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BEHAVIOR OF SOLUTIONS OF A RATIONAL THIRD ORDER DIFFERENCE EQUATION

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Abstract. In this paper, we solve the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{a x_n - b x_{n-2}}, \quad n = 0, 1, ...,$$

where a and b are positive real numbers and the initial values x_{-2}, x_{-1} and x_0 are real numbers. We also find invariant sets and discuss the global behavior of the solutions of aforementioned equation..

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1. Introduction

The behavior of the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{b x_n - c x_{n-2}}, \quad n = 0, 1, ...,$$

was studied in [11]. In [12], we studied the behavior of the solutions of the two difference equations

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-2}}, \quad n = 0, 1, ...,$$

and

$$x_{n+1} = \frac{x_n x_{n-1}}{-x_n + x_{n-2}}, \quad n = 0, 1, ...$$

In [3], we studied the global behavior of the fourth order difference equation

$$x_{n+1} = \frac{a x_n x_{n-2}}{-b x_n + c x_{n-3}}, \quad n = 0, 1, ...$$

For more publications on global behavior of the solutions and forbidden sets, one can see [1], [2], [4]-[10], [13]-[35].

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In this paper, we shall determine the forbidden set, find the solution and investigate the behavior of the solutions of the equation

$$x_{n+1} = \frac{x_n x_{n-2}}{a x_n - b x_{n-2}}, \quad n = 0, 1, ..., \tag{1}$$

where a and b are positive real numbers and the initial values x_{-2}, x_{-1} and x_0 are real numbers.

2. Solution of equation (1)

The reciprocal transformation

$$x_n = \frac{1}{y_n}$$

reduces equation (1) into the third order linear homogeneous difference equation

$$y_{n+1} + b y_n - a y_{n-2} = 0, \quad n = 0, 1.... \tag{2}$$

The characteristic equation of equation (2) is

$$\lambda^3 + b\lambda^2 - a = 0. \tag{3}$$

Clear that equation (3) has a positive real root λ_0 for all values of $(a, b > 0)$. Equation (3) can be written as

$$\lambda^3 + b\lambda^2 - a = (\lambda - \lambda_0)(\lambda^2 + (b + \lambda_0)\lambda + \lambda_0(b + \lambda_0)) = 0.$$

Therefore, the roots of equation (3) are

$$\lambda_0, \quad \lambda_{\pm} = -\frac{b + \lambda_0}{2} \pm \frac{\sqrt{(b + \lambda_0)^2 - 4\lambda_0(b + \lambda_0)}}{2}.$$

The roots of equation (3) depends on the relation between a and b.

Lemma 2.1. *For equation (3), we have the following:*

(1) *If $a > \frac{4}{27}b^3$, then equation (3) has one positive real root and two complex conjugate roots.*
(2) *If $a = \frac{4}{27}b^3$, then equation (3) has one positive real root and a repeated negative real root.*
(3) *If $a < \frac{4}{27}b^3$, then equation (3) has three real different roots, one of them is positive and two negative roots.*

Proof. It is sufficient to see that, the discriminant of the polynomial

$$p(\lambda) = \lambda^3 + b\lambda^2 - a = 0$$

is

$$\triangle = 4b^3 a - 27a^2.$$

□

We shall consider the three cases given in lemma (2.1).

Case $a > \frac{4}{27}b^3$:

When $a > \frac{4}{27}b^3$, the roots of equation (3) are

$$\lambda_0 > \frac{b}{3}, \quad \lambda_\pm = -\frac{b+\lambda_0}{2} \pm i\frac{\sqrt{4\lambda_0(b+\lambda_0)-(b+\lambda_0)^2}}{2}.$$

Then the solution of equation (2) is

$$y_n = c_1\lambda_0^n + (\frac{a}{\lambda_0})^{\frac{n}{2}}(c_2\cos n\theta + c_3\sin n\theta), \tag{4}$$

where

$$|\lambda_\pm| = \sqrt{\lambda_0(b+\lambda_0)} = \sqrt{\frac{a}{\lambda_0}} \quad \text{and} \quad \theta = tan^{-1}(-\sqrt{\frac{3\lambda_0-b}{b+\lambda_0}}) \in]\frac{\pi}{2},\pi[.$$

