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SOME EXPLICIT PROPERTIES OF (p, q)-ANALOGUE EULER SUM USING (p, q)-SPECIAL POLYNOMIALS†

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ABSTRACT. In this paper we discuss some interesting properties of (p, q)-special polynomials and derive various relations. We gain some relations between (p, q)-zeta function and (p, q)-special polynomials by considering (p, q)-analogue Euler sum types. In addition, we derive the relationship between (p, q)-polylogarithm function and (p, q)-special polynomials.

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1. Introduction

Riemann zeta function is classically defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

at the positive integers. This zeta function converges in $Re(s) > 1$ and in particular, when $s = 1$, we can see that $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$. A class of analytic functions of a complex variable contains zeta-functions in number theory, and many mathematicians have studied this zeta function for zero-free regions, average values, counting zeros, and the Riemann hypothesis. They also have started expanding Riemann zeta functions, which are famous for the generalized Hurwitz zeta function and the Dedekind's zeta function, and several research results have been obtained. Furthermore, they have calculated even values of the Riemann zeta function using Fourier transformation and found that it is a very important function possessing various properties(see [4, 7]).

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Euler studied the linear sum and derived that the linear sums have evaluations in terms of zeta values in specific cases. This linear sum is defined as

$$S(s,t) = \sum_{n=1}^{\infty} \frac{H_n^{(s)}}{m^t}, \qquad \text{where} \quad H_n^{(r)} := \sum_{j=1}^{n} \frac{1}{j^r}.$$

From the above equation we can see typical evaluations of Euler sums as

$$\text{(i)} \quad \sum_{n\geq 1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \quad \sum_{n\geq 1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4), \quad \sum_{n\geq 1} \frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3),$$

$$\text{(ii)} \quad \sum_{n\geq 1}^{\infty} \frac{H_n^{(2)}}{n^4} = \zeta(3)^2 - \frac{1}{3}\zeta(6),$$

$$\text{(iii)} \quad \sum_{n\geq 1}^{\infty} \frac{H_n^{(2)}}{n^5} = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 10\zeta(7),$$

$$\text{(iv)} \quad \sum_{n\geq 1}^{\infty} \frac{(H_n)^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4),$$

$$\text{(v)} \quad \sum_{n\geq 1}^{\infty} \frac{(H_n)^3}{n^4} = \frac{231}{16}\zeta(7) - \frac{51}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5),$$

$\cdots$.

We can learn that the linear Euler sums have the symmetric property and some relations of zeta function and study the nonlinear sums which involve products of at least two harmonic numbers in [4,7,18]. Furthermore, in [8], we can find some properties of quadratic Euler sums and cubic and higher order Euler sums. In 2017, an expanded Euler sum combining q-numbers appeared in a paper by Xu and Zhang [18], in which we can see some evaluations of q-Euler sums. Research on q-difference equations appeared in intensive works by F. H. Jackson[12], R. D. Carmichael, T. E. Mason, and other authors[14-16].

In [1,3,5], R. Chakrabarti and R. Jagannathan, G. Brodimas, et al. and M. Arik, et al. introduced the concept of the (p,q)-number in order to unify various forms of q-oscillator algebras.

For any $n \in \mathbb{C}$, the (p,q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It is clear that (p,q)-number possesses the symmetric property and this number reduces to q-number when $p = 1$. In particular, we can see that $\lim_{q\to 1}[n]_{p,q} = n$ with $p = 1$(see [6-7,12]). By using the above numbers, many researchers have studied (p,q)-calculus(see [9-11,13-15]).

Definition 1.1. We define the (p,q)-derivative operator as

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \qquad x \neq 0,$$

and $D_{p,q}f(0) = f'(0)$.

The following properties of (p,q)-derivative operator are immediate.

Theorem 1.2. *For the operator $D_{p,q}$, the following hold :*
(i) Derivative of a product
$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

(ii) Derivative of a ratio
$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)} = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

From Definition 1.1, one has

$$\frac{1 - T_{p,q}}{\left(1 - \frac{q}{p}\right)x} g(x) = f\left(\frac{1}{p}x\right), \qquad T_{p,q}g(x) = g\left(\frac{q}{p}x\right).$$

Thus, we can see that

$$g(x) = \left(1 - \frac{q}{p}\right)\sum_{i=0}^{\infty} T_{p,q}^{i}\left\{xf\left(\frac{1}{p}x\right)\right\} = \left(1 - \frac{q}{p}\right)x\sum_{i=0}^{\infty}\left(\frac{q}{p}\right)^{i} f\left(\frac{q^{i}}{p^{i+1}}x\right).$$

If the series in the right hand side of the above is convergent, then we can see that the previous calculus is obviously valid. Let f be an arbitrary function. In [16], we note that the definition of (p,q)- integral is

$$\int f(x)d_{p,q}x = (p-q)x\sum_{k=0}^{\infty}\frac{q^{k}}{p^{k+1}}f\left(\frac{q^{k}}{p^{k+1}}x\right).$$

Since the product of differentiable functions is also differentiable in ordinary calculus, we can obtain

$$\int_a^b f(px)\left(D_{p,q}g(x)\right)d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)\left(D_{p,q}f(x)\right)d_{p,q}x,$$

which is the formula of (p,q)-integration by part. We note that $b = \infty$ is allowed.

Since (p,q)-numbers have some different properties from q-numbers although they are similar to q-numbers, we need to study (p,q)-numbers in detail. We hypothesized that the Euler sum would have some relations to Riemann zeta function and special polynomials when we combined (p,q)-numbers respectively. In addition, we conjectured the (p,q)-Euler sums will be evaluated from (p,q)-special polynomials.

Based on this idea, the main concern of this paper is to define Euler sum combining (p,q)-numbers and study some of their formulae. In addition, we find some relations between (p,q)-zeta function and (p,q)-special polynomial. Our paper is organised as follows: in Section 2, we define (p,q)-special polynomials. From this definition, we investigate some interesting properties of polynomials and derive some relations. In Section 3, we consider (p,q)-analogue Euler sum types and gain some relations between (p,q)-special polynomials and (p,q)-zeta function. Furthermore, we derive relationships between (p,q)-polylogarithm function and (p,q)-special polynomials.

2. Some properties of (p,q)-special polynomials

In this section, we define a (p,q)-special polynomial, $Li_{p,q}(m;x)$, which is related to (p,q)-polylogarithmic function and (p,q)-Riemann zeta function. This polynomial contains some interesting properties on $|q/p|<1$ and we find a few generalized theorems.

