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PSEUDO P-CLOSURE WITH RESPECT TO IDEALS IN PSEUDO BCI-ALGEBRAS[†]

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ABSTRACT. In this paper, for any non-empty subsets A, I of a pseudo BCI-algebra X, we introduce the concept of pseudo p-closure of A with respect to I, denoted by A_I^{pc}, and investigate some related properties. Applying this concept, we state a necessary and sufficient condition for a pseudo BCI-algebra 1) to be a p-semisimple pseudo BCI-algebra; 2) to be a pseudo BCK-algebra. Moreover, we show that $A_{\{0\}}^{pc}$ is the least positive pseudo ideal of X containing A, and characterize it by the union of some branches. We also show that the set of all pseudo ideals of X which $A_I^{pc} = A$, is a complete lattice. Finally, we prove that this notion can be used to define a closure operation.

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1. Introduction

The notion of BCI-algebras has been introduced by K. Iséki in 1966 (see [8]). BCI-algebras are algebraic formulation of the BCI-system in combinatory logic which has application in the language of functional programming. The name of BCI-algebras originates from the combinatories B, C, I in combinatory logic.

The notion of pseudo-BCI-algebras has been introduced by W. A. Dudek and Y. B. Jun in [2] as an extension of BCI-algebras and it was investigated by several authors in [3], [10] and [12]. These algebras have connections with pseudo BCK-algebras, pseudo BL-algebras and pseudo MV-algebras introduced by G. Georgescu and A. Iorgulescu in [4], [5] and [6], respectively. More about those algebras the reader can find in [7].

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Ideals of algebras are important algebraic notion and for pseudo BCI-algebras, and they have been extensively investigated by many authors. Y. B. Jun et al in [10] introduced the concepts of pseudo-atoms, pseudo ideals and pseudo BCI-homomorphisms in pseudo BCI-algebras. They displayed characterizations of a pseudo ideal, and provided conditions for a subset to be a pseudo ideal. They also introduced the notion of a $\circ$-medial pseudo BCI-algebra, and gave its characterization.

The aim of this paper is to introduce and study the concept of pseudo p-closure with respect to any non-empty subset of a pseudo BCI-algebra. This paper is organized as follow: in section 2, we recall the notions of BCI-algebras and pseudo BCI-algebras; and some properties of pseudo BCI-algebras. In section 3, we introduce the concept of p-closure with respect to a non-empty subset in a pseudo BCI-algebra and study some related properties. Also, using the mentioned concept, we give a necessary and sufficient condition for a pseudo BCI-algebra to be a p-semisimple pseudo BCI-algebra. We show that $A^{pc}_{\{0\}}$ is the least positive ideal of X containing A. We prove that the set of all ideals A of X which $I \subseteq A$ and $A^{pc}_I = A$, is a complete lattice. For the first time, Moore in [15] introduced a closure operation on a set. Using the concept of p-closure, we introduce a closure operation on the set of all ideals of X. Finally, we investigate the quotient algebra of X, induced by A^{pc}_I, and obtain some related results.

2. Preliminary

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X, *, 0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions:

- $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,
- $(\forall x \in X)$ $(x * 0 = x)$,
- $(\forall x, y \in X)$ $(x * y = 0$ and $y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

- $(\forall x \in X)$ $(0 * x = 0)$,

then we say that X is a BCK-algebra. Any BCI-algebra X satisfies the following conditions: [16]

(a_1) $(\forall x \in X)$ $(x * x{=}0)$,
(a_2) $(\forall x, y, z \in X)$ $((x * y) * z = (x * z) * y)$,
(a_3) $(\forall x, y, z \in X)$ $(x \leq y \Rightarrow x * z \leq y * z$, $z * y \leq z * x)$,
(a_4) $(\forall x, y \in X)$ $(x * (x * (x * y)) = x * y)$,
(a_5) $(\forall x, y \in X)$ $(0 * (x * y) = (0 * x) * (0 * y))$,

where $x \leq y$ if and only if $x * y = 0$.

A non-empty subset A of a BCI-algebra X is called a BCI-ideal of X if it satisfies:

- $0 \in A$,

- $(\forall x, y \in X)\ y * x \in A,\ x \in A \Rightarrow y \in A$.

Definition 2.1. A pseudo BCI-algebra is a structure $\mathfrak{X} = (X, \preceq, *, \circ, 0)$, where "$\preceq$" is a binary on a set X, "$*$", and "$\circ$" are binary operations on X and "0" is an element of X, verifying the axioms: for all $x, y, z \in X$,

(b_1) $(x * y) \circ (x * z) \preceq z * y$, $(x \circ y) * (x \circ z) \preceq z \circ y$,
(b_2) $x * (x \circ y) \preceq y$, $x \circ (x * y) \preceq y$,
(b_3) $x \preceq x$,
(b_4) $x \preceq y,\ y \preceq x \Longrightarrow x = y$,
(b_5) $x \preceq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \circ y = 0$.

Note that every pseudo BCI-algebra satisfying $x * y = x \circ y$ for all $x, y \in X$ is a BCI-algebra. Every pseudo BCK-algebra is a pseudo BCI-algebra.