Using the initials y_{-2}, y_{-1} and y_0, the values of c_1, c_2 and c_3 are:

$$\begin{array}{lll} c_1 & = & \frac{1}{\Delta_1}(y_0c_{11}+y_{-1}c_{12}+y_{-2}c_{13}), \\ c_2 & = & \frac{1}{\Delta_1}(y_0c_{21}+y_{-1}c_{22}+y_{-2}c_{23}) \\ \text{and} & & \\ c_3 & = & \frac{1}{\Delta_1}(y_0c_{31}+y_{-1}c_{32}+y_{-2}c_{33}), \end{array} \tag{5}$$

where

$$\begin{array}{l} c_{11} = -\frac{\lambda_0}{a}\sqrt{\frac{\lambda_0}{a}}\sin\theta, \quad c_{12} = \frac{\lambda_0}{a}\sin 2\theta, \quad c_{13} = -\sqrt{\frac{\lambda_0}{a}}\sin\theta, \\ c_{21} = \frac{1}{a}\sin 2\theta - \frac{1}{\lambda_0^2}\sqrt{\frac{\lambda_0}{a}}\sin\theta, \quad c_{22} = -\frac{\lambda_0}{a}\sin 2\theta, \quad c_{23} = \sqrt{\frac{\lambda_0}{a}}\sin\theta, \\ c_{31} = \frac{1}{a}\cos 2\theta - \frac{1}{\lambda_0^2}\sqrt{\frac{\lambda_0}{a}}\cos\theta, \; c_{32} = -\frac{\lambda_0}{a}\cos 2\theta + \frac{1}{\lambda_0^2}, \; c_{33} = \sqrt{\frac{\lambda_0}{a}}\cos\theta - \frac{1}{\lambda_0} \end{array} \tag{6}$$

and

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ \frac{1}{\lambda_0} & \sqrt{\frac{\lambda_0}{a}}\cos\theta & -\sqrt{\frac{\lambda_0}{a}}\sin\theta \\ \frac{1}{\lambda_0^2} & \frac{\lambda_0}{a}\cos 2\theta & -\frac{\lambda_0}{a}\sin 2\theta \end{vmatrix}. \tag{7}$$

By simple calculations, we can write the solution of equation (1) as

$$x_n = \frac{1}{\frac{\alpha_{1n}}{x_0}+\frac{\alpha_{2n}}{x_{-1}}+\frac{\alpha_{3n}}{x_{-2}}}, \tag{8}$$

where

$$\begin{array}{lll} \alpha_{1n} & = & \frac{1}{\Delta_1}(c_{11}\lambda_0^n + c_{21}(\frac{a}{\lambda_0})^{\frac{n}{2}}\cos n\theta + c_{31}(\frac{a}{\lambda_0})^{\frac{n}{2}}\sin n\theta), \\ \alpha_{2n} & = & \frac{1}{\Delta_1}(c_{12}\lambda_0^n + c_{22}(\frac{a}{\lambda_0})^{\frac{n}{2}}\cos n\theta + c_{32}(\frac{a}{\lambda_0})^{\frac{n}{2}}\sin n\theta) \\ \text{and} & & \\ \alpha_{3n} & = & \frac{1}{\Delta_1}(c_{13}\lambda_0^n + c_{23}(\frac{a}{\lambda_0})^{\frac{n}{2}}\cos n\theta + c_{33}(\frac{a}{\lambda_0})^{\frac{n}{2}}\sin n\theta) \end{array} \tag{9}$$

are such that c_{ij}, $i,j = 1,2,3$ are given in (6).