Definition 2.1. We define (p,q)-special polynomials as

$$Li_{p,q}(m;x)=\sum_{k=1}^{\infty}\frac{p^{m(k-1)}}{[k]_{p,q}^{m}}x^{k}.$$

From $m=1$ in Definition 2.1, we can define (p,q)-logarithmic function as

$$\ln_{p,q}(1-x)=-\sum_{k=1}^{\infty}\frac{p^{(k-1)}}{[k]_{p,q}}x^{k}=-Li_{p,q}(1;x).$$

From the above equation, we note that

$$\ln_{q}(1-x)=-Li_{1,q}(1;x)=-\mathrm{Li}_{q}(1;x).$$

Clearly, the (p,q)-polylogarithmic function tends to the classical polylogarithmic function when $p=1$ and q approches to 1.

$$-Li(1;x)=\ln(1-x).$$

We can also define the relation between (p,q)-special polynomials and (p,q)-Riemann zeta functions as follows:

$$Li_{p,q}(s;q/p)=\sum_{k=1}^{\infty}\frac{p^{s(k-1)}}{[k]_{p,q}^{s}}\left(\frac{q}{p}\right)^{k}=\zeta_{p,q}(s).$$

From the above equation, we can consider the q-special polynomials and q-Riemann zeta functions as

$$Li_{1,q}(s;q) = Li_q(s;q) = \sum_{k=1}^{\infty} \frac{1}{[k]_q^s} q^k = \zeta_q(s).$$

Definition 2.2. Let n be a nonnegative integer. Then we define

$$Li_{n,p,q}(m;x) = \sum_{k=1}^{n} \frac{p^{m(k-1)}}{[k]_{p,q}^m} x^k.$$

From Definition 2.2, we note that

$$Li_{n,p,q}(s;q/p) = \sum_{k=1}^{n} \frac{p^{s(k-1)}}{[k]_{p,q}^s} \left(\frac{q}{p}\right)^k = \zeta_{n,p,q}(s),$$

and

$$Li_{n,p,q}(1;1) = \sum_{k=1}^{n} \frac{p^{k-1}}{[k]_{p,q}} = Li_{n,p,q} = \zeta_{n,p,q}.$$

Definition 2.3. We define the transformed (p,q)-special polynomials as

$$\widetilde{Li}_{p,q}(m;x) = \sum_{k=1}^{\infty} \frac{(p-q)n}{[k+1]_{p,q}^m} p^{m(k-1)} x^{k+1}, \qquad |x| < 1.$$

The integral form of $\widetilde{Li}_{p,q}(m;t)$ on the interval $[0,x]$ is easy to show and can be used to define the analytic continuation of $\widetilde{Li}_{p,q}(m;x)$.

Theorem 2.4. *Let $m \geq 1$ and $|q/p| < 1$. Then we have*

$$D_{p,q} Li_{p,q}(m;x) = \frac{1}{px} Li_{p,q}(m-1;px).$$

Proof. Using the definition of the (p,q)-derivative, we find

$$\begin{aligned} D_{p,q} Li_{p,q}(m;x) &= \sum_{k=1}^{\infty} \frac{p^{m(k-1)}}{[k]_{p,q}^m} [k]_{p,q} x^{k-1} \\ &= \frac{1}{px} \sum_{k=1}^{\infty} \frac{p^{(m-1)(k-1)}}{[k]_{p,q}^{m-1}} (px)^k \\ &= \frac{1}{px} Li_{p,q}(m-1;px). \end{aligned}$$

Therefore, we complete the proof of Theorem 2.4. □

Corollary 2.5. *From Theorem 2.4, we can easily verify*

$$Li_{p,q}(m;x)=\int_0^x \frac{1}{pt}Li_{p,q}(m-1;pt)d_{p,q}t.$$

Lemma 2.6. *Let s,t be nonnegative integers. Then we have*

$$A_{p,q}(s,t;x)=A_{p,q}(s-1,t+1;x)+\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{(t+1)(m-1)+s(n-1)}x^{m+n}}{[m]_{p,q}^{t+1}[n]_{p,q}^{s}}.$$

Proof. Consider that

$$A_{p,q}(s,t;x)=\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{t(m-1)+s(n-1)}(p-q)p^{m+n-1}x^{m+n}}{[m]_{p,q}^{t}[n]_{p,q}^{s}(p^mq^n-p^nq^m)}.$$

Then we can transform the above equation as

$$A_{p,q}(s,t;x)=\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{(t+1)(m-1)+s(n-1)}(p-q)p^{m+n-1}x^{m+n}[m]_{p,q}}{p^{m-1}[m]_{p,q}^{t+1}[n]_{p,q}^{s}(p^mq^n-p^nq^m)}.$$

Here, we note that

$$\frac{1}{p^{m-1}}[m]_{p,q}=\frac{1}{p^{n-1}}[n]_{p,q}+\frac{p^mq^n-p^nq^m}{p^{m+n-1}(p-q)}.$$

Applying the above formula, we can find

$$\begin{aligned}A_{p,q}(s,t;x)&=\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{(t+1)(m-1)+(s-1)(n-1)}(p-q)p^{m+n-1}x^{m+n}}{[m]_{p,q}^{t+1}[n]_{p,q}^{s-1}(p^mq^n-p^nq^m)}\\&\quad+\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{(t+1)(m-1)+s(n-1)}x^{m+n}}{[m]_{p,q}^{t+1}[n]_{p,q}^{s}}.\\&=A_{p,q}(s-1,t+1;x)+\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{(t+1)(m-1)+s(n-1)}x^{m+n}}{[m]_{p,q}^{t+1}[n]_{p,q}^{s}},\end{aligned}$$

and complete the proof of Lemma 2.6. □

Lemma 2.7. *For $|q/p|<1$, one has*

(i) $A_{p,q}(s,t;x)=-A_{p,q}(t,s;x)$.