Proposition 2.2. [4] *In a pseudo BCI-algebra* $\mathfrak{X}$ *the following holds:*

(p_1) $x \preceq 0 \Rightarrow x = 0$.
(p_2) $x \preceq y \Rightarrow z * y \preceq z * x,\ z \circ y \preceq z \circ x$.
(p_3) $x \preceq y,\ y \preceq z \Rightarrow x \preceq z$.
(p_4) $(x * y) \circ z = (x \circ z) * y$.
(p_5) $x * y \preceq z \Leftrightarrow x \circ z \preceq y$.
(p_6) $(x * y) * (z * y) \preceq x * z,\ (x \circ y) \circ (z \circ y) \preceq x \circ z$.
(p_7) $x \preceq y \Rightarrow x * z \preceq y * z,\ x \circ z \preceq y \circ z$.
(p_8) $x * 0 = x = x \circ 0$.
(p_9) $x * (x \circ (x * y)) = x * y$ *and* $x \circ (x * (x \circ y)) = x \circ y$.
(p_{10}) $0 * (x \circ y) \preceq y \circ x$.
(p_{11}) $0 \circ (x * y) \preceq y * x$.
(p_{12}) $0 * (x * y) = (0 \circ x) \circ (0 * y)$.
(p_{13}) $0 \circ (x \circ y) = (0 * x) * (0 \circ y)$.
(p_{14}) $0 * x = 0 \circ x$.

Example 2.3. [10] Let $X = [0, \infty)$ and $\preceq$ be the usual order on X. Define binary operation $*$ and $\circ$ on X by

$$x * y = \begin{cases} 0 & \text{if } x \preceq y \\ \frac{2x}{\pi} \arctan(\ln(\frac{x}{y})) & \text{if } y \prec x, \end{cases}$$

$$x \circ y = \begin{cases} 0 & \text{if } x \preceq y \\ x e^{-\tan(\frac{\pi y}{2x})} & \text{if } y \prec x, \end{cases}$$

for all $x, y \in X$. Then $\mathfrak{X} = (X, \preceq, *, \circ, 0)$ is a pseudo BCK-algebra, and hence it is a pseudo BCI-algebra.

By a *subalgebra* of a pseudo BCI-algebra $\mathfrak{X}$, we mean a non-empty subset S of $\mathfrak{X}$ which satisfies

$$x * y \in S \text{ and } x \circ y \in S,$$

for all $x, y \in S$.

A subset A of X is called a pseudo ideal of $\mathfrak{X}$ if it satisfies for all $x, y \in X$:

- $0 \in A$,
- if $x * y, x \circ y \in A$ and $y \in A$, then $x \in A$.

A pseudo ideal A of a pseudo BCI-algebra $\mathfrak{X}$ is called closed if A is a subalgebra of $\mathfrak{X}$.

Theorem 2.4. *An ideal A of a pseudo BCI-algebra $\mathfrak{X}$ is closed if and only if for any $x \in A$, $0 * x = 0 \circ x \in A$.*

Proposition 2.5. [10] *For any pseudo BCI-algebra $\mathfrak{X}$ the set*

$$K(X)=\{x \in X \mid 0 \preceq x\}$$

is a subalgebra of $\mathfrak{X}$, and so it is a pseudo BCK-algebra. Any subset or element of $K(X)$ is called positive.

Definition 2.6. [10] A pseudo BCI-algebra $\mathfrak{X}$ is said to be $\circ$-medial if it satisfies the following identity:

$$(x * y) \circ (z * u) = (x * z) \circ (y * u)$$

for all $x, y, z, u \in X$.

Proposition 2.7. [10] *Every $\circ$-medial pseudo BCI-algebra $\mathfrak{X}$ satisfies the following identities:*

(i) $x * y = 0 \circ (y * x)$.
(ii) $0 \circ (0 * x) = x$.
(iii) $x \circ (x * y) = y$.

An element a of a pseudo BCI-algebra $\mathfrak{X}$ is called a pseudo-atom of $\mathfrak{X}$ if for every $x \in X$ the following holds:

$$x \preceq a \Rightarrow x = a.$$

We will denote by $M(X)$ the set of all atoms of $\mathfrak{X}$. Obviously,

$$0 \in M(X) \cap K(X).$$

Notice that $M(X) \cap K(X) = \{0\}$ and for every $x \in X$, $0 * x \in M(X)$.

A pseudo BCI-algebra $\mathfrak{X}$ is said to be *p-semisimple* if it satisfies for all $x \in X$.

$$0 \preceq x \Rightarrow x = 0.$$

Note that if $\mathfrak{X}$ is a *p-semisimple* pseudo BCI-algebra, then $K(X) = 0$.

Let $\mathfrak{X}$ be a pseudo BCI-algebra. For $a \in M(X)$, define

$$V(a) = \{x \in X \mid a \preceq x\}.$$

$V(a)$ is called a branch of $\mathfrak{X}$. Notice also that $V(0) = K(X)$ and it is a pseudo BCK-part of $\mathfrak{X}$.

Proposition 2.8. [3] *Let $\mathfrak{X}$ be a pseudo BCI-algebra. Then*

$$X = \bigcup_{a \in M(X)} V(a).$$

A mapping $f : E \to E$ is said to be a closure operation on an ordered set $(E, \leq)$ if it satisfies the following properties:
(i) $x \leq f(x)$ (extensivity),
(ii) $x \leq y \Rightarrow f(x) \leq f(y)$, (isotony),
(iii) $f(f(x)) = f(x)$ (idempotence).