Case $a=\frac{4}{27}b^3$:
When $a=\frac{4}{27}b^3$, the roots of equation (3) are

$$\lambda_0=\frac{b}{3},\quad -\frac{2b}{3},\quad -\frac{2b}{3}.$$

Then the solution of equation (2) is

$$y_n=c_1(\frac{b}{3})^n+c_2(-\frac{2b}{3})^n+c_3(-\frac{2b}{3})^n n. \tag{10}$$

Using the initials y_{-2},y_{-1} and y_0, the values of c_1,c_2 and c_3 in this case are:

$$\begin{array}{lll} c_1 & = & \frac{1}{\Delta_2}(y_0c_{11}+y_{-1}c_{12}+y_{-2}c_{13}),\\ c_2 & = & \frac{1}{\Delta_2}(y_0c_{21}+y_{-1}c_{22}+y_{-2}c_{23})\\ \text{and} & & \\ c_3 & = & \frac{1}{\Delta_2}(y_0c_{31}+y_{-1}c_{32}+y_{-2}c_{33}), \end{array} \tag{11}$$

where

$$\begin{array}{l} c_{11}=\dfrac{27}{8b^3},\quad c_{12}=\dfrac{9}{2b^2},\quad c_{13}=\dfrac{3}{2b},\\ c_{21}=\dfrac{27}{b^3},\quad c_{22}=-\dfrac{9}{2b^2},\quad c_{23}=-\dfrac{3}{2b},\\ c_{31}=\dfrac{81}{4b^3},\quad c_{32}=\dfrac{27}{4b^2},\quad c_{33}=-\dfrac{9}{2b} \end{array} \tag{12}$$

and

$$\Delta_2=\begin{vmatrix} 1 & 1 & 0\\ (\frac{3}{b}) & (-\frac{3}{2b}) & -(-\frac{3}{2b})\\ (\frac{3}{b})^2 & (-\frac{3}{2b})^2 & -2(-\frac{3}{2b})^2 \end{vmatrix}.$$

By simple calculations, we can write the solution of equation (1) in this case as

$$x_n=\frac{1}{\frac{\alpha_{1n}}{x_0}+\frac{\alpha_{2n}}{x_{-1}}+\frac{\alpha_{3n}}{x_{-2}}}, \tag{13}$$

where

$$\begin{array}{lll} \alpha_{1n} & = & \frac{1}{\Delta_2}(c_{11}(\frac{b}{3})^n+c_{21}(-\frac{2b}{3})^n+c_{31}(-\frac{2b}{3})^n n),\\ \alpha_{2n} & = & \frac{1}{\Delta_2}(c_{12}(\frac{b}{3})^n+c_{22}(-\frac{2b}{3})^n+c_{32}(-\frac{2b}{3})^n n)\\ \text{and} & & \\ \alpha_{3n} & = & \frac{1}{\Delta_2}(c_{13}(\frac{b}{3})^n+c_{23}(-\frac{2b}{3})^n+c_{33}(-\frac{2b}{3})^n n) \end{array} \tag{14}$$

are such that $c_{ij},\ i,j=1,2,3$ are given in (12).

Case $a<\frac{4}{27}b^3$:
When $a<\frac{4}{27}b^3$, the roots of equation (3) are

$$\lambda_0<\frac{b}{3},\quad \lambda_{\pm}=-\frac{b+\lambda_0}{2}\pm\frac{\sqrt{(b+\lambda_0)^2-4\lambda_0(b+\lambda_0)}}{2},$$

where

$$0<\lambda_0<|\lambda_+|<|\lambda_-|.$$

Then the solution of equation (2) is

$$y_n = c_1\lambda_0^n + c_2\lambda_-^n + c_3\lambda_+^n. \tag{15}$$

Using the initials y_{-2}, y_{-1} and y_0, the values of c_1, c_2 and c_3 in this case are:

$$\begin{array}{lcl} c_1 & = & \frac{1}{\Delta_3}(y_0c_{11} + y_{-1}c_{12} + y_{-2}c_{13}), \\ c_2 & = & \frac{1}{\Delta_3}(y_0c_{21} + y_{-1}c_{22} + y_{-2}c_{23}) \\ \text{and} & & \\ c_3 & = & \frac{1}{\Delta_3}(y_0c_{31} + y_{-1}c_{32} + y_{-2}c_{33}), \end{array} \tag{16}$$

