_n and I_m are the identity matrices of order n and m, respectively, and $\alpha, \beta > 0$. Ren et al. in [9] presented an spectral analysis of the this method. In [13], Salkuyeh et al. applied the same method for the generalized saddle point problems, when the $(1,1)$ and the $(2,2)$ blocks of $\mathcal{A}$ are, respectively, PD and SPD. Then, they generalized the method to the case that the matrix A is PD (see [14]). Semi-convergence of this method has been investigated in [6] and [15] for the cases that the matrix A is SPD and PD, respectively.

Recently, Salkuyeh et al. in [12] presented a modification of GSS (MGSS) method for solving singular nonsymmetric saddle point problems. They used

$$\Omega = \begin{pmatrix} H & 0 \\ 0 & Q \end{pmatrix}, \tag{2}$$

as the shift matrix in the shift-splitting method for solving the singular saddle point problem (1) with the matrix A being PD, where $H \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{m \times m}$ are SPD. Moreover, they proved the semi-convergence of the proposed method. For the convergence of the method for nonsingular saddle point problems see [2]. Since the eigenvalue distribution of the coefficient matrices of the systems is very important in the convergence analysis of the Krylov subspace methods like GMRES, in this paper we present eigenvalue bounds for the spectrum of the MGSS preconditioned singular saddle point matrices for the case that the matrix A is symmetric.

Throughout the paper, for a complex matrix A, the conjugate transpose of A is denoted by A^*. For two square matrices A and B, we write $A \succ B$ (resp. $A \succeq B$) if $A - B$ is SPD (resp. symmetric positive semidefinite). In the same way,

$A \prec B$ and $A \preceq B$ are defined. For a nonsigular matrix C, the spectral condition number of C is denoted by $\kappa(C)$, i.e., $\kappa(C) = \|C\|_2\|C^{-1}\|_2$. For a square matrix Z, the spectral radius of Z is defined by $\rho(Z)$, i.e., $\rho(Z) = \max_{\lambda \in \sigma(Z)} |\lambda|$, where $\sigma(Z)$ is the spectrum of Z. Throughout the paper, the vector $z = (x^T, y^T)^T$ is denoted by $z = (x; y)$.

This paper is organized as follows. The MGSS iterative method and its semi-convergence properties are described in Section 2. Section 3 is devoted to the spectral analysis of the preconditioned matrix. Numerical illustration is presented in 4. The paper is ended by some concluding remarks in Section 5.

2. A brief description of MGSS iteration method

Considering the modified generalized shift-splitting

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{2}(\Omega + \mathcal{A}) - \frac{1}{2}(\Omega - \mathcal{A}) \\
&= \frac{1}{2}\begin{pmatrix} H + A & B^T \\ -B & Q \end{pmatrix} - \frac{1}{2}\begin{pmatrix} H - A & -B^T \\ B & Q \end{pmatrix} \\
&= \mathcal{M} - \mathcal{N},
\end{aligned}
$$

the MGSS iteration method can be written as (see [12])

$$\mathcal{M}u^{(k+1)} = \mathcal{N}u^{(k)} + b, \tag{3}$$

where $u^{(0)}$ is an initial guess. Denoting $\Gamma = \mathcal{M}^{-1}\mathcal{N}$ and $c = \mathcal{M}^{-1}b$, the iterative method (3) can be rewritten as

$$u^{(k+1)} = \Gamma u^{(k)} + c. \tag{4}$$

We have

$$\Gamma = \mathcal{M}^{-1}\mathcal{N} = \mathcal{I} - \mathcal{M}^{-1}\mathcal{A}. \tag{5}$$

From the singularity of $\mathcal{A}$, we have $1 \in \sigma(\Gamma)$ and as a result we $\rho(\Gamma) \geqslant 1$. Hence, we need to investigate the semi-convergence of the method.

Definition 2.1. (see [4]) The iterative method (4) is semi-convergent if, for any initial guess $(x_0; y_0)$, the iteration sequence $(x_k; y_k)$ produced by (4) converges to a solution $(x_\star; y_\star)$ of (1). Moreover, it holds

$$\begin{bmatrix} x_\star \\ y_\star \end{bmatrix} = (I - \Gamma)^D c + (I - E)\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \tag{6}$$

with $E = (I - \Gamma)(I - \Gamma)^D$, where I is the identity matrix and $(I - \Gamma)^D$ denotes the Drazin inverse of $I - \Gamma$.