Theorem 2.9. [1] *Let L be a lattice and let $f : L \to L$ be a closure. Then Imf is a lattice in which the lattice operations are given by*

$$inf\{a, b\} = a \wedge b \quad , \; sup\{a, b\} = f(a \vee b).$$

3. pseudo p-closure with respect to ideals

In this section, we introduce the concept of p-closure of A with respect to I, for any non-empty subsets A and I of $\mathfrak{X}$ and establish some useful related properties. In what follows, let $\mathfrak{X}$ denote a pseudo BCI-algebra unless otherwise specified.

Definition 3.1. For any non-empty subsets I and A of $\mathfrak{X}$, we define the p-closure of A with respect to I by

$$A_I^{pc} = \{x \in X \mid a * x \in I, \, a \circ x \in I \text{ for some } a \in A\}.$$

Note that in special case, when $I = A$, we write $A_I^{pc} = A^{pc}$.

The following lemma is an immediate consequence from Definition 3.1.

Lemma 3.2. *For any subsets I, J, A, B of $\mathfrak{X}$, the following hold:*
(i) *$I \cap A \neq \emptyset$ if and only if $0 \in A_I^{pc}$,*
(ii) *if $0 \in I$, then $A \subseteq A_I^{pc}$.*
(iii) *if $A \subseteq B$, then $A_I^{pc} \subseteq B_I^{pc}$,*
(iv) *if $I \subseteq J$, then $A_I^{pc} \subseteq A_J^{pc}$.*

In the following theorem, we introduce some subsets of $\mathfrak{X}$ whose p-closure with respect to a subset of X, is equal to the pseudo BCK-part of $\mathfrak{X}$.

Theorem 3.3. *Let I, A be non-empty subsets of $\mathfrak{X}$. Then the following hold:*
(i) *if I is positive containing 0, then $(K(X))_I^{pc} = K(X)$,*
(ii) *if A is positive and $0 \in I \subseteq A$, then $A_I^{pc} = K(X)$,*
(iii) *for any pseudo-atom element a of X, $\{V(a)\}_{\{a\}}^{pc} = K(X)$.*

Proof. (i) By Lemma 3.2, $K(X) \subseteq (K(X))_I^{pc}$. To show the reverse inclusion, let $x \in (K(X))_I^{pc}$. Thus there exists $a \in K(X)$ such that $a * x \in I$ and $a \circ x \in I$. It follows that $0 * (a * x) = 0$. Hence by (p_{14}) we have

$$0 * (0 * x) = (0 \circ a) * (0 \circ x) = 0 \circ (a \circ x) = 0,$$

that is, $0 \preceq 0 * x$. Since $0 * x$ is a pseudo-atom we get, $0 * x = 0$ and so $x \in K(X)$. Therefore $(K(X))_I^{pc} = K(X)$.

(ii) Since $A \subseteq K(X)$, it follows from (i) and Lemma 3.2 that $A_I^{pc} \subseteq (K(X))_I^{pc} = K(X)$. On the other hand, by $0 \in I \cap A$, we can see that $K(X) \subseteq A_I^{pc}$. Therefore (ii) holds.

(iii) Let $x \in K(X)$. Then $0 * x = 0$. Now, since

$$(a * x) \circ a = (a \circ a) * x = 0 * x = 0,$$

we get $a * x \preceq a$ and so we have $a * x = a$. Similarly, $a \circ x = a$. This implies that $x \in \{V(a)\}^{pc}_{\{a\}}$. In order to show the reverse inclusion, let $x \in \{V(a)\}^{pc}_{\{a\}}$. Then $t * x = a$ for some $a \preceq t$. Thus by (p_7), $a * x \preceq t * x = a$ and so we get $a * x = a$. Hence we have

$$0 * x = (a \circ a) * x = (a * x) \circ a = a \circ a = 0,$$

that is, $x \in K(X)$. Therefore $\{V(a)\}^{pc}_{\{a\}} = K(X)$. □

In the following example, we show that the condition 0 belong to I in Theorem 3.3 (ii) is necessary.

Example 3.4. Let $X = \{0, a, b, c, d\}$ be a pseudo BCI-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	b	b	d
b	b	0	0	b	d
c	c	0	0	0	d
d	d	d	d	d	0

$\circ$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	c	a	d
b	b	0	0	b	d
c	c	0	0	0	d
d	d	d	d	d	0

Taking $A := \{a, b\}$ and $I := \{b\}$. It can be check that $A^{pc}_I = \{0, c\}$ while $K(X) = \{0, a, b, c\}$. Therefore $A^{pc}_I \neq K(X)$.

Proposition 3.5. *For any element c of $\mathfrak{X}$, $(A(c))^{pc}_{\{c\}}$ is a positive closed ideal of $\mathfrak{X}$, where $A(c) = \{x \in X \mid x \preceq c\}$.*

Proof. Because of $c \circ 0 = c * 0 = c$, we have $0 \in (A(c))^{pc}_{\{c\}}$. We assert that any element x in $(A(c))^{pc}_{\{c\}}$ is positive. In fact, let $x \in (A(c))^{pc}_{\{c\}}$. Then there exists $t \preceq c$ such that $t \circ x = t * x = c$. Now

$$0 * x = (t \circ c) * x = (t * x) \circ c = c \circ c = 0,$$

as asserted. Now, for any $x, y * x \in (A(c))^{pc}_{\{c\}}$, there exist $t_1, t_2 \preceq c$ such that $t_1 * x = t_1 \circ x = c$ and $t_2 * (y * x) = t_2 \circ (y * x) = c$. By the positivity of x and $y * x$, we have

$$(t_1 * y) \circ c = (t_1 \circ c) * y = 0 * y = (0 * y) \circ (0 * x) = 0 * (y * x) = 0,$$

that is, $t_1 * y \preceq c$. Also, from $t_1 * x = c$ and $t_2 * (y * x) = c$, it yields

$$c = t_2 * (y * x) \preceq c * (y * x) = (t_1 * x) * (y * x) \preceq t_1 * y.$$

Hence $t_1 * y = c$ and so $y \in (A(c))^{pc}_{\{c\}}$. We have shown that $(A(c))^{pc}_{\{c\}}$ is a positive ideal of X. Also, since for any $x \in (A(c))^{pc}_{\{c\}}$, we have $0 * x = 0 \in (A(c))^{pc}_{\{c\}}$, it follows from Theorem 2.4 that $(A(c))^{pc}_{\{c\}}$ is closed. □