where

$$\begin{array}{lll} c_{11} = \dfrac{\lambda_- - \lambda_+}{\lambda_-^2\lambda_+^2}, & c_{12} = \dfrac{-\lambda_-^2 + \lambda_+^2}{\lambda_-^2\lambda_+^2}, & c_{13} = \dfrac{\lambda_- - \lambda_+}{\lambda_-\lambda_+}, \\ c_{21} = \dfrac{\lambda_+ - \lambda_0}{\lambda_+^2\lambda_0^2}, & c_{22} = \dfrac{\lambda_0^2 - \lambda_+^2}{\lambda_+^2\lambda_0^2}, & c_{23} = \dfrac{\lambda_+ - \lambda_0}{\lambda_+\lambda_0}, \\ c_{31} = \dfrac{\lambda_0 - \lambda_-}{\lambda_0^2\lambda_-^2}, & c_{32} = \dfrac{\lambda_-^2 - \lambda_0^2}{\lambda_0^2\lambda_-^2}, & c_{33} = \dfrac{\lambda_0 - \lambda_-}{\lambda_0\lambda_-} \end{array} \tag{17}$$

and

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{\lambda_0} & \frac{1}{\lambda_-} & \frac{1}{\lambda_+} \\ \frac{1}{\lambda_0^2} & \frac{1}{\lambda_-^2} & \frac{1}{\lambda_+^2} \end{vmatrix}.$$

By simple calculations, we can write the solution of equation (1) in this case as

$$x_n = \frac{1}{\frac{\alpha_{1n}}{x_0} + \frac{\alpha_{2n}}{x_{-1}} + \frac{\alpha_{3n}}{x_{-2}}}, \tag{18}$$

where

$$\begin{array}{lcl} \alpha_{1n} & = & \frac{1}{\Delta_3}(c_{11}\lambda_0^n + c_{21}\lambda_-^n + c_{31}\lambda_+^n), \\ \alpha_{2n} & = & \frac{1}{\Delta_3}(c_{12}\lambda_0^n + c_{22}\lambda_-^n + c_{32}\lambda_+^n) \\ \text{and} & & \\ \alpha_{3n} & = & \frac{1}{\Delta_3}(c_{13}\lambda_0^n + c_{23}\lambda_-^n + c_{33}\lambda_+^n) \end{array} \tag{19}$$

are such that c_{ij}, $i, j = 1, 2, 3$ are given in (17).

Using equations (8), (13) and (18), we can write the forbidden set of equation (1) as

$$F = \bigcup_{n=-2}^{\infty} \{(x_0, x_{-1}, x_{-2}) \in \mathbb{R}^3 : \frac{\alpha_{1n}}{x_0} + \frac{\alpha_{2n}}{x_{-1}} + \frac{\alpha_{3n}}{x_{-2}} = 0\},$$

where α_{1n}, α_{2n} and α_{3n} are given as follows:

$$\begin{cases} \alpha_{1n}, \alpha_{2n} \text{ and } \alpha_{3n} \text{ are given in (9)}, & a > \frac{4}{27}b^3; \\ \alpha_{1n}, \alpha_{2n} \text{ and } \alpha_{3n} \text{ are given in (14)}, & a = \frac{4}{27}b^3; \\ \alpha_{1n}, \alpha_{2n} \text{ and } \alpha_{3n} \text{ are given in (19)}, & a < \frac{4}{27}b^3. \end{cases}$$

3. Global behavior of equation (1)

Consider the set $D = \{(x,y,z) \in \mathbb{R}^3 : \frac{\lambda_0^2}{x} + \frac{a}{y} + \frac{a\lambda_0}{z} = 0\}$.