(ii) $A_{p,q}(s,0;x)=Li_{p,q}\left(s;px^2/q\right)Li_{p,q}(1;x)-\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^n}{[n]_{p,q}^{s}}\sum_{k=1}^{n-1}\frac{p^{k-1}}{[k]_{p,q}}\left(\frac{px}{q}\right)^{n-k}.$

Proof. (i) In proof of Lemma 2.6, we think the following hold:

$$-A_{p,q}(t,s;x)=-\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{s(m-1)+t(n-1)}(p-q)p^{m+n-1}x^{m+n}}{[m]_{p,q}^{s}[n]_{p,q}^{t}(p^{m}q^{n}-p^{n}q^{m})}=A_{p,q}(t,s;x).$$

(ii) Substituting 0 instead of t, we can see

$$A_{p,q}(s,0;x)=\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{s(n-1)}(p-q)p^{m+n-1}x^{m+n}}{[n]_{p,q}^{s}(p^{m}q^{n}-p^{n}q^{m})}=\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{s(n-1)+(m-1)}x^{m+n}}{q^{n}[n]_{p,q}^{s}[m-n]_{p,q}}.$$

From the above equation, since the condition is $m\neq n$, we can rewrite as

$$\begin{aligned}A_{p,q}(s,0;x)&=\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^{n}}{q^{n}[n]_{p,q}^{s}}\left(\sum_{m=1}^{n-1}\frac{p^{m-1}x^{m}}{[m-n]_{p,q}}+\sum_{m=n+1}^{\infty}\frac{p^{m-1}x^{m}}{[m-n]_{p,q}}\right)\\&=\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^{n}}{q^{n}[n]_{p,q}^{s}}\sum_{k=1}^{n-1}\frac{p^{n-k-1}x^{n-k}}{[-k]_{p,q}}+\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^{n}}{q^{n}[n]_{p,q}^{s}}\sum_{m=1}^{\infty}\frac{p^{m+n-1}x^{m+n}}{[m]_{p,q}}\\&=-\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^{n}}{[n]_{p,q}^{s}}\sum_{k=1}^{n-1}\frac{p^{k-1}}{[k]_{p,q}}\left(\frac{px}{q}\right)^{n-k}+Li_{p,q}(s;px^{2}/q)Li_{p,q}(1;x).\end{aligned}$$

The required relation now follows immediately. □

Theorem 2.8. *Let s,t be nonnegative integers. For $|q/p|<1$, the following hold:*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^{n}}{[n]_{p,q}^{s}}\sum_{k=1}^{n-1}\frac{p^{k-1}}{[k]_{p,q}}\left(\frac{px}{q}\right)^{n-k}\\&=Li_{p,q}\left(s;px^{2}/q\right)Li_{p,q}(1;x)+\frac{s}{2}Li_{p,q}\left(s+1;x^{2}\right)\\&-\frac{1}{2}\sum_{k=1}^{s}Li_{p,q}(k;x)Li_{p,q}(s+1-k;x).\end{aligned}$$

Proof. From Lemma 2.6, we can find

$$\begin{aligned}&\sum_{\substack{n,m=1\\ n\neq m}}^{\infty}\frac{p^{(t+1)(m-1)+s(n-1)}x^{m+n}}{[m]_{p,q}^{t+1}[n]_{p,q}^{s-1}}\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{p^{(t+1)(m-1)+s(n-1)}x^{m+n}}{[m]_{p,q}^{t+1}[n]_{p,q}^{s}}-\sum_{n=1}^{\infty}\frac{p^{(t+1)(n-1)+s(n-1)}x^{2n}}{[n]_{p,q}^{t+1}[n]_{p,q}^{s}}.\\&=\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^{n}}{[n]_{p,q}^{s}}\left(\sum_{m=1}^{\infty}\frac{p^{(t+1)(m-1)}x^{m}}{[m]_{p,q}^{t+1}}-\frac{p^{(t+1)(n-1)}x^{n}}{[n]_{p,q}^{t+1}}\right).\\&=Li_{p,q}(s;x)Li_{p,q}(t+1;x)-Li_{p,q}(s+t+1;x^{2}).\end{aligned}$$

Therefore, we derive that

$$\begin{aligned}&A_{p,q}(s,t;x)\\&=A_{p,q}(s-1,t+1;x)+Li_{p,q}(s;x)Li_{p,q}(t+1;x)-Li_{p,q}\left(s+t+1;x^2\right).\end{aligned}$$

Using telescoping method for the above equation, we can make

$$\begin{aligned}&A_{p,q}(s,t;x)-A_{p,q}(0,s+t;x)\\&=\sum_{k=1}^{s}Li_{p,q}(k;x)Li_{p,q}(s+t+1-k;x)-sLi_{p,q}\left(s+t+1;x^2\right).\end{aligned}$$

Setting $t=0$ from Lemma 2.7 (i), we also have

$$2A_{p,q}(s,0;x)=\sum_{k=1}^{s}Li_{p,q}(k;x)Li_{p,q}(s+1-k;x)-sLi_{p,q}\left(s+1;x^2\right).$$

Applying Lemma 2.7 (ii) on the left-hand side in the above equation, we can investigate

$$\begin{aligned}&2\left(Li_{p,q}\left(s;px^2/q\right)Li_{p,q}(1;x)-\sum_{n=1}^{\infty}\frac{p^{s(n-1)}x^n}{[n]_{p,q}^s}\sum_{k=1}^{n-1}\frac{p^{k-1}}{[k]_{p,q}}\left(\frac{px}{q}\right)^{n-k}\right)\\&=\sum_{k=1}^{s}Li_{p,q}(k;x)Li_{p,q}(s+1-k;x)-sLi_{p,q}\left(s+1;x^2\right).\end{aligned}$$

The required relation now follows at once. □

Lemma 2.9. *Let m be a nonnegative integer. Then, we have*

(i) $I_{p,q}(m;x)$

$$=\sum_{k=1}^{m-1}\left(-\frac{q^n}{p[n]_{p,q}}\right)^{k-1}\frac{Li_{p,q}(m+1-k;x)}{[n]_{p,q}}x^n+\left(-\frac{q^n}{p[n]_{p,q}}\right)^{m-1}I_{p,q}(1;x).$$

(ii) $I_{p,q}(1;x)=-\dfrac{1}{[n]_{p,q}}$

$$\left(x^n ln_{p,q}(1-x)-\left(\frac{q}{p}\right)^n ln_{p,q}(1-x)-\left(\frac{q}{p}\right)^n\sum_{k=1}^{n}\frac{1}{[k]_{p,q}}p^{k-1}x^k\right).$$