Lemma 2.2. (see [4]) The iterative method (4) is semi-convergent if and only if the following conditions hold:
(1) $Index(I - \Gamma) = 1$, i.e., $rank(I - \Gamma) = rank(I - \Gamma)^2$;
(2) The pseudo-spectral radius of Γ satisfies

$$\vartheta(\Gamma) = \max\{|\lambda|, \lambda \in \sigma(\Gamma), \lambda \neq 1\} < 1.$$

Theorem 2.3. *Assume that the matrices $A, H \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{m \times m}$ are SPD. $B \in \mathbb{R}^{m \times n}$ $(m \leqslant n)$ is rank-deficient. Then, $\vartheta(\Gamma) < 1$.*

Proof. It is a result of Theorem 1 in [12]. □

Theorem 2.4. *Suppose that $A, H \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{m \times m}$ are symmetric positive definite and the matrix $B \in \mathbb{R}^{m \times n}(m \leqslant n)$ is rank-deficient. Then, $rank(I - \Gamma) = rank(I - \Gamma)^2$, where Γ is the iteration matrix of the MGSS method.*

Proof. It is a result of Theorem 2 in [12]. □

According to Lemma 2.2 and Theorems 2.3 and 2.4 the semi-convergence of the MGSS method is deduced for the saddle point problem (1). The MGSS method induces the preconditioner $\mathcal{M}$ to the saddle point problem $\mathcal{A}u = b$. From the semi-convergence of the method and Eq. (5), we conclude that the eigenvalues of the matrix $\mathcal{M}^{-1}\mathcal{A}$ are clustered in the circle $|z-1| \leqslant 1$. Therefore, a Krylov subspace method like GMRES or its restarted version GMRES(ℓ) would be quite suitable for solving the preconditioned system $\mathcal{M}^{-1}\mathcal{A}u = \mathcal{M}^{-1}b$ (See [3, Page 420]). In the next section we investigate the eigenvalue distribution of the preconditioned system in more details.

3. Eigenvalue distribution of the preconditioned matrix

Since the multiplicative factor $1/2$ in the preconditioner matrix $\mathcal{M}$, has no effect on the preconditioned system, we drop it and use $\mathcal{K} = \Omega + \mathcal{A}$ as a preconditioner.

Let λ be an eigenvalue of the preconditioned matrix $\mathcal{K}^{-1}\mathcal{A}$. From Eq. (5), each eigenvalue μ of Γ can be written as $\mu = 1 - 2\lambda$, where $\lambda \in \sigma(\Gamma)$. Therefore,

$$|1 - 2\lambda| = |\mu| \leqslant \rho(\Gamma) \leqslant 1,$$

which is equivalent to

$$|\lambda - \frac{1}{2}| \leqslant \frac{1}{2}. \tag{7}$$

This shows that the eigenvalues of the preconditioned matrix are located in a circle centered at the point $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

In continuation we investigate the eigenvalues of the matrix $\mathcal{K}^{-1}\mathcal{A}$. To do so, we first note that the matrix $\mathcal{K}$ can be factorized as

$$\mathcal{K} = \begin{pmatrix} I & 0 \\ -B(H+A)^{-1} & I \end{pmatrix} \begin{pmatrix} H+A & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & (H+A)^{-1}B^T \\ 0 & I \end{pmatrix},$$

where $S = Q + B(H+A)^{-1}B^T$. Hence,

$$\mathcal{K}^{-1} = \begin{pmatrix} I & -(H+A)^{-1}B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} (H+A)^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ B(H+A)^{-1} & I \end{pmatrix}. \tag{8}$$

Therefore, using the this equation we get

$$\mathcal{K}^{-1}\mathcal{A} = \begin{pmatrix} L & (H+A)^{-1}B^T(I-K) \\ S^{-1}B((H+A)^{-1}A - I) & K \end{pmatrix},$$

where

$$L = (H + A)^{-1}(A - B^T S^{-1} B(H + A)^{-1} A + B^T S^{-1} B),$$
$$K = S^{-1} B(H + A)^{-1} B^T.$$

Now, similar to Theorems 3.2 and 3.3 in [9], we present the following results.

Theorem 3.1. *Suppose that the matrices $A, H \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{m \times m}$ are SPD and $B \in \mathbb{R}^{m \times n}$ is rank-deficient. Then all the nonzero eigenvalues having nonzero imaginary parts of the preconditioned matrix $\mathcal{K}^{-1}\mathcal{A}$ are located in a circle centered at $(1, 0)$ with radius $\sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\max}(H) + \lambda_{\min}(A)}}$.*