Theorem 3.6. *A pseudo BCI-algebra $\mathfrak{X}$ is a pseudo BCK-algebra if and only if $\{0\}_I^{pc} = X$ for any subset I containing* 0.

Proof. Straightforward. □

Theorem 3.7. *A pseudo BCI-algebra $\mathfrak{X}$ is p-semisimple if and only if $\{0\}_I^{pc} = \{0\}$, for any positive subset I of $\mathfrak{X}$ containing* 0.

Proof. Let $x \in \{0\}_I^{pc}$. Thus $0 * x = 0 \circ x \in I$ and so $0 * (0 \circ x) = 0 \circ (0 * x) = 0$. But $0 \circ (0 * x) = x$ and so we have $x = 0$. Therefore $\{0\}_I^{pc} = \{0\}$.

Conversely, assume that $\{0\}_I^{pc} = \{0\}$. For any $x \in K(X)$, we have $0 * x = 0 \circ x = 0$. But $0 \in I$ and so $x \in \{0\}_I^{pc}$. Thus $x = 0$, and this implies that $K(X) = \{0\}$. Now let $x \in X$. Since

$$0 \circ (x * (0 \circ (0 * x))) = (0 \circ x) \circ (0 \circ (0 \circ (0 * x))) = (0 \circ x) \circ (0 * x) = 0,$$

we have $x * (0 \circ (0 * x)) \in K(X)$ and so $x * (0 \circ (0 * x)) = 0$. Obviously $(0 \circ (0 * x)) * x = 0$ and so $0 \circ (0 * x) = x$. Therefore $x \in M(X)$ and we get $X = M(X)$. Thus X is a p-semisimple. □

Lemma 3.8. *For any subset I of pseudo BCI-algebra $\mathfrak{X}$ containing 0,*

$$(M(X))_I^{pc} = X.$$

Proof. Let $x \in X$. It follows from Proposition 2.8 that $x \in V(t)$ for some pseudo-atom element t of X. Hence $t * x = t \circ x = 0$. This implies that $x \in (M(X))_I^{pc}$ and so the proof is completed. □

Theorem 3.9. *Let A be a subalgebra of $\mathfrak{X}$ and $0 \in I \subseteq A$. Then*

(i) *$x \in A_I^{pc}$ if and only if $0 * x \in A$,*
(ii) *A_I^{pc} is a subalgebra of X containing A.*

Proof. (i) ($\Rightarrow$) Let $x \in A_I^{pc}$. Then there exists $a \in A$ such that $a * x \in I$ and $a \circ x \in I$. Since A is a subalgebra of X, we get $(a * x) \circ a \in A$ and $(a \circ x) * a \in A$. Therefore $0 * x = 0 \circ x \in A$.

($\Leftarrow$) Let $0 * x \in A$. By (p_4) and (p_{14}), $(0 * (0 * x)) \circ x = (0 \circ x) * (0 * x) = 0$ and similarly $(0 * (0 * x)) * x = 0$. It follows from $0 \in I$ and $0 * (0 * x) \in A$ that $x \in A_I^{pc}$.

(ii) Since $0 \in I$, by Lemma 3.2, we have $A \subseteq A_I^{pc}$ and so it remains to show that A_I^{pc} is a subalgebra of X. Let $x, y \in A_I^{pc}$. Then there exist $a, b \in A$ such that

$$\begin{cases} a * x \in I \\ a \circ x \in I, \end{cases} \qquad \begin{cases} b * y \in I \\ b \circ y \in I. \end{cases}$$

Thus by the closeness of A and $I \subseteq A$, we have

$$\begin{cases} 0 * x \in A \\ 0 \circ x \in A, \end{cases} \qquad \begin{cases} 0 * y \in A \\ 0 \circ y \in A. \end{cases}$$

Now we show that $x*y \in A_I^{pc}$ and $x \circ y \in A_I^{pc}$. It follows by (p_{12}) and (p_{14}) that $0*(y*x) = (0*y)\circ(0*x) \in A$ and hence, we get

$$(0*(y*x))*(x*y) = ((0*y)\circ(0*x))*(x*y) = 0 \in I.$$

Also $(0*(y*x))\circ(x*y) = (0\circ(x*y))*(y*x) = 0 \in I$. Therefore $x*y \in A_I^{pc}$. Similarly, since $0\circ(y\circ x) \in A$ we can show that $x\circ y \in A_I^{pc}$. Therefore A_I^{pc} is a subalgebra of X. □

Theorem 3.10. *Let I, A be pseudo ideals of $\circ$-medial pseudo BCI-algebra $\mathfrak{X}$. Then A_I^{pc} is a pseudo ideal of $\mathfrak{X}$. Moreover, if I, A are closed, then so is A_I^{pc}.*