Theorem 3.1. *The set D is an invariant for equation (1).*

Proof. Let $(x_0, x_{-1}, x_{-2}) \in D$. We show that $(x_k, x_{k-1}, x_{k-2}) \in D$ for each $k \in N$. The proof is by induction on k. The point $(x_0, x_{-1}, x_{-2}) \in D$, implies

$$\frac{\lambda_0^2}{x_0} + \frac{a}{x_{-1}} + \frac{a\lambda_0}{x_{-2}} = 0.$$

Now for $k = 1$, we have

$$\frac{\lambda_0^2}{x_1} + \frac{a}{x_0} + \frac{a\lambda_0}{x_{-1}} = \frac{\lambda_0^2}{x_0 x_{-2}}(ax_0 - bx_{-2}) + \frac{a}{x_0} + \frac{a\lambda_0}{x_{-1}}$$

$$= \frac{1}{x_0 x_{-1} x_{-2}}(a\lambda_0^2 x_0 x_{-1} - b\lambda_0^2 x_{-1} x_{-2} + ax_{-1}x_{-2} + a\lambda_0 x_0 x_{-2})$$

$$= \frac{1}{x_0 x_{-1} x_{-2}}(a\lambda_0^2 x_0 x_{-1} + (\lambda_0^3 - a)x_{-1} x_{-2} + ax_{-1}x_{-2} + a\lambda_0 x_0 x_{-2})$$

$$= \frac{1}{x_0 x_{-1} x_{-2}}(a\lambda_0^2 x_0 x_{-1} + \lambda_0^3 x_{-1} x_{-2} + a\lambda_0 x_0 x_{-2})$$

$$= \lambda_0(\frac{\lambda_0^2}{x_0} + \frac{a}{x_{-1}} + \frac{a\lambda_0}{x_{-2}}) = 0.$$

This implies that $(x_1, x_0, x_{-1}) \in D$.
Suppose that the $(x_k, x_{k-1}, x_{k-2}) \in D$. That is

$$\frac{\lambda_0^2}{x_k} + \frac{a}{x_{k-1}} + \frac{a\lambda_0}{x_{k-2}} = 0.$$

Then

$$\frac{\lambda_0^2}{x_{k+1}} + \frac{a}{x_k} + \frac{a\lambda_0}{x_{k-1}} = \frac{\lambda_0^2}{x_k x_{k-2}}(ax_k - bx_{k-2}) + \frac{a}{x_k} + \frac{a\lambda_0}{x_{k-1}}$$

$$= \frac{1}{x_k x_{k-1} x_{k-2}}(a\lambda_0^2 x_k x_{k-1} - b\lambda_0^2 x_{k-1} x_{k-2} + ax_{k-1}x_{k-2} + a\lambda_0 x_k x_{k-2})$$

$$= \frac{1}{x_k x_{k-1} x_{k-2}}(a\lambda_0^2 x_k x_{k-1} + (\lambda_0^3 - a)x_{k-1} x_{k-2} + ax_{k-1}x_{k-2} + a\lambda_0 x_k x_{k-2})$$

$$= \frac{1}{x_k x_{k-1} x_{k-2}}(a\lambda_0^2 x_k x_{k-1} + \lambda_0^3 x_{k-1} x_{k-2} + a\lambda_0 x_k x_{k-2})$$

$$= \lambda_0(\frac{\lambda_0^2}{x_k} + \frac{a}{x_{k-1}} + \frac{a\lambda_0}{x_{k-2}}) = 0.$$

Therefore, $(x_{k+1}, x_k, x_{k-1}) \in D$.
This completes the proof. ☐

Note that, for the point $(x,y,z) \in \mathbb{R}^3$, the relation $\frac{\lambda_0^2}{x} + \frac{a}{y} + \frac{a\lambda_0}{z} = 0$ is equivalent to $c_1(x,y,z) = 0$.