Proof. Let

$$\mathcal{T}_0 = \begin{pmatrix} H + A & 0 \\ 0 & Q \end{pmatrix}, \quad \mathcal{I}_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Obviously, the matrix $\mathcal{T}_0$ is SPD. Hence, the matrix $\mathcal{K}^{-1}\mathcal{A}$ is similar to

$$\begin{aligned}
\mathcal{T}_0^{\frac{1}{2}}\mathcal{K}^{-1}\mathcal{A}\mathcal{T}_0^{-\frac{1}{2}} &= \left(\mathcal{T}_0^{-\frac{1}{2}}\mathcal{I}_0\mathcal{K}\mathcal{T}_0^{-\frac{1}{2}}\right)^{-1}\left(\mathcal{T}_0^{-\frac{1}{2}}\mathcal{I}_0\mathcal{A}\mathcal{T}_0^{-\frac{1}{2}}\right) \\
&= \begin{pmatrix} I & \bar{B}^T \\ \bar{B} & -I \end{pmatrix}^{-1} \begin{pmatrix} \bar{A} & \bar{B}^T \\ \bar{B} & 0 \end{pmatrix},
\end{aligned} \tag{9}$$

where $\bar{A} = (H + A)^{-\frac{1}{2}} A (H + A)^{-\frac{1}{2}}$ and $\bar{B} = Q^{-\frac{1}{2}} B (H + A)^{-\frac{1}{2}}$. Eq. (9) can be rewritten as

$$\mathcal{T}_0^{\frac{1}{2}}\mathcal{K}^{-1}\mathcal{A}\mathcal{T}_0^{-\frac{1}{2}} = \begin{pmatrix} I & 0 \\ -\bar{B} & I \end{pmatrix}^{-1} \begin{pmatrix} I + \bar{B}^T\bar{B} & 0 \\ 0 & -I \end{pmatrix}^{-1} \begin{pmatrix} I & -\bar{B}^T \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} \bar{A} & \bar{B}^T \\ \bar{B} & 0 \end{pmatrix},$$

which is similar to

$$\begin{aligned}
&\left[\begin{pmatrix} I & -\bar{B}^T \\ 0 & I \end{pmatrix}\begin{pmatrix} I + \bar{B}^T\bar{B} & 0 \\ 0 & -I \end{pmatrix}\right]^{-1} \begin{pmatrix} \bar{A} & \bar{B}^T \\ \bar{B} & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ -\bar{B} & I \end{pmatrix}^{-1} \\
&= \begin{pmatrix} I + \bar{B}^T\bar{B} & \bar{B}^T \\ 0 & -I \end{pmatrix}^{-1} \begin{pmatrix} \bar{A} + \bar{B}^T\bar{B} & \bar{B}^T \\ \bar{B} & 0 \end{pmatrix} \\
&=: \hat{\mathcal{K}}^{-1}\hat{\mathcal{A}}.
\end{aligned}$$

Evidently, the matrix

$$\mathcal{T}_1 = \begin{pmatrix} I + \bar{B}^T\bar{B} & 0 \\ 0 & I \end{pmatrix}.$$

is SPD. Therefore, the matrix $\hat{\mathcal{K}}^{-1}\hat{\mathcal{A}}$ is similar to

$$\begin{aligned}
\mathcal{T}_1^{\frac{1}{2}}\hat{\mathcal{K}}^{-1}\hat{\mathcal{A}}\mathcal{T}_1^{-\frac{1}{2}} &= \left(\mathcal{T}_1^{-\frac{1}{2}}\hat{\mathcal{K}}\mathcal{T}_1^{-\frac{1}{2}}\right)^{-1}\left(\mathcal{T}_1^{-\frac{1}{2}}\hat{\mathcal{A}}\mathcal{T}_1^{-\frac{1}{2}}\right) \\
&= \begin{pmatrix} I & \tilde{B}^T \\ 0 & -I \end{pmatrix}^{-1} \begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \\
&=: \tilde{\mathcal{K}}^{-1}\tilde{\mathcal{A}},
\end{aligned}$$

where $\tilde{A} = (I + \bar{B}^T\bar{B})^{-\frac{1}{2}}(\bar{A} + \bar{B}^T\bar{B})(I + \bar{B}^T\bar{B})^{-\frac{1}{2}}$ and $\tilde{B} = \bar{B}(I + \bar{B}^T\bar{B})^{-\frac{1}{2}}$. Hence, we deduce that $\mathcal{K}^{-1}\bar{\mathcal{A}}$ is similar to $\hat{\mathcal{K}}^{-1}\hat{\mathcal{A}}$. Hence, it is enough to analyze the eigenvalue distribution of $\tilde{\mathcal{K}}^{-1}\tilde{\mathcal{A}}$.

Let $(\lambda, (u; v))$ be an eigenpair of the matrix $\tilde{\mathcal{K}}^{-1}\tilde{\mathcal{A}}$. Then, we have

$$\begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} I & \tilde{B}^T \\ 0 & -I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

which can be rewritten as

$$\begin{cases} \tilde{A}u + \tilde{B}^T v = \lambda u + \lambda\tilde{B}^T v, \\ \tilde{B}u = -\lambda v. \end{cases} \tag{10}$$

If $v = 0$, then from the first equality of (10), we obtain $\tilde{A}u = \lambda u$. This shows that λ is real, because $\tilde{A}$ is SPD. If $u = 0$, then from the second equality of (10) we get $-\lambda v = 0$. Therefore, we have $\lambda = 0$, since v cannot be zero.