Proof. Obviously $0 \in A_I^{pc}$. Let $x, y*x \in A_I^{pc}$. Then there exist $a, b \in A$ such that

$$\begin{cases} a*x \in I \\ a\circ x \in I, \end{cases} \qquad \begin{cases} b*(y*x) \in I \\ b\circ(y*x) \in I. \end{cases}$$

Since $(b*(0*a))\circ b = (b\circ b)*(0*a) = 0*(0*a) \preceq a \in A$ and $b \in A$, we get $b*(0*a) \in A$. Applying (p_4) and (b_1) we have

$$((b*(0*a))*y)\circ(b*(y*x)) \preceq ((b*(0*a))\circ(b*(y*x))*y \preceq ((y*x)*(0*a))*y.$$

Now we show $((y*x)*(0*a))*y \preceq a*x$. For this,

$$\begin{aligned} (((y*x)*(0*a))*y))\circ(a*x) &= (((y*x)\circ(a*x))*(0*a))*y \\ &\preceq (((y*a)\circ(x*x))*(0*a))*y \\ &\preceq ((y*a)*(0*a))*y \\ &= (y*0)*y \\ &= y*y \\ &= 0 \in I, \end{aligned}$$

and from the definition of pseudo ideal we conclude that $(b*(0*a))*y \in I$. Now we show $(b*(0*a))\circ y \in I$. But,

$$\begin{aligned} ((b*(0*a))\circ y)*(b\circ(y*x)) &= ((b*y)\circ((0*a)*0))*((b*y)\circ(0*x)) \\ &= ((b*y)\circ(0*a))*((b*y)\circ(0*x)) \\ &\preceq (0*x)\circ(0*a) \\ &\preceq a*x \\ &\in I. \end{aligned}$$

Since I is a pseudo ideal and $b\circ(y*x) \in I$, we get $(b*(0*a))\circ y \in I$. Therefore $y \in A_I^{pc}$, and so A_I^{pc} is a pseudo ideal of $\mathfrak{X}$. Now we show that A_I^{pc} is closed. Let $x \in A_I^{pc}$. Then there exists $a \in A$ such that $a*x, a\circ x \in I$. Thus we have $0*(a*x) \in I$ and $0*a \in A$. On the other hand, by (p_{12}), we have

$$(0*a)\circ(0*x) = 0*(a*x).$$

Therefore $0*x \in A_I^{pc}$ and so the result is obtained. □

Example 3.11. Consider the $\circ$-medial pseudo BCI-algebra $\mathfrak{X} = (\mathbb{Z}, -, 0)$ which $x * y = x \circ y = x - y$, and note that $A = \mathbb{N}$ is a pseudo ideal of $\mathfrak{X}$ where $\mathbb{N}$ is the set of non-negative integers. Taking $I := \{0\}$, by some calculations, we can see that $A_I^{pc} = \mathbb{N}$. Thus A_I^{pc} is an ideal of $\mathfrak{X}$ which is not closed because $1 * 2 = -1 \notin A_I^{pc}$.

Remark 3.1. For subsets A and I of $\mathfrak{X}$ with $I \subseteq A$, A_I^{pc} is not necessary to be an ideal of $\mathfrak{X}$ in general as seen in the following example.

Example 3.12. Let $X = \{0, a, b, c\}$ be a pseudo BCI-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	b	b
b	b	0	0	b
c	c	0	0	0

$\circ$	0	a	b	c
0	0	0	0	0
a	a	0	c	a
b	b	0	0	b
c	c	0	0	0

Taking $A := \{b, c\}$ and $I := \{b\}$, by routine calculations, we can see that $A_I^{pc} = \{0, b\}$, which is not a pseudo ideal of $\mathfrak{X}$, because $c * b = 0 \in A_I^{pc}$ and $c \notin A_I^{pc}$.

Lemma 3.13. *For any two subsets I and A of X with $0 \in I \cap X$, A_I^{pc} contains $K(X)$.*

Proof. Let $x \in K(X)$. Then $0 \circ x = 0 * x = 0 \in I$. But $0 \in A$. It implies that $x \in A_I^{pc}$. Therefore $K(X) \subseteq A_I^{pc}$. $\square$

Now, we characterization the $A_{\{0\}}^{pc}$ by some branches.

Theorem 3.14. *Let A be a pseudo ideal of $\mathfrak{X}$. Then $A_{\{0\}}^{pc} = \bigcup_{x \in A \cap M(X)} V(x)$.*

Proof. Assume that $y \in \bigcup_{x \in A \cap M(X)} V(x)$. Then there exists $x \in A \cap M(X)$ such that $y \in V(x)$. Hence, we have $x \preceq y$ and so $x * y = x \circ y = 0$. Thus, by $x \in A$, we get $y \in A_{\{0\}}^{pc}$. Therefore $\bigcup_{x \in A \cap M(X)} V(x) \subseteq A_{\{0\}}^{pc}$. To show the reverse inclusion, let $z \in A_{\{0\}}^{pc}$. Then there exists $a \in A$ such that $a * z = a \circ z = 0$. But by Proposition 2.8, $a \in V(b)$ for some pseudo-atom element b of X. Hence $b \preceq a$ and so $b \in A$. Also, we have $b * z \preceq a * z$. Thus, $b * z = 0$ and similarly $b \circ z = 0$. It follows that $z \in V(b)$. Therefore $z \in \bigcup_{x \in A \cap M(X)} V(x)$ and the proof is completed. $\square$

Corollary 3.15. *Let A be a pseudo ideal of $\mathfrak{X}$. Then the following statements are equivalent:*

(i) $K(X) \subseteq A$,
(ii) $A = A_{\{0\}}^{pc}$,
(iii) $A = \bigcup_{x \in A \cap M(X)} V(x)$.