Theorem 3.2. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F \cup D$. If $a > \frac{4}{27}b^3$, then we have the following:*

(1) *If $a \geq b+1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
(2) *If $a < b+1$, then we have the following:*
  (a) *If $a \geq 1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
  (b) *If $a < 1$, then we have the following:*
    (i) *If $a^2+ab-1>0$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
    (ii) *If $a^2+ab-1=0$, then $\{x_n\}_{n=-2}^{\infty}$ is bounded.*
    (iii) *If $a^2+ab-1<0$, then $\{x_n\}_{n=-2}^{\infty}$ is unbounded.*

Proof. The solution of equation (1) when $a > \frac{4}{27}b^3$ is

$$x_n = \frac{1}{c_1\lambda_0^n + (\frac{a}{\lambda_0})^{\frac{n}{2}}(c_2\cos n\theta + c_3 \sin n\theta))}.$$

(1) When $a > b+1$, we have that $\lambda_0 > 1$ and $\lambda_0 < \sqrt[3]{a} < a$. That is $(\frac{a}{\lambda_0})^n \to \infty$ and $\lambda_0^n \to \infty$ as $n \to \infty$.
If $a = b+1$, then we have that $\lambda_0 = 1$ and $\lambda_0 = 1 < \sqrt[3]{a} < a$. That is $(\frac{a}{\lambda_0})^n \to \infty$ as $n \to \infty$ and the result follows.
(2) When $a < b+1$, we have that $\lambda_0 < 1$.
  (a) If $a \geq 1$, then $\lambda_0 < 1 \leq \sqrt[3]{a} \leq a$. That is $\lambda_0^n \to 0$ and $(\frac{a}{\lambda_0})^n \to \infty$, from which the result follows.
  (b) If $a < 1$, then $a < \sqrt[3]{a}$ and we have the following:
    (i) If $a^2+ab-1>0$, then $\lambda_0 < a < \sqrt[3]{a} < 1$. This implies that $\lambda_0^n \to 0$ and $(\frac{a}{\lambda_0})^n \to \infty$, from which the result follows.
    (ii) If $a^2+ab-1=0$, then $\lambda_0 = a < \sqrt[3]{a} < 1$. That is $\lambda_0^n \to 0$. But as

$$|c_1\lambda_0^n + c_2\cos n\theta + c_3 \sin n\theta| \neq 0 \text{ for all } n \geq 0, \tag{20}$$

    the quantity (20) attains its infemum value say $\epsilon > 0$ and the result follows.
    (iii) If $a^2+ab-1<0$, then $a < \lambda_0 < \sqrt[3]{a} < 1$. This implies that $\lambda_0^n \to 0$ and $(\frac{a}{\lambda_0})^n \to 0$, from which the result follows.

□

When $a = \frac{4}{27}b^3$, we have that $\lambda_0 = \frac{b}{3}$. So the set D can be written as

$$D = \{(x,y,z) \in \mathbb{R}^3 : \frac{9}{x} + \frac{12b}{y} + \frac{4b^2}{z} = 0\}.$$

Theorem 3.3. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F \cup D$. If $a = \frac{4}{27}b^3$, then we have the following:*

(1) *If $a \geq b+1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
(2) *If $a < b+1$, then we have the following:*
  (a) *If $0 < b < \frac{3}{2}$, then $\{x_n\}_{n=-2}^{\infty}$ is unbounded.*
  (b) *If $b = \frac{3}{2}$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*

(c) *If* $\frac{3}{2} < b < 3$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to zero.*

Proof. The solution of equation (1) when $a = \frac{4}{27}b^3$ is

$$x_n = \frac{1}{c_1(\frac{b}{3})^n + c_2(-\frac{2b}{3})^n + c_3(-\frac{2b}{3})^n n}.$$

(1) When $a \geq b+1$, it is sufficient to see that $\lambda_0 = \frac{b}{3} \geq 1$ and the result follows.
(2) When $a < b+1$, we have that $\lambda_0 = \frac{b}{3} < 1$.
  (a) If $0 < b < \frac{3}{2}$, then $\frac{b}{3} < \frac{1}{2}$ and $\frac{2b}{3} < 1$, from which the result follows.
  (b) If $b = \frac{3}{2}$, then $\frac{b}{3} = \frac{1}{2}$ and $\frac{2b}{3} = 1$, from which the result follows.
  (c) If $\frac{3}{2} < b < 3$, then $\frac{1}{2} < \frac{b}{3} < 1$ and $1 < \frac{2b}{3} < 2$, from which the result follows.