Now, we assume that $u \neq 0$ and $v \neq 0$ with $\|u\|_2^2 + \|v\|_2^2 = 1$. Multiplying both sides of the first equality of (10) by u^* gives

$$u^*\tilde{A}u - \lambda\|u\|_2^2 = (\lambda - 1)u^*\tilde{B}^T v. \tag{11}$$

Multiplying the second equation of Eq. (10) by v^* results in $u^*\tilde{B}^T v = -\bar{\lambda}\|v\|_2^2$. Substituting this into Eq. (11) yields

$$u^*\tilde{A}u + |\lambda|^2\|v\|_2^2 - \lambda + (\lambda - \bar{\lambda})\|v\|_2^2 = 0. \tag{12}$$

Defining $\lambda = a + ib$, the imaginary part of Eq. (12) is written as

$$b(2\|v\|_2^2 - 1) = 0.$$

From this equation we deduce that $b = 0$ or $\|v\|_2^2 = \frac{1}{2}$. If $b = 0$, then λ is real. If $b \neq 0$, then we get $\|v\|_2^2 = \|u\|_2^2 = \frac{1}{2}$. This, together with Eq. (12) gives

$$2u^*\tilde{A}u + |\lambda|^2 - \lambda - \bar{\lambda} = 0.$$

By some computations and using the Courant-Fisher min-max theorem [10], we can write

$$|\lambda - 1|^2 = 1 - 2u^*\tilde{A}u = 1 - \frac{u^*\tilde{A}u}{u^*u} \leqslant 1 - \lambda_{\min}(\tilde{A}). \tag{13}$$

It is not difficult to verify that the matrix $\tilde{A}$ is similar to

$$J = (H + A + B^T Q^{-1}B)^{-1}(A + B^T Q^{-1}B).$$

Suppose that $(\tilde{\lambda}, \tilde{x})$ is an eigenpair of the matrix J. Therefore, we have

$$(A + B^T Q^{-1}B)\tilde{x} = \tilde{\lambda}(H + A + B^T Q^{-1}B)\tilde{x}. \tag{14}$$

Multiplying both sides of Eq. (14) by $\tilde{x}^*$ and some simplifications we obtain

$$\tilde{\lambda} = \frac{x^*Ax + x^*B^T Q^{-1}Bx}{x^*Hx + x^*Ax + x^*B^T Q^{-1}Bx}$$

$$\geqslant \frac{x^*Ax}{x^*Hx + x^*Ax} = \frac{\frac{x^*Ax}{x^*x}}{\frac{x^*Hx}{x^*x} + \frac{x^*Ax}{x^*x}}$$

$$\geqslant \frac{\lambda_{\min}(A)}{\lambda_{\max}(H) + \lambda_{\min}(A)}. \tag{15}$$

It follows from Eqs. (13) and (15) that

$$|\lambda - 1|^2 \leqslant 1 - \frac{\lambda_{\min}(A)}{\lambda_{\max}(H) + \lambda_{\min}(A)} = \frac{\lambda_{\max}(H)}{\lambda_{\max}(H) + \lambda_{\min}(A)} < 1,$$

which shows that all nonzero eigenvalues having nonzero imaginary parts of the preconditioned matrix $\mathcal{K}^{-1}\mathcal{A}$ are located in a circle centered at (1,0) with radius $\sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\max}(H) + \lambda_{\min}(A)}}$ which is strictly less than one. $\qquad\square$

Corollary 3.2. *Let the matrices* $A, H \in \mathbb{R}^{n \times n}$ *and* $Q \in \mathbb{R}^{m \times m}$ *be SPD and* $B \in \mathbb{R}^{m \times n}$ *be rank-deficient. Then, all the nonzero eigenvalues having nonzero imaginary parts of the preconditioned matrix* $\mathcal{K}^{-1}\mathcal{A}$ *are located in the following domain*

$$\mathcal{D} = \left\{ \lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leqslant \frac{1}{2} \right\} \cap \left\{ \lambda \in \mathbb{C} : |\lambda - 1| \leqslant \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\max}(H) + \lambda_{\min}(A)}} \right\}.$$