Proof. (i)$\Rightarrow$ (ii) Let $K(X) \subseteq A$. By Lemma 3.2, $A \subseteq A^{pc}_{\{0\}}$. Now, let $x \in A^{pc}_{\{0\}}$. Then there exists $a \in A$ such that $a * x = a \circ x = 0$ and so $a \preceq x$. Thus, by (p_7), we get $0 \preceq x * a$ and $0 \preceq x \circ a$ which implies that $x * a, x \circ a \in K(X)$. Hence $x * a, x \circ a \in A$ and so from $a \in A$, we conclude that $x \in A$. Therefore $A^{pc}_{\{0\}} \subseteq A$, and so (i) holds.

(ii)$\Rightarrow$(iii) By Theorem 3.14, the result is obvious.

(iii)$\Rightarrow$ (i) Since $0 \in A \cap M(X)$, we get $V(0) \subseteq \bigcup_{x \in A \cap M(X)} V(x)$ and so $V(0) \subseteq A$. Hence $K(X) \subseteq A$. □

Theorem 3.16. *For any pseudo ideal A of $\mathfrak{X}$, $A^{pc}_{\{0\}}$ is a pseudo ideal of $\mathfrak{X}$.*

Proof. Obviously $0 \in A^{pc}_{\{0\}}$. Let $x, y * x \in A^{pc}_{\{0\}}$. Then there exist $a, b \in A$ such that $a \preceq x$ and $b \preceq y * x$. Since $(b * (0 * a)) \circ b = (b \circ b) * (0 * a) = 0 * (0 * a) \preceq a \in A$ and $b \in A$, we get $b * (0 * a) \in A$. From $(p_7), (p_6)$ and $b \preceq y * x$, we get $b * (0 * x) \preceq (y * x) * (0 * x) \preceq y$, and so $(b * (0 * x)) * y = 0$. On the other hand, by $a \preceq x$ and (p_2), we have $0 * x = 0 * a$, which implies that

$$(b * (0 * a)) * y = 0. \tag{1}$$

Also, from (p_7) and $b \preceq y * x$, we get $b \circ y \preceq 0 * x$, and so $(b * (0 * a)) \circ y = (b \circ y) * (0 * a) \preceq (0 * x) * (0 * a) = (0 \circ x) * (0 \circ a) \preceq a \circ x = 0$, which implies that

$$(b * (0 * a)) \circ y = 0. \tag{2}$$

Using (1) and (2), we get $y \in A^{pc}_{\{0\}}$ and the proof is completed. □

Theorem 3.17. *For any pseudo closed ideal A of $\mathfrak{X}$, $A^{pc}_{\{0\}}$ is closed.*

Proof. Let $x \in A^{pc}_{\{0\}}$. Then there exists $a \in A$ such that $a * x = a \circ x = 0$ and so $0 * (a * x) = 0$. By (p_{12}), we have $(0 \circ a) \circ (0 * x) = 0 * (a * x) = 0$. Similarly, by (p_{13}) and (p_{14}), we get $(0 \circ a) * (0 * x) = 0 \circ (a \circ x) = 0$, , which the closeness of A implies that $0 * x \in A^{pc}_{\{0\}}$. Using Theorem 2.4 we get $A^{pc}_{\{0\}}$ is closed. □

Remark 3.2. The closed condition of ideal A in Theorem 3.17 is necessary as we see in Example 3.11.

Theorem 3.18. *For any pseudo ideal A of $\mathfrak{X}$, $A^{pc}_{\{0\}}$ is the least positive pseudo ideal containing A.*

Proof. By Lemmas 3.2 and 3.13, $A \subseteq A^{pc}_{\{0\}}$ and $K(X) \subseteq A^{pc}_{\{0\}}$. Let C be another positive pseudo ideal of $\mathfrak{X}$ containing A. Now, let $x \in A^{pc}_{\{0\}}$. By Lemma 3.2, we have $x \in C^{pc}_{\{0\}}$, and so by Corollary 3.15, we get $x \in C$. Therefore $A^{pc}_{\{0\}} \subseteq C$ and so $A^{pc}_{\{0\}}$ is the least positive pseudo ideal containing A. □

In the following, we establish another important property of the p-closure of an ideal with respect to an ideal.

Theorem 3.19. *For any two pseudo ideals I and A of $\mathfrak{X}$, $(A^{pc}_I)^{pc}_I = A^{pc}_I$.*

Proof. Using Lemma 3.2, we have $A_I^{pc} \subseteq (A_I^{pc})_I^{pc}$. Let $x \in (A_I^{pc})_I^{pc}$. Then there exist $a \in A_I^{pc}$ and $b \in A$ such that $a * x, a \circ x \in I$ and $b * a, b \circ a \in I$. Now, since $(b * x) \circ (b * a) \preceq a * x \in I$, we have $b * x \in I$ and similarly $b \circ x \in I$. Therefore $x \in A_I^{pc}$ and we get $(A_I^{pc})_I^{pc} = A_I^{pc}$. □