Proof. Suppose that

$$I_{p,q}(m;x)=\int_0^x t^{n-1}Li_{p,q}(m;pt)d_{p,q}t.$$

(i) Applying (p,q)-integral by parts in the above equation we have

$$\begin{aligned}I_{p,q}(m;x)&=\frac{1}{[n]_{p,q}}Li_{p,q}(m;x)x^n-\frac{q^n}{p[n]_{p,q}}\int_0^x t^{n-1}Li_{p,q}(m-1;pt)d_{p,q}t\\&=\frac{1}{[n]_{p,q}}Li_{p,q}(m;x)x^n-\frac{q^n}{p[n]_{p,q}}I_{p,q}(m-1;x).\end{aligned}$$

Using telescoping method during $(m-1)$-times, we can derive

$$I_{p,q}(m;x)=\frac{1}{[n]_{p,q}}Li_{p,q}(m;x)x^n-\frac{q^n}{p[n]_{p,q}}I_{p,q}(m-1;x)$$
$$I_{p,q}(m-1;x)=\frac{1}{[n]_{p,q}}Li_{p,q}(m-1;x)x^n-\frac{q^n}{p[n]_{p,q}}I_{p,q}((m-1)-1;x)$$
$$I_{p,q}(m-2;x)=\frac{1}{[n]_{p,q}}Li_{p,q}(m-2;x)x^n-\frac{q^n}{p[n]_{p,q}}I_{p,q}((m-1)-2;x)$$
$$\vdots$$
$$I_{p,q}(m-(m-2);x)=\frac{1}{[n]_{p,q}}Li_{p,q}(m-(m-2);x)x^n-\frac{q^n}{p[n]_{p,q}}I_{p,q}((m-1)-(m-2);x).$$

Therefore, we can obtain

$$\begin{aligned}&I_{p,q}(m;x)\\&=\sum_{k=1}^{m-1}\left(-\frac{q^n}{p[n]_{p,q}}\right)^{k-1}\frac{Li_{p,q}(m+1-k;x)}{[n]_{p,q}}x^n\\&+\left(-\frac{q^n}{p[n]_{p,q}}\right)^{m-1}I_{p,q}(m-(m-1);x),\end{aligned}$$

and we complete the proof of Lemma 2.9.(i).
(ii) Setting $m=1$ for $I_{p,q}(m;x)$, we can find

$$\begin{aligned}I_{p,q}(1;x)&=-\int_0^x t^{n-1}\ln_{p,q}(1-pt)d_{p,q}t\\&=-\frac{1}{[n]_{p,q}}\left(x^n\ln_{p,q}(1-x)+q^n\sum_{k=1}^{\infty}p^{k-1}\int_0^x t^{k+n-1}d_{p,q}t\right)\\&=-\frac{1}{[n]_{p,q}}\\&\left(x^n\ln_{p,q}(1-x)-\left(\frac{q}{p}\right)^n\ln_{p,q}(1-x)-\left(\frac{q}{p}\right)^n\sum_{k=1}^{n}\frac{1}{[k]_{p,q}}p^{k-1}x^k\right),\end{aligned}$$

or equivalently,

$$\begin{aligned}&I_{p,q}(1;x)\\&=-\frac{1}{[n]_{p,q}}\left(x^n\ln_{p,q}(1-x)-\left(\frac{q}{p}\right)^n\ln_{p,q}(1-x)-\left(\frac{q}{p}\right)^nLi_{n,p,q}(1;x)\right).\end{aligned}$$

The required relation now follows at once. □

Lemma 2.10. *For $|q/p| < 1$, the following hold:*

$$I_{p,q}(m;q/p) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1} q^{kn}}{p^{k+n-1}[n]_{p,q}^k} \zeta_{p,q}(m+1-k) + \frac{(-1)^{m-1} q^{mn}}{p^{m+n-1}[n]_{p,q}^m} Li_{n,p,q}(1;q/p).$$

Proof. Putting (ii) on (i) in Lemma 2.9, one has

$$\begin{aligned} & I_{p,q}(m;x) \\ &= \sum_{k=1}^{m-1} \left(-\frac{q^n}{p[n]_{p,q}}\right)^{k-1} \frac{Li_{p,q}(m+1-k;x)}{[n]_{p,q}} x^n + \left(-\frac{q^n}{p[n]_{p,q}}\right)^{m-1} \\ & \left[-\frac{1}{[n]_{p,q}} \left(x^n \ln_{p,q}(1-x) - \left(\frac{q}{p}\right)^n \ln_{p,q}(1-x) - \left(\frac{q}{p}\right)^n \sum_{k=1}^{n} \frac{1}{[k]_{p,q}} p^{k-1} x^k \right)\right]. \end{aligned}$$

Substituting x by q/p and calculating the above equation, we obtain

$$\begin{aligned} & I_{p,q}(m;q/p) \\ &= \sum_{k=1}^{m-1} \left(-\frac{q^n}{p[n]_{p,q}}\right)^{k-1} \left(\frac{q}{p}\right)^n \frac{Li_{p,q}(m+1-k;q/p)}{[n]_{p,q}} \\ &+ (-1)^{m-1} \frac{q^{mn}}{p^{m+n-1}[n]_{p,q}^m} Li_{n,p,q}(1;q/p) \\ &= \sum_{k=1}^{m-1} \frac{(-1)^{k-1} q^{kn}}{p^{k+n-1}[n]_{p,q}^k} \zeta_{p,q}(m+1-k) + \frac{(-1)^{m-1} q^{mn}}{q^{m+n-1}[n]_{p,q}^m} Li_{n,p,q}(1;q/p), \end{aligned}$$

which immediately gives the required result. □

Theorem 2.11. *Let a, b be any nonnegative integers. Then, we have*

$$\begin{aligned} & \int_0^{\frac{q}{p}} \frac{Li_{p,q}(a;px) Li_{p,q}(b;px)}{x} d_{p,q}x \\ &= p \sum_{k=1}^{b-1} (-1)^{k-1} Li_{p,q}(a+k;(q/p)^k) \zeta_{p,q}(b+1-k) \\ &+ (-1)^{b-1} p \sum_{n=1}^{\infty} \left(\frac{p^{n-1}}{[n]_{p,q}}\right)^{a+b} \left(\frac{q}{p}\right)^{bn} Li_{n,p,q}(1;q/p). \end{aligned}$$

Proof. To prove the relation, we note that

$$\int_0^{\frac{q}{p}} \frac{Li_{p,q}(a;px) Li_{p,q}(b;px)}{x} d_{p,q}x = \sum_{n=1}^{\infty} \frac{p^{a(n-1)+n}}{[n]_{p,q}^a} \int_0^{\frac{q}{p}} x^{n-1} Li_{p,q}(b;px) d_{p,q}x.$$