Theorem 3.3. *Let* $A, H \in \mathbb{R}^{n \times n}$ *and* $Q \in \mathbb{R}^{m \times m}$ *be SPD matrices and* $B \in \mathbb{R}^{m \times n}$ *be rank-deficient. Let also* $\sigma_{\min}$ *and* $\sigma_{\max}$ *be the smallest and the largest nonzero singular values of the matrix* B, *respectively. Then, all the nonzero real eigenvalues of the matrix* $\mathcal{K}^{-1}\mathcal{A}$ *are located in*

$$\left[\min \left\{ \frac{\lambda_{\min}(A)}{\lambda_{\max}(H) + \lambda_{\min}(A)}, \frac{\sigma_{\min}^2}{\lambda_{\max}(Q)(\lambda_{\max}(H) + \kappa(H)\lambda_{\max}(A)) + \sigma_{\min}^2} \right\}, \right.$$
$$\left. \frac{\lambda_{\min}(Q)\lambda_{\max}(A) + \sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \lambda_{\max}(A)) + \sigma_{\max}^2} \right].$$

Proof. Since $\mathcal{K}^{-1}\mathcal{A}$ is similar to $\tilde{\mathcal{K}}^{-1}\tilde{\mathcal{A}}$, we only need to study the nonzero real eigenvalues of the matrix $\tilde{\mathcal{K}}^{-1}\tilde{\mathcal{A}}$ which are the same as those of the matrix $\tilde{\mathcal{A}}\tilde{\mathcal{K}}^{-1}$. Since $\tilde{A}$ is symmetric positive definite and it is similar to $(H + A + B^TQ^{-1}B)^{-1}(A + B^TQ^{-1}B)$, we deduce that all the eigenvalues of $\tilde{A}$ are positive and less than one. On the other hand,

$$\tilde{\mathcal{A}}\tilde{\mathcal{K}}^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} I & \tilde{B}^T \\ 0 & -I \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \tilde{A}(\tilde{A} - I) & (\tilde{A} - I)\tilde{B}^T \\ \tilde{B}(\tilde{A} - I) & \tilde{B}\tilde{B}^T \end{pmatrix} \begin{pmatrix} \tilde{A} - I & 0 \\ 0 & I \end{pmatrix}^{-1}$$

$$=: \breve{\mathcal{A}}\breve{\mathcal{S}}^{-1},$$

where $\breve{\mathcal{A}}$ and $\breve{\mathcal{S}}$ are symmetric and $\breve{\mathcal{S}}$ is nonsingular. Then, the eigenvalues of $\tilde{\mathcal{K}}^{-1}\tilde{\mathcal{A}}$ and $\breve{\mathcal{S}}^{-1}\breve{\mathcal{A}}$ are the same. Assume that $\tilde{A} = X\Lambda X^T$ with $I - \Lambda \succ 0$, where

X is an orthogonal matrix. Define

$$\mathcal{Z} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} I - \Lambda & 0 \\ 0 & I \end{pmatrix} \succ 0. \tag{16}$$

Since $\mathcal{Z}$ is orthogonal and $\mathcal{D}$ is SPD, we deduce that the matrix $\breve{\mathcal{S}}^{-1}\breve{\mathcal{A}}$ is similar to $\mathcal{D}^{\frac{1}{2}}\mathcal{Z}^T\breve{\mathcal{S}}^{-1}\breve{\mathcal{A}}\mathcal{Z}\mathcal{D}^{-\frac{1}{2}}$. We have

$$\mathcal{D}^{\frac{1}{2}}\mathcal{Z}^T\breve{\mathcal{S}}^{-1}\breve{\mathcal{A}}\mathcal{Z}\mathcal{D}^{-\frac{1}{2}} = \left(\mathcal{D}^{-\frac{1}{2}}\mathcal{Z}^T\breve{\mathcal{S}}\mathcal{Z}\mathcal{D}^{-\frac{1}{2}}\right)^{-1}\left(\mathcal{D}^{-\frac{1}{2}}\mathcal{Z}^T\breve{\mathcal{A}}\mathcal{Z}\mathcal{D}^{-\frac{1}{2}}\right). \tag{17}$$

Let (λ, w) be an eigenpair of the matrix $\mathcal{D}^{\frac{1}{2}}\mathcal{Z}^T\breve{\mathcal{S}}^{-1}\breve{\mathcal{A}}\mathcal{Z}\mathcal{D}^{-\frac{1}{2}}$ such that $\lambda \neq 0$. Thus, from Eq. (17) it holds that

$$\mathcal{D}^{-\frac{1}{2}}\mathcal{Z}^T\breve{\mathcal{A}}\mathcal{Z}\mathcal{D}^{-\frac{1}{2}}w = \lambda\mathcal{D}^{-\frac{1}{2}}\mathcal{Z}^T\breve{\mathcal{S}}\mathcal{Z}\mathcal{D}^{-\frac{1}{2}}w,$$

which is equivalent to

$$\begin{pmatrix} -\Lambda & Q^T \\ Q & P \end{pmatrix} w = \lambda \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} w, \tag{18}$$

where $Q = -\tilde{B}X(I - \Lambda)^{-\frac{1}{2}}$ and $P = \tilde{B}\tilde{B}^T$. Without loss of generality, assume that $w = (u; v)$ such that $\|u\|_2^2 + \|v\|_2^2 = 1$. Therefore, we can rewrite Eq. (18) as

$$\begin{cases} -\Lambda u + Q^T v = -\lambda u, \\ Qu + Pv = \lambda v. \end{cases} \tag{19}$$