Theorem 3.20. *For any pseudo ideals I, A, B of $\mathfrak{X}$, if $I \subseteq A, B$, then*

$$(A \cap B)_I^{pc} = A_I^{pc} \cap B_I^{pc}.$$

Proof. By Lemma 3.2, we have $(A \cap B)_I^{pc} \subseteq A_I^{pc} \cap B_I^{pc}$. Let $x \in A_I^{pc} \cap B_I^{pc}$. Then there exist $a \in A$ and $b \in B$ such that

$$\begin{cases} a * x \in I \\ a \circ x \in I, \end{cases} \qquad \begin{cases} b * x \in I \\ b \circ x \in I. \end{cases}$$

First, we show that $(b * x) \circ (x * a) \in I$. For this, we have

$$\begin{aligned} ((b * x) \circ (x * a)) * (b * x) &= ((b * x) * (b * x)) \circ (x * a) \\ &= 0 \circ (x * a) \\ &\preceq a * x \in I. \end{aligned}$$

Thus, since I is an ideal of X, we get $(b * x) \circ (x * a) \in I$. Taking $y = b \circ (x * a)$, we get

$$y * b = (b \circ (x * a)) * b = (b * b) \circ (x * a) = 0 \circ (x * a) \preceq a * x \in I \subseteq B$$

and so $y \in B$. Similarly, $y * (b \circ x) = (b \circ (x * a)) * (b \circ x) \preceq x \circ (x * a) \preceq a \in A$ and so we have $y \in A$. Thus $y \in A \cap B$. But $y * x = (b \circ (x * a)) * x = (b * x) \circ (x * a) \in I$. Therefore $x \in (A \cap B)_I^{pc}$ and so the proof is completed. □

Theorem 3.21. *Let I be a pseudo ideal of $\mathfrak{X}$ and define*

$$\mathcal{A}(I) := \{I \subseteq A \mid A \textit{ is a pseudo ideal which } \quad A_I^{pc} = A\}.$$

Then $(\mathcal{A}(I), \subseteq)$ is a complete lattice.

Proof. Clearly, $X \in \mathcal{A}(I)$ and $(\mathcal{A}(I), \subseteq)$ is a partially ordered set. Let $A, B \in \mathcal{A}(I)$. Then, by Theorem 3.20, $A \cap B \in \mathcal{A}(I)$ and by using Theorem 3.19, $\langle A \cup B \rangle_I^{pc} \in \mathcal{A}(I)$. Define $A \wedge B = A \cap B$ and $A \vee B = \langle A \cup B \rangle_I^{pc}$. Let $C \in \mathcal{A}(I)$ such that $A, B \subseteq C$. Then, $\langle A \cup B \rangle \subseteq C$ and hence $\langle A \cup B \rangle_I^{pc} \subseteq C_I^{pc} = C$. Now, $\langle A \cup B \rangle_I^{pc}$ is a l.u.b of A, B. Hence, $(\mathcal{A}(I), \wedge, \vee, \subseteq)$ is a lattice. Now, let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of ideals of $\mathcal{A}(I)$. By simple calculation we can get that $\bigwedge_{\alpha \in \Lambda} A_\alpha = \bigcap_{\alpha \in \Lambda} A_\alpha$ and $\bigvee_{\alpha \in \Lambda} A_\alpha = \langle \bigcup_{\alpha \in \Lambda} A_\alpha \rangle_I^{pc}$, hence $\mathcal{A}(I)$ is a complete lattice. □

In the following theorem, we show that the notion of p-closure ideals introduces a closure operation on $(\mathcal{I}(X), \subseteq)$, where $\mathcal{I}(X)$ is denoted the set of all ideals of X.

Theorem 3.22. *For any pseudo ideal I of $\mathfrak{X}$, $f_I : \mathcal{I}(X) \to \mathcal{I}(X)$ defined by $f_I(A) = A_I^{pc}$ is a closure operation.*

Proof. Combining Lemma 3.2 and Theorem 3.19, the result is obvious. □

Let $L = (\mathcal{I}(X), \subseteq, \wedge, \vee)$ be the lattice of all pseudo ideals of $\mathfrak{X}$ where $A \wedge B = A \cap B$ and $A \vee B = \langle A \cup B \rangle$. Then we have the following theorem.

Theorem 3.23. *Let $L = (\mathcal{I}(X), \subseteq, \wedge, \vee)$ and let $f_I : L \to L$ be the closure operation as in Theorem 3.22. Then Imf is a lattice in which the lattice operations are given by $inf\{A, B\} = A \cap B$ and $sup\{A, B\} = \langle A \cup B \rangle_I^{pc}$.*

Proof. By Theorem 2.9 the result is obvious. □

Theorem 3.24. *Let I, A, B be pseudo ideals of $\mathfrak{X}$ with $I \subseteq A \subseteq B$. Then*

$$(B/I)^{pc}_{A/I} = B^{pc}_A/I.$$

Proof. By $I \subseteq A \subseteq B$, we get $A/I \subseteq B/I$. Now we have

$$\begin{aligned}
(B/I)^{pc}_{A/I} &= \{I_x \in X/I \mid I_b * I_x \in A/I, I_b \circ I_x \in A/I \text{ for some } I_b \in B/I\} \\
&= \{I_x \in X/I \mid I_{b*x} \in A/I, I_{b\circ x} \in A/I \text{ for some } I_b \in B/I\} \\
&= \{I_x \in X/I \mid b * x \in A, b \circ x \in A \text{ for some } b \in B\} \\
&= \{I_x \in X/I \mid x \in B^{pc}_A\} \\
&= B^{pc}_A/I.
\end{aligned}$$

□

Theorem 3.25. *Let I, A and J, B be pseudo ideals of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Then*

(i) $A^{pc}_I \times B^{pc}_J = (A \times B)^{pc}_{I\times J}$,

(ii) $(X/A^{pc}_I) \times (Y/B^{pc}_J) \simeq (X \times Y)/((A^{pc}_I)_0 \times (B^{pc}_J)_0)$.