Using the process of Lemma 2.10, we find

$$\begin{aligned}
&\int_0^{\frac{q}{p}} \frac{Li_{p,q}(a;px)Li_{p,q}(b;px)}{x} d_{p,q}x \\
&= \sum_{n=1}^{\infty} \frac{p^{a(n-1)+n}}{[n]_{p,q}^a} \\
&\left(\sum_{k=1}^{b-1} \frac{(-1)^{k-1} q^{kn}}{q^{k+n-1}[n]_{p,q}^k} \zeta_{p,q}(b+1-k) + \frac{(-1)^{b-1} q^{bn}}{p^{b+n-1}[n]_{p,q}^b} Li_{n,p,q}(1;q/p) \right) \\
&= p \sum_{k=1}^{b-1} (-1)^{k-1} Li_{p,q}(a+k;(q/p)^k) \zeta_{p,q}(b+1-k) \\
&+ (-1)^{b-1} \sum_{n=1}^{\infty} \frac{p^{a(n-1)-(b-1)} q^{bn}}{[n]_{p,q}^{a+b}} Li_{n,p,q}(1;q/p) \\
&= p \sum_{k=1}^{b-1} (-1)^{k-1} Li_{p,q}(a+k;(q/p)^k) \zeta_{p,q}(b+1-k) \\
&+ (-1)^{b-1} p \sum_{n=1}^{\infty} \left(\frac{p^{n-1}}{[n]_{p,q}} \right)^{a+b} \left(\frac{q}{p} \right)^{bn} Li_{n,p,q}(1;q/p).
\end{aligned}$$

Therefore, the required relation follows. □

Theorem 2.12. *For $|q/p| < 1$, the following relation hold:*

$$\begin{aligned}
&(\zeta_{p,q}(m))^a \\
&= \sum_{n=1}^{\infty} \left[\sum_{b=1}^{a-1} \binom{a}{b} (Li_{n,p,q}(m;q/p))^b \left(\frac{p^{mn} q^{n+1}}{p^{n+1}[n+1]_{p,q}^m} \right)^{a-b} \right] + Li_{p,q}(am;(q/p)^a).
\end{aligned}$$

Proof. Construct

$$B = \sum_{n=1}^{\infty} (Li_{n,p,q}(m;q/p))^a x^n.$$

The form B turns into

$$\begin{aligned}
B &= (Li_{1,p,q}(m;q/p))^a x + \sum_{n=2}^{\infty} (Li_{n,p,q}(m;q/p))^a x^n \\
&= \left(\frac{q}{p} \right)^a x + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{p^{m(k-1)}}{[k]_{p,q}^m} \left(\frac{q}{p} \right)^k + \frac{p^{mn}}{[n+1]_{p,q}^m} \left(\frac{q}{p} \right)^{n+1} \right)^a x^n.
\end{aligned}$$

Using the process of Lemmas 2.8 and 2.9, we have

$$B = \left(\frac{q}{p}\right)^a x + \sum_{n=1}^{\infty}\left[\sum_{b=0}^{a}\binom{a}{b}\left(Li_{n,p,q}(m;q/p)\right)^b\left(\frac{1}{[n+1]_{p,q}^m}\left(\frac{q}{p}\right)^{n+1}\right)^{a-b}\right]x^{n+1}$$

$$= \sum_{n=1}^{\infty}\left(\frac{p^{m(n-1)}}{[n]_{p,q}^m}\left(\frac{q}{p}\right)^n\right)^a x^n$$

$$+ \sum_{n=1}^{\infty}\left[\sum_{b=1}^{a-1}\binom{a}{b}\left(Li_{n,p,q}(m;q/p)\right)^b\left(\frac{p^{mn}}{[n+1]_{p,q}^m}\left(\frac{q}{p}\right)^{n+1}\right)^{a-b}\right]x^{n+1} + xB.$$

Hence, we investigate

$$(1-x)B = Li_{p,q}(am;(q/p)^a)x^n$$

$$+ \sum_{n=1}^{\infty}\left[\sum_{b=1}^{a-1}\binom{a}{b}\left(Li_{n,p,q}(m;q/p)\right)^b\left(\frac{p^{mn}}{[n+1]_{p,q}^m}\left(\frac{q}{p}\right)^{n+1}\right)^{a-b}\right]x^{n+1}.$$

Using Abel's continuity theorem in the left-hand side we can represent this as

$$\lim_{n\to 1^-}(1-x)B = \lim_{n\to 1^-}(1-x)\sum_{n=1}^{\infty}\left(Li_{n,p,q}(m;q/p)\right)^a x^n = \left(\zeta_{p,q}(m)\right)^a,$$

which leads to the required relation immediately. □

Corollary 2.13. *Setting $a = m = 1$ in Theorem 2.12, the following hold :*

$$\text{(i)} \qquad \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)x^n = -\frac{ln_{p,q}(1-(qx/p))}{1-x},$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)(px)^n = -\frac{ln_{p,q}(1-qx)}{1-px}.$$

Theorem 2.14. *Let $|q/p| < 1$. Then, the following holds :*

$$\sum_{n=1}^{\infty} Li_{n,p,q}x^n = -\frac{ln_{p,q}(1-x)}{1-x}.$$

Proof. Suppose that

$$C = \sum_{n=1}^{\infty} Li_{n,p,q}x^n.$$

The form C can turn into

$$C = Li_{1,p,q}x + \sum_{n=2}^{\infty} Li_{n,p,q}x^n = x + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{p^{k-1}x^{n+1}}{[k]_{p,q}} + \sum_{n=1}^{\infty}\frac{p^n x^{n+1}}{[n+1]_{p,q}}$$

$$= \frac{Li_{p,q}(1;x)}{1-x} = -\frac{\ln_{p,q}(1-x)}{1-x},$$

which immediately gives the required result. □

Corollary 2.15. *Substituting x by px in Theorem 2.14, the following holds :*

$$\sum_{n=1}^{\infty} p^n x^n Li_{n,p,q} = -\frac{ln_{p,q}(1-px)}{1-px}.$$

Lemma 2.16. *Let $|q/p|<1$. Then, one has*

$$D_{p,q} ln_{p,q}^2(1-x) = -\frac{ln_{p,q}(1-px)+ln_{p,q}(1-qx)}{1-px}.$$

Proof. Using a formula of the (p,q)-derivative, we easily see that

$$\begin{aligned} D_{p,q}\ln_{p,q}^2(1-x) &= D_{p,q}\ln_{p,q}(1-x)\big(\ln_{p,q}(1-px)+\ln_{p,q}(1-qx)\big) \\ &= -\frac{\big(\ln_{p,q}(1-px)+\ln_{p,q}(1-qx)\big)}{1-px}. \end{aligned}$$