Multiplying both sides of the first and second equality in Eq. (19) by u^* and v^*, respectively, leads to

$$u^*\Lambda u - u^*Q^T v = \lambda\|u\|_2^2 \quad \text{and} \quad v^*Qu = \lambda\|v\|_2^2 - v^*Pv. \tag{20}$$

Combining the two equations in (20) and using $\|u\|_2^2 + \|v\|_2^2 = 1$, eventuate

$$u^*\Lambda u + v^*Pv - \bar{\lambda} + (\bar{\lambda} - \lambda)\|v\|_2^2 = 0. \tag{21}$$

By considering the real part of Eq. (21), we see that

$$a = u^*\Lambda u + v^*Pv \leqslant \|u\|_2^2\lambda_{\max}(\Lambda) + \|v\|_2^2\lambda_{\max}(P) \leqslant \max\{\lambda_{\max}(\Lambda), \lambda_{\max}(P)\}. \tag{22}$$

In the same way, we deduce that

$$a \geqslant \min\{\lambda_{\min}(\Lambda), \lambda_{\min}(P)\}. \tag{23}$$

On the other hand, the eigenvalues of Λ and $\tilde{A}$ are the same. Using the proof of Theorem 3.1 we want to study the upper bound of the eigenvalues of Λ. To do this, since Λ is symmetric, using the Courant-Fisher min-max theorem we conclude that

$$\begin{aligned} \lambda(\Lambda) &= \frac{x^*Ax + x^*B^TQ^{-1}Bx}{x^*Hx + x^*Ax + x^*B^TQ^{-1}Bx} \\ &\leqslant \frac{\lambda_{\max}(A)x^*x + \lambda_{\max}(Q^{-1})x^*B^TBx}{\lambda_{\min}(H)x^*x + \lambda_{\max}(A)x^*x + \lambda_{\max}(Q^{-1})x^*B^TBx} \end{aligned}$$

$$\leqslant \frac{\lambda_{\max}(A)x^*x + \lambda_{\max}(Q^{-1})\lambda_{\max}(B^T B)x^*x}{\lambda_{\min}(H)x^*x + \lambda_{\max}(A)x^*x + \lambda_{\max}(Q^{-1})\lambda_{\max}(B^T B)x^*x}$$

$$= \frac{\lambda_{\min}(Q)\lambda_{\max}(A) + \sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \lambda_{\max}(A)) + \sigma_{\max}^2}. \tag{24}$$

It follows from the proof of Theorem 3.1 and Eq. (24) that

$$\frac{\lambda_{\min}(A)}{\lambda_{\max}(H) + \lambda_{\min}(A)}I \preceq \Lambda \preceq \frac{\lambda_{\min}(Q)\lambda_{\max}(A) + \sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \lambda_{\max}(A)) + \sigma_{\max}^2}I. \tag{25}$$

We can rewrite the matrix P as $P = \tilde{B}\tilde{B}^T = \bar{B}(I + \bar{B}^T \bar{B})^{-1}\bar{B}^T$. Let $\bar{B} = U[\Sigma, 0]V^T$ be the singular value decomposition of the matrix $\bar{B}$ such that $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices and the matrix Σ is of the form $\Sigma = \mathrm{diag}(\tau_1, \tau_2, \cdots, \tau_r, 0, \ldots, 0) \in \mathbb{R}^{m\times m}$, where $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_r > 0$ are the nonzero singular values of the matrix $\bar{B}$. Hence,

$$P = U\Sigma(I + \Sigma^2)^{-1}\Sigma U^T$$

$$= U\mathrm{diag}(\frac{\tau_1^2}{1 + \tau_1^2}, \ldots, \frac{\tau_r^2}{1 + \tau_r^2}, 0, \ldots, 0)U^T.$$