Proof. (i) Let $(x, y) \in A^{pc}_I \times B^{pc}_J$. Then $x \in A^{pc}_I$ and $y \in B^{pc}_J$. Thus there exist $a \in A$ and $b \in B$ such that $a * x, a \circ x \in I$ and $b * y, b \circ y \in J$. It follows that $(a, b)*(x, y) = (a*x, b*y) \in I\times J$ and $(a, b)\circ(x, y) = (a\circ x, b\circ y) \in I\times J$ for some $(a, b) \in A \times B$. Therefore $(x, y) \in (A \times B)^{pc}_{I\times J}$ and so $A^{pc}_I \times B^{pc}_J \subseteq (A \times B)^{pc}_{I\times J}$. The proof of reverse inclusion is similar.

(ii) Consider the natural homomorphisms $\pi_X : X \to X/A^{pc}_I$ and $\pi_Y : Y \to Y/B^{pc}_J$ with $\pi_X(x) = (A^{pc}_I)_x$ and $\pi_Y(y) = (B^{pc}_J)_y$. Define the mapping $f : X \times Y \to X/A^{pc}_I \times Y/B^{pc}_J$ by $f(x, y) = (\pi_X(x), \pi_Y(y)) = ((A^{pc}_I)_x, (B^{pc}_J)_y)$. Clearly, f is an epimorphism. Moreover,

$$\begin{aligned}
kerf &= \{(x, y) \in X \times Y \mid f(x, y) = ((A^{pc}_I)_0, (B^{pc}_J)_0)\} \\
&= \{(x, y) \in X \times Y \mid (A^{pc}_I)_x = (A^{pc}_I)_0 \text{ and } (B^{pc}_J)_y = (B^{pc}_J)_0\} \\
&= \{(x, y) \in X \times Y \mid x, 0 * x \in A^{pc}_I \text{ and } y, 0 * y \in B^{pc}_J\} \\
&= (A^{pc}_I)_0 \times (B^{pc}_J)_0.
\end{aligned}$$

Therefore by the first isomorphism theorem, we have $(X\times Y)/((A^{pc}_I)_0\times(B^{pc}_J)_0) \simeq (X/A^{pc}_I) \times (Y/B^{pc}_J)$. □

References

1. T.S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer-Verlag, London, 2005.
2. W.A. Dudek, Y.B. Jun, *Pseudo BCI-algebras*, East Asian Math. J. **24** (2008), 187-190.
3. G. Dymek, *p-semisimple pseudo BCI-algebras*, J. Mult-Valued Logic Soft Comput. **19** (2012), 461-474.
4. G. Georgescu, A. Iorgulescu, *Pseudo BCK-algebras: an extension of BCK-algebras*, Combinatorics, computability and logic (Constant¸a, 2001), 97-114, Springer Ser. Discrete Math. Theor. Comput. Sci. Springer, London, 2001.
5. G. Georgescu and A. Iorgulescu, *Pseudo-BL algebras: a noncommutative extension of BL-algebras*, Abstracts of The Fifth International Conference FSTA 2000, Slovakia, February (2000), 90-92.
6. G. Georgescu and A. Iorgulescu, *Pseudo MV-algebras: a noncommutative extension of MV-algebras*, The Proceedings The Fourth International Symposium on Economic Informatics, INFOREC Printing House, Bucharest, Romania, May (1999), 961-968.
7. A. Iorgulescu, *Algebras of logic as BCK algebras*, Editura ASE, Bucharest, 2008.
8. K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), 26-29.
9. Y.B. Jun, *Characterizations of pseudo BCK-algebras*, Sci. Math. Jpn. **57**, (2003), 265-270.
10. Y.B. Jun, H.S. Kim, J. Neggers, *On pseudo BCI-ideals of pseudo BCI-algebras*, Mat. Vesnik **58** (2006), 39-46.
11. Y.B. Jun, M. Kondo, K.H. Kim, *Pseudo ideals of pseudo BCK-algebras*, Sci. Math. Jpn. **58** (2003), 93-97.
12. K.J. Lee, C.H. Park, *Some ideals of pseudo BCI-algebras*, J. Appl. Math. Informatics **27** (2009), 217-231.
13. Y.L. Liu, S.Y. Liu, Y. Xu, *Pseudo BCK-algebras and PD-posets*, Soft Comput **11** (2007), 91-101.
14. Y.L. Liu, J. Meng, X.H. Zhang, Z.C. Yue, *q-ideals and a-ideals in BCI-algebras*, Southeast Asian Bull. Math. **24**, (2000), 243-253.
15. E.H. Moore *Introduction to a form of general analysis*, The New Haven mathematical colloquium. Yale University Press, New Haven, 1910.
16. H. Yisheng, *BCI-Algebra*, Published by Science Press, 2006.
17. X.H. Zhang, H. Jiang, S.A. Bhatti,*On p-ideals of a BCI-algebra*, Punjab Univ. J. Math. **27** (1994), 121-128.

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