- □

Theorem 2.17. *Let $|q/p|<1$. Then we get*

$$ln_{p,q}^2(1-x) = \sum_{n=1}^{\infty} (\zeta_{n,p,q}+\zeta_{n,p,q}(1)) \frac{p^{n-1}x^n}{[n]_{p,q}} - Li_{p,q}(2;x) - Li_{p,q}(2;qx/p).$$

Proof. Combining Corollaries 2.13 (ii) and 2.15 in Lemma 2.16, we derive

$$D_{p,q}\ln_{p,q}^2(1-x) = \sum_{n=1}^{\infty} p^n x^n \left(Li_{n,p,q} + Lin,p,q(1;q/p)\right).$$

Using (p,q)-integral in the above equation, the left-hand side can be transformed as follows:

$$\int_0^x D_{p,q}\ln_{p,q}^2(1-t)d_{p,q}t = \ln_{p,q}^2(1-x).$$

We also can transform the right-hand side as follows:

$$\begin{aligned} &\int_0^x \sum_{n=1}^{\infty} p^n t^n \left(Li_{n,p,q}+Li_{n,p,q}(1;q/p)\right) d_{p,q}t \\ &= \sum_{n=1}^{\infty} p^n \left(Li_{n,p,q}+Li_{n,p,q}(1;q/p)\right) \frac{x^{n+1}}{[n+1]_{p,q}} \\ &= \sum_{n=1}^{\infty} \left(Li_{n,p,q}+Li_{n,p,q}(1;q/p)\right) \frac{p^{n-1}x^n}{[n]_{p,q}} - \sum_{n=1}^{\infty} \frac{p^{2(n-1)}}{[n]_{p,q}^2} x^n - \sum_{n=1}^{\infty} \frac{p^{2(n-1)}}{[n]_{p,q}^2} \left(\frac{qx}{p}\right)^n \\ &= \sum_{n=1}^{\infty} \left(Li_{n,p,q}+Li_{n,p,q}(1;q/p)\right) \frac{p^{n-1}x^n}{[n]_{p,q}} - Li_{p,q}(2;x) - Li_{p,q}(2;(qx/p)). \end{aligned}$$

From relation between $Li_{n,p,q}$ and $\zeta_{n,p,q}$ the required relation now follows at once. □

Lemma 2.18. *For $|q/p| < 1$, the following holds :*

$$ln_{p,q}(1-qx) = \frac{(p-q)x}{1-px} + ln_{p,q}(1-px).$$

Proof. Using the definition of (p,q)-logarithmic function, we can find

$$\begin{aligned}\ln_{p,q}(1-qx/p) &= -\sum_{n=1}^{\infty}\frac{p^{n-1}}{[n]_{p,q}}\left(\frac{q}{p}x\right)^n + \sum_{n=1}^{\infty}\frac{p^{n-1}}{[n]_{p,q}}x^n - \sum_{n=1}^{\infty}\frac{p^{n-1}}{[n]_{p,q}}x^n \\ &= \sum_{n=1}^{\infty}\frac{p-q}{p}x^n - \sum_{n=1}^{\infty}\frac{p^{n-1}}{[n]_{p,q}}x^n \\ &= \frac{(p-q)x}{p(1-x)} + \ln_{p,q}(1-x).\end{aligned}$$

Substituting $x = px$ in the above equation, we can see the required relation. □

Lemma 2.19. *Let $|q/p| < 1$. Then we have*

$$ln^2_{p,q}(1-x) + \widetilde{Li}_{p,q}(1;x) = 2\sum_{n=1}^{\infty} Li_{n,p,q}\frac{p^{n-1}x^n}{[n]_{p,q}} - 2Li_{p,q}(2;x).$$

Proof. Combining Lemma 2.18 into Lemma 2.16, we get

$$\begin{aligned}D_{p,q}\ln^2_{p,q}(1-x) &= -\frac{\ln_{p,q}(1-px)}{1-px} - \frac{1}{1-px}\left(\frac{(p-q)x}{1-px} + \ln_{p,q}(1-px)\right) \\ &= -\frac{(p-q)x}{(1-px)^2} - 2\frac{\ln_{p,q}(1-px)}{1-px}.\end{aligned}$$

From Corollary 2.15 we obtain

$$\begin{aligned}D_{p,q}\ln^2_{p,q}(1-x) &= -\frac{(p-q)x}{(1-px)^2} + 2\sum_{n=1}^{\infty}p^n x^n Li_{n,p,q} \\ &= -(p-q)\sum_{n=0}^{\infty}(n+1)p^n x^{n+1} + 2\sum_{n=1}^{\infty}(px)^n Li_{n,p,q}.\end{aligned}$$

Using (p,q)-integral in the above equation, the left-hand side can be transformed as

$$\int_0^x D_{p,q}\ln^2_{p,q}(1-t)d_{p,q}t = \ln^2_{p,q}(1-x),$$

and the right-hand side can be transformed as

$$\int_0^x \left(-(p-q) \sum_{n=0}^{\infty} (n+1) p^n t^{n+1} + 2 \sum_{n=1}^{\infty} p^n Li_{n,p,q} t^n \right) d_{p,q} t$$

$$= -(p-q) \sum_{n=0}^{\infty} (n+1) p^n \frac{x^{n+2}}{[n+2]_{p,q}} + 2 \sum_{n=1}^{\infty} \frac{p^n x^{n+1}}{[n+1]_{p,q}} \sum_{k=1}^{n} \frac{p^{k-1}}{[k]_{p,q}}$$

$$= -(p-q) \sum_{n=1}^{\infty} \frac{n p^{n-1} x^{n+1}}{[n+1]_{p,q}} + 2 \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{p^{k-1}}{[k]_{p,q}} \right) \frac{p^{n-1} x^n}{[n]_{p,q}} - 2 \sum_{n=1}^{\infty} \frac{p^{2(n-1)} x^n}{[n]_{p,q}^2}$$

$$= -\widetilde{Li}_{p,q}(1; x) + 2 \sum_{n=1}^{\infty} Li_{n,p,q} \frac{p^{n-1} x^n}{[n]_{p,q}} - 2 Li_{p,q}(2; x).$$

Therefore, the proof is completed. □

Corollary 2.20. *For $|q/p| < 1$, the following holds :*

$$ln_{p,q}^2(1-x) - \widetilde{Li}_{p,q}(1; x) = 2 \sum_{n=1}^{\infty} Li_{n,p,q}(1; q/p) \frac{p^{n-1} x^n}{[n]_{p,q}} - 2 Li_{p,q}(2; qx/p).$$

3. Some evaluation of (p, q)-analogue Euler sum types

In this section, we define (p, q)-analogue Euler sum types that are related to (p, q)-Riemann zeta functions and finite (p, q)-special polynomials. We also give some explicit formulae using theorems in Section 2.