Therefore, the nonzero eigenvalues of P satisfy

$$\frac{\tau_r^2}{1 + \tau_r^2} \leqslant \lambda(P) \leqslant \frac{\tau_1^2}{1 + \tau_1^2}, \tag{26}$$

where $\lambda(P)$ is a nonzero eigenvalue of P. Obviously, τ_1^2 is the largest eigenvalue of the matrix $\bar{B}\bar{B}^T = Q^{-\frac{1}{2}}B(H + A)^{-1}B^T Q^{-\frac{1}{2}}$. By using Courant-Fisher Min-Max theorem we obtain

$$x^*Q^{-\frac{1}{2}}B(H + A)^{-1}B^T Q^{-\frac{1}{2}}x = x^*Q^{-\frac{1}{2}}BH^{-\frac{1}{2}}(I + H^{-\frac{1}{2}}AH^{-\frac{1}{2}})^{-1}H^{-\frac{1}{2}}B^T Q^{-\frac{1}{2}}$$

$$\leqslant \lambda_{\max}(I + S)^{-1}x^*Q^{-\frac{1}{2}}BH^{-1}B^T Q^{-\frac{1}{2}}x$$

$$\leqslant \frac{1}{1 + \lambda_{\min}(S)}\frac{1}{\lambda_{\min}(H)}x^*Q^{-\frac{1}{2}}BB^T Q^{-\frac{1}{2}}x$$

$$\leqslant \frac{1}{1 + \lambda_{\min}(S)}\frac{1}{\lambda_{\min}(H)}\sigma_{\max}^2 x^*Q^{-1}x$$

$$\leqslant \frac{\sigma_{\max}^2}{(1 + \lambda_{\min}(S))\lambda_{\min}(H)\lambda_{\min}(Q)}x^*x, \tag{27}$$

where $S = H^{-\frac{1}{2}}AH^{-\frac{1}{2}}$. Furthermore, since S is symmetric, we can write that

$$x^*Sx = x^*H^{-\frac{1}{2}}AH^{-\frac{1}{2}}x$$

$$\geqslant \lambda_{\min}(A)x^*H^{-1}x$$

$$\geqslant \lambda_{\min}(H^{-1})\lambda_{\min}(A)\,x^*x$$

$$= \frac{\lambda_{\min}(A)}{\lambda_{\max}(H)}\,x^*x.$$

Thus

$$\lambda_{\min}(S) \geqslant \frac{\lambda_{\min}(A)}{\lambda_{\max}(H)}. \tag{28}$$

Form Eqs. (27) and (28), it is straightforward to see that

$$\tau_1^2 \leqslant \frac{\sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \frac{1}{\kappa(H)}\lambda_{\min}(A))}.$$

In the same way, we derive that

$$\frac{\sigma_{\min}^2}{\lambda_{\max}(Q)(\lambda_{\max}(H) + \kappa(H)\lambda_{\max}(A))} \leqslant \tau_r^2$$

$$\leqslant \tau_1^2 \leqslant \frac{\sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \frac{1}{\kappa(H)}\lambda_{\min}(A))}. \tag{29}$$

From Eq. (26) and (29) for the nonzero eigenvalues of P we have

$$\begin{cases} \lambda_{\min}(P) \geqslant \dfrac{\sigma_{\min}^2}{\lambda_{\max}(Q)(\lambda_{\max}(H) + \kappa(H)\lambda_{\max}(A)) + \sigma_{\min}^2}, \\ \lambda_{\max}(P) \leqslant \dfrac{\sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \frac{1}{\kappa(H)}\lambda_{\min}(A)) + \sigma_{\max}^2}. \end{cases} \tag{30}$$

Using Eqs. (22), (23), (25) and (30), for the nonzero real eigenvalues of $\mathcal{K}^{-1}\mathcal{A}$ we evaluate that

$$\min\left\{ \frac{\lambda_{\min}(A)}{\lambda_{\max}(H) + \lambda_{\min}(A)}, \frac{\sigma_{\min}^2}{\lambda_{\max}(Q)(\lambda_{\max}(H) + \kappa(H)\lambda_{\max}(A)) + \sigma_{\min}^2} \right\} \leqslant a$$

$$\leqslant \max\{\ell_1, \ell_2\},$$

where

$$\ell_1 = \frac{\lambda_{\min}(Q)\lambda_{\max}(A) + \sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \lambda_{\max}(A)) + \sigma_{\max}^2},$$

$$\ell_2 = \frac{\sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \frac{1}{\kappa(H)}\lambda_{\min}(A)) + \sigma_{\max}^2}.$$

Now, since $\ell_1 > \ell_2$, the desired result is obtained. □

4. Numerical illustration

In this section, to illustrate the theoretical results presented in Sections 2 and 3 we have taken the following example from [16]. In this example, we consider the saddle point problem (1) with

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2l^2 \times 2l^2},$$

$$B = (E, b_1, b_2)^T \in \mathbb{R}^{(l^2+1) \times 2l^2},$$

where

$$T = \frac{1}{h^2} \cdot \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{l \times l}, \qquad E = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2l^2 \times l^2},$$

$$b_1 = E \begin{pmatrix} e \\ 0 \end{pmatrix}, \qquad b_2 = E \begin{pmatrix} 0 \\ e \end{pmatrix}, \qquad e = (1, 1, \dots, 1)^T \in \mathbb{R}^{l^2/2},$$

and

$$F = \frac{1}{h} \mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{l \times l}.$$

Here, $h = 1/(l+1)$, $n = 2l^2$ and $m = l^2 + 2$. We choose $l = 16$.