□

Now assume that $a < \frac{4}{27}b^3$. We shall consider the three sets

$$D_i = \{(x, y, z) \in \mathbb{R}^3 : \frac{\lambda^2}{x} + \frac{a}{y} + \frac{a\lambda}{z} = 0\}, \quad i = 1, 2, 3,$$

where

$$\begin{cases} \lambda = \lambda_0, & \text{i=1;} \\ \lambda = \lambda_-, & \text{i=2;} \\ \lambda = \lambda_+, & \text{i=3.} \end{cases}$$

By simple calculations, we can see that:

$$\begin{cases} D_i \text{ is equivalent to } c_1(x, y, z) = 0, & \text{i=1;} \\ D_i \text{ is equivalent to } c_2(x, y, z) = 0, & \text{i=2;} \\ D_i \text{ is equivalent to } c_3(x, y, z) = 0, & \text{i=3.} \end{cases}$$

Theorem 3.4. *Each set of the sets* D_i, $i = 1$ *,2 and* 3 *is an invariant for equation (1).*

Proof. The proof is similar to that of theorem (3.1) and will be omitted. □

Theorem 3.5. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation (1) such that* $(x_0, x_{-1}, x_{-2}) \notin F \cup D_2$. *If* $a < \frac{4}{27}b^3$, *then we have the following:*

(1) *If* $a > -1 + b$, *then* $\{x_n\}_{n=-2}^{\infty}$ *is unbounded.*
(2) *If* $a = -1 + b$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to the period-*2 *solution*

$$\{..., -\frac{1}{c_2}, \frac{1}{c_2}, -\frac{1}{c_2}, \frac{1}{c_2}, ...\}.$$

(3) *If* $a < -1 + b$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to zero.*

Proof. The solution of equation (1) when $a < \frac{4}{27}b^3$ is

$$x_n = \frac{1}{c_1\lambda_0^n + c_2\lambda_-^n + c_3\lambda_+^n}.$$

Clear that

$$\lambda_- < -\frac{2b}{3} < \lambda_+ < -\frac{b}{3} < 0 < \lambda_0 \text{ and } 0 < \lambda_0 < |\lambda_+| < |\lambda_-|.$$

The condition $(x_0, x_{-1}, x_{-2}) \notin F \cup D_2$ ensures that $c_2 \neq 0$.

(1) If $a > -1 + b$, then $\lambda_- > -1$. This implies that $\lambda_0 < |\lambda_+| < |\lambda_-| < 1$, from which the result follows.

(2) If $a = -1 + b$, then $\lambda_- = -1$. This implies that $\lambda_0 < |\lambda_+| < |\lambda_-| = 1$. Then

$$x_{2n} \to \frac{1}{c_2} \text{ and } x_{2n+1} \to -\frac{1}{c_2}.$$

Clear that

$$\{..., -\frac{1}{c_2}, \frac{1}{c_2}, -\frac{1}{c_2}, \frac{1}{c_2}, ...\}$$

is a period-2 solution of equation (1).

(3) If $a < -1+b$, then $\lambda_- < -1$. The solution of equation (1) can be written

$$x_n = \frac{1}{\lambda_-^n (c_1 (\frac{\lambda_0}{\lambda_-})^n + c_2 + c_3 (\frac{\lambda_+}{\lambda_-})^n)}.$$

Clear that $\frac{\lambda_0}{\lambda_-} > -1$ and $\frac{\lambda_+}{\lambda_-} < 1$, from which the result follows.

□

In the following results, we show that when $a > \frac{4}{27}b^3$, under certain conditions there exist solutions, either periodic or converge to periodic solutions for equation (1).

Suppose that $\theta = \frac{p}{q}\pi$, where p and q are positive relatively prime integers such that $\frac{q}{2} < p < q$.