Definition 3.1. (p, q)-analogue Euler sum is defined by

$$S_{p,q}(a, b, c, d, e; s, t) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{p^{dk} q^{ek}}{[k]_{p,q}^t} \right) \frac{p^{an+b} q^{cn}}{[n]_{p,q}^s},$$

where a, c, d, e, s, t are positive integers and b is an integer.

From Definition 3.1, we note that (p, q)-analogue Euler sum is q-Euler sum when $p = 1$. According to conditions that $p = 1$ and $q \to 1$, we can see linear Euler sum(see [4, 7, 8, 18]).

Theorem 3.2. *For $|q/p| < 1$, we have*

$$\text{(i)} \quad \sum_{n=1}^{\infty} \zeta_{n,p,q} \frac{q^n}{p[n]_{p,q}} = \frac{1}{2} Li_{p,q}\left(2; (q/p)^2\right) + \frac{1}{2}\left(\zeta_{p,q}(1)\right)^2 + \zeta_{p,q}(2),$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \zeta_{n,p,q} \frac{p^{n-2}q^n}{[n]^2_{p,q}} = Li_{p,q}(3; (q/p)^2) + \zeta_{p,q}(3),$$

$$\text{(iii)} \quad \sum_{n=1}^{\infty} \zeta_{n,p,q} \frac{p^{2n-3}q^n}{[n]^3_{p,q}} = \frac{3}{2} Li_{p,q}(4; (q/p)^2) - \frac{1}{2}\left(\zeta_{p,q}(2)\right)^2 + \zeta_{p,q}(4).$$

Proof. (i) Replacing $1, q/p$ with s, x, respectively, in Theorem 2.8, we get

$$\sum_{n=1}^{\infty} \frac{p^{n-1}}{[n]_{p,q}} \left(\frac{q}{p}\right)^n \sum_{k=1}^{n-1} \frac{p^{k-1}}{[k]_{p,q}} = \frac{1}{2}\left(Li_{p,q}(1; q/p)\right)^2 + \frac{1}{2} Li_{p,q}(2; (q/p)^2).$$

The left-hand side is represented as

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{p^{n-1}}{[n]_{p,q}} \left(\frac{q}{p}\right)^n \sum_{k=1}^{n-1} \frac{p^{k-1}}{[k]_{p,q}} &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{p^{k-1}}{[k]_{p,q}} \frac{q^n}{p[n]_{p,q}} - \sum_{n=1}^{\infty} \frac{p^{2(n-1)}}{[n]^2_{p,q}} \left(\frac{q}{p}\right)^n \\ &= \sum_{n=1}^{\infty} \zeta_{n,p,q} \frac{q^n}{p[n]_{p,q}} - Li_{p,q}(2; (q/p).\end{aligned}$$

We omit the proofs of (ii) and (iii) since we can derive them by the same method as (i) when $s = 2$ and $s = 3$, respectively. □

Corollary 3.3. *From Theorem 2.19, we find*

$$\sum_{n=1}^{\infty} \zeta_{n,p,q} \frac{q^n}{p[n]_{p,q}} = \frac{1}{2} \widetilde{Li}_{p,q}(1; q/p) + \frac{1}{2}\left(Li_{p,q}(1; q/p)\right)^2 + Li_{p,q}(2; q/p).$$

Corollary 3.4. *In Corollary 3.3 and Theorem 3.2, the following holds:*

$$\widetilde{Li}_{p,q}(1; q/p) = Li_{p,q}(2; (q/p))^2.$$

Theorem 3.5. *Let $|q/p| < 1$. Then we have*

$$\left(\zeta_{p,q}(2)\right)^2 = \sum_{n=1}^{\infty} \frac{(pq)^n (p^{n+1} + q^n)\zeta_{n,p,q}(1)}{p^4 [n]^3_{p,q}}.$$

Proof. Setting $a = 2, b = 1$ in Theorem 2.11, we can get

$$\int_0^{\frac{q}{p}} \frac{Li_{p,q}(2; px) Li_{p,q}(1; px)}{x} d_{p,q}x = p \sum_{n=1}^{\infty} Li_{n,p,q}(1; q/p) \frac{p^{2n-3}q^n}{[n]^3_{p,q}}.$$

Using Lemma 2.9 (i) and (ii) in the above equation, the left-hand side transforms as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{p^{2n-1}}{[n]_{p,q}}\int_0^{\frac{q}{p}} x^{n-1}Li_{p,q}(2;px)d_{p,q}(x)\\
&=\sum_{n=1}^{\infty}\frac{p^{2n-1}}{[n]_{p,q}}\left[\frac{Li_{p,q}(2;q/p)}{[n]_{p,q}}\left(\frac{q}{p}\right)^n-\frac{q^n}{p[n]_{p,q}}I_{p,q}(1;q/p)\right]\\
&=p\left(\zeta_{p,q}(2)\right)^2-\sum_{n=1}^{\infty}Li_{n,p,q}(1;q/p)\frac{p^{n-2}q^{2n}}{[n]_{p,q}^3}.
\end{aligned}$$

Therefore,

$$\left(\zeta_{p,q}(2)\right)^2=\sum_{n=1}^{\infty}Li_{n,p,q}(1;q/p)\frac{p^{n-2}1q^{2n}}{[n]_{p,q}^3}+\sum_{n=1}^{\infty}Li_{n,p,q}(1;q/p)\frac{p^{2n-3}q^{n}}{[n]_{p,q}^3},$$

which immediately gives the required relation. □

Theorem 3.6. *For $|q/p|<1$, the following relation holds :*

$$\left(\zeta_{p,q}(1)\right)^3=Li_{p,q}(3;(q/p)^3)+3\sum_{n=1}^{\infty}\left(p[n]_{p,q}\zeta_{n,p,q}(1)-q^n\right)\zeta_{n,p,q}(1))\,\frac{q^n}{p^2[n]_{p,q}^2}.$$