We set $H = \alpha \mathrm{diag}(A)$ and $Q = \alpha I + \beta BB^T$. Obviously, for $\alpha, \beta > 0$, both of these matrices are SPD. The parameters α and β are set to be $\alpha = 0.05$ and $\beta = 0.01$. From Corollary 3.2 the eigenvalues of the matrix $\mathcal{K}^{-1}\mathcal{A}$ are contained in the intersection of the disks $|\lambda - 0.5| \leqslant 0.5$ and

$$|\lambda - 1| \leqslant \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\max}(H) + \lambda_{\min}(A)}} \approx 0.864.$$

These disks along with the eigenvalues of $\mathcal{K}^{-1}\mathcal{A}$ have been displayed in Figure 1 (right). Also, the eigenvalue distribution of the matrix $\mathcal{A}$ have been presented in Figure 1 (left). As we observe, all the nonzero eigenvalues having nonzero imaginary parts of the preconditioned matrix $\mathcal{K}^{-1}\mathcal{A}$ are located in the intersection of the disks $|\lambda - 0.5| \leqslant 0.5$ and $|\lambda - 1| \leqslant 1$. This confirms the Corollary 3.2. Another observation which can be posed here is the semi-convergence of the GHSS method for this example.

Now, we consider the nonzero real eigenvalues of the preconditioned matrix $\mathcal{K}^{-1}\mathcal{A}$. The eigenvalues of $\mathcal{K}^{-1}\mathcal{A}$ are contained in the interval $[0.042, 0.975]$. On the other hand, we have

$$t_1 = \frac{\lambda_{\min}(A)}{\lambda_{\max}(H) + \lambda_{\min}(A)} \approx 0.254,$$

$$t_2 = \frac{\sigma_{\min}^2}{\lambda_{\max}(Q)(\lambda_{\max}(H) + \kappa(H)\lambda_{\max}(A)) + \sigma_{\min}^2} \approx 4.660 \times 10^{-5},$$

$$t_3 = \frac{\lambda_{\min}(Q)\lambda_{\max}(A) + \sigma_{\max}^2}{\lambda_{\min}(Q)(\lambda_{\min}(H) + \lambda_{\max}(A)) + \sigma_{\max}^2} \approx 1.$$

Hence the interval provided by Theorem 3.3 is equal to $[\min\{t_1, t_2\}, t_2] = [4.660 \times 10^{-5}, 1]$. As we see $[0.042, 0.975] \subseteq [4.660 \times 10^{-5}, 1]$. This confirms the result presented in Theorem 3.3.
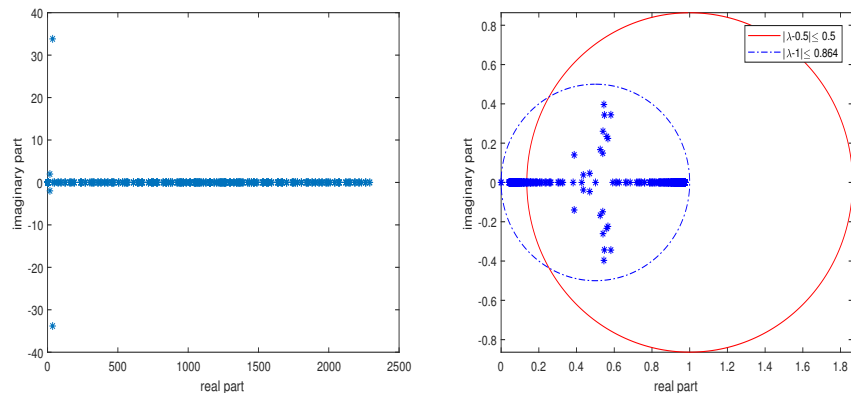
FIGURE 1. Eigenvalue distribution of the saddle point matrix
$\mathcal{A}$ (left) and the preconditioned matrices $\mathcal{K}^{-1}\mathcal{A}$ (right).

5. Conclusion

We have presented the spectral analysis of the matrix $\mathcal{K}^{-1}\mathcal{A}$ where the matrix
$\mathcal{A}$ is the singular saddle point matrix and $\mathcal{K}$ is the modified generalized shift-
splitting preconditioner. We have provided some bounds for the nonzero complex
eigenvalues and nonzero real eigenvalues of the matrix $\mathcal{K}^{-1}\mathcal{A}$. Using an example
we have illustrated the presented theoretical results.

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