Theorem 3.6. *Assume that $a > \frac{4}{27}b^3$, $a < b+1$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin D \cup F$. If $a^2 + ba - 1 = 0$, then $\{x_n\}_{n=-2}^{\infty}$ converges to a periodic solution with prime period $2q$.*

Proof. Assume that $\{x_n\}_{n=-2}^{\infty}$ is a solution of equation (1) such that (x_0, x_{-1}, x_{-2}) $\notin D \cup F$ and let the angle $\theta = \frac{p}{q}\pi \in]\frac{\pi}{2}, \pi[$.

When $a > \frac{4}{27}b^3$ and $a^2 + ba - 1 = 0$ ($\lambda_0 = a < 1$), the solution of equation (1) is

$$x_n = \frac{1}{c_1 \lambda_0^n + c_2 \cos n\theta + c_3 \sin n\theta}.$$

Then we can write

$$\begin{aligned} x_{2qm+l} &= \frac{1}{c_1 \lambda_0^{2qm+l} + c_2 \cos(2qm+l)\theta + c_3 \sin(2qm+l)\theta} \\ &= \frac{1}{c_1 \lambda_0^{2qm+l} + c_2 \cos l\theta + c_3 \sin l\theta}, \ l = 1, 2, ..., 2q. \end{aligned}$$

As $m \to \infty$, we get

$$x_{2qm+l} \to \mu_l = \frac{1}{c_2 \cos l\theta + c_3 \sin l\theta}, \ l = 1, 2, ..., 2q.$$

Therefore, the solution $\{x_n\}_{n=-2}^{\infty}$ converges to

$$\{..., \mu_1, \mu_2, ..., \mu_{2q-1}, \mu_{2q}, \mu_1, \mu_2, ..., \mu_{2q-1}, \mu_{2q}, ...\}. \quad (21)$$

Simple calculations show that the solution (21) is a period-$2q$ solution for equation (1) and will be omitted.
This completes the proof. □

Theorem 3.7. *Assume that $a > \frac{4}{27}b^3$, $a < b+1$ and $a^2 + ba - 1 = 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F$. If $(x_0, x_{-1}, x_{-2}) \in D$, then $\{x_n\}_{n=-2}^{\infty}$ is a periodic solution with prime period $2q$.*

Proof. Assume that $\{x_n\}_{n=-2}^{\infty}$ is a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F$ and let the angle $\theta = \frac{p}{q}\pi \in]\frac{\pi}{2}, \pi[$.
When $(x_0, x_{-1}, x_{-2}) \in D$, we have that $c_1 = 0$ and the solution of equation (1) is

$$x_n = \frac{1}{c_2 \cos n\theta + c_3 \sin n\theta}.$$

Then we have

$$\begin{aligned} x_{n+2q} &= \frac{1}{c_2 \cos(n+2q)\theta + c_3 \sin(n+2q)\theta} \\ &= \frac{1}{c_2 \cos(n\theta + 2p\pi) + c_3 \sin(n\theta + 2p\pi)} \\ &= \frac{1}{c_2 \cos(n\theta) + c_3 \sin(n\theta)} \\ &= x_n. \end{aligned}$$

This completes the proof. □

Example (1) Figure 1. shows that if $a = b = \frac{1}{\sqrt{2}}$ ($a > \frac{4}{27}b^3$, $a < b+1$, $a^2 + ab - 1 = 0$ and $\theta = \frac{3}{4}\pi$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1) with initial conditions $x_{-2} = 1$, $x_{-1} = 1.2$ and $x_0 = -1$ converges to a period-8 solution.

Example (2) Figure 2. shows that if $a = b = \frac{1}{\sqrt{2}}$ ($a > \frac{4}{27}b^3$, $a < b+1$, $a^2 + ab - 1 = 0$ and $\theta = \frac{3}{4}\pi$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1) with initial conditions $x_{-2} = 2$, $x_{-1} = -\frac{2\sqrt{2}}{3}$ and $x_0 = 1$ ($(x_{-2}, x_{-1}, x_0) \in D$) is periodic with prime period-8 solution.