Proof. Putting $a=3, m=1$ on Theorem 2.12, we have

$$\begin{aligned}
&\left(\zeta_{p,q}(1)\right)^3-Li_{p,q}(3;(q/p)^3)\\
&=3\sum_{n=1}^{\infty}Li_{n,p,q}(1;q/p)\frac{q^{2(n+1)}}{p^2[n+1]_{p,q}^2}+3\sum_{n=1}^{\infty}\left(Li_{n,p,q}(1;q/p)\right)^2\frac{q^{n+1}}{p[n+1]_{p,q}}\\
&=3\sum_{n=2}^{\infty}Li_{n-1,p,q}(1;q/p)\frac{q^{2n}}{p^2[n]_{p,q}^2}+3\sum_{n=2}^{\infty}\left(Li_{n-1,p,q}(1;q/p)\right)^2\frac{q^{n}}{p[n]_{p,q}}\\
&=3\sum_{n=1}^{\infty}\left(Li_{n,p,q}(1;q/p)\right)^2\frac{q^{n}}{p[n]_{p,q}}-3\sum_{n=1}^{\infty}Li_{n,p,q}(1;q/p)\frac{q^{2n}}{p^2[n]_{p,q}^2},
\end{aligned}$$

and it is proved. □

Theorem 3.7. *Let $|q/p| < 1$. Then one has*

$$\text{(i)}\quad \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)\frac{q^n}{p[n]_{p,q}} = \frac{1}{2}\left(\zeta_{p,q}(1)\right)^2 + \frac{1}{2}Li_{p,q}(2;(q/p)^2),$$

$$\text{(ii)}\quad \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)\frac{q^n}{p[n]_{p,q}^2}$$
$$= \frac{1}{2}\int_0^{\frac{q}{p}} \frac{ln_{p,q}^2(1-x)}{x}d_{p,q}x - \frac{p^2}{2}\widetilde{Li}_{p,q}(2;q/p^2) + pLi_{p,q}(3;q/p^2),$$

$$\text{(iii)}\quad \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)\frac{p^{2n-3}q^n}{[n]_{p,q}^3}$$
$$= \sum_{n=1}^{\infty} Li_{n,p,q}\frac{p^{2n-3}q^n}{[n]_{p,q}^3} - \frac{\widetilde{Li}_{p,q}(3;q/p)}{p^2} - \frac{(p-q)Li_{p,q}(3;q/p)}{p}.$$

Proof. (i) Taking $x = q/p$ on Theorem 2.17, we obtain

$$\left(Li_{p,q}(1;q/p)\right)^2$$
$$= \sum_{n=1}^{\infty} Li_{n,p,q}\frac{q^n}{p[n]_{p,q}} + \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)\frac{q^n}{p[n]_{p,q}} - Li_{p,q}(2;q/p) - Li_{p,q}(2;(q/p)^2).$$

By using Lemma 2.9 (i) and (ii) in the above equation the left-hand side turns into

$$\sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)\frac{q^n}{p[n]_{p,q}} = \frac{1}{2}\left(Li_{p,q}(1;q/p)\right)^2 + \frac{1}{2}\left(Li_{p,q}(2;(q/p)^2\right).$$

(ii) Dividing x and (p,q)-integral over the interval $[0, q/p]$ in Corollary 2.20, the left-hand side turns into

$$\int_0^{\frac{q}{p}} \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)\frac{q^{n-1}}{[n]_{p,q}}x^{n-1}d_{p,q}x = \sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p)\frac{q^n}{p[n]_{p,q}^2},$$

and the right-hand side is transformed as

$$\int_0^{\frac{q}{p}} \frac{1}{2}\frac{\ln_{p,q}^2(1-x)}{x}d_{p,q}x - \frac{1}{2}\int_0^{\frac{q}{p}} \frac{\widetilde{Li}_{p,q}(1;x)}{x}d_{p,q}x + \int_0^{\frac{q}{p}} \frac{Li_{p,q}(2;(q/p)x)}{x}d_{p,q}x$$
$$= \frac{1}{2}\int_0^{\frac{q}{p}} \frac{\ln_{p,q}^2(1-x)}{x}d_{p,q}x - \frac{p^2}{2}\sum_{k=1}^{\infty}\frac{(p-q)n}{[k+1]_{p,q}^2}p^{2(k-1)}\left(\frac{q}{p^2}\right)^{k+1}$$
$$+ p\sum_{n=0}^{\infty}\frac{p^{3(n-1)}}{[n]_{p,q}^3}\left(\frac{q}{p^2}\right)^n$$
$$= \frac{1}{2}\int_0^{\frac{q}{p}} \frac{\ln_{p,q}^2(1-x)}{x}d_{p,q}x - \frac{p^2}{2}\widetilde{Li}_{p,q}(2;q/p^2) + pLi_{p,q}(3;q/p^2).$$

(iii) From the definition of (p,q)-special polynomials, we have

$$Li_{n,p,q}(1;q/p) - Li_{n,p,q} = \sum_{k=1}^{n} \frac{q-p}{q^k - p^k} p^{k-1} \left(\left(\frac{q}{p} \right)^k - 1 \right) = \frac{(q-p)n}{p}.$$

We can turn the above equation into

$$\begin{aligned} &\sum_{n=1}^{\infty} Li_{n,p,q}(1;q/p) \frac{p^{2n-3}q^n}{[n]_{p,q}^3} \\ &= -\sum_{n=1}^{\infty} \frac{(p-q)(n-1)}{[n]_{p,q}^3} p^{2n-4} q^n \ - \sum_{n=1}^{\infty} \frac{(p-q)}{[n]_{p,q}^3} p^{2n-4} q^n \\ &= -\frac{1}{p^2} \widetilde{Li}_{p,q}(3;q/p) - \frac{p-q}{p} Li_{p,q}(3;q/p) + \sum_{n=1}^{\infty} Li_{n,p,q} \frac{p^{2n-3}q^n}{[n]_{p,q}^3}, \end{aligned}$$

which gives the required relation at once. □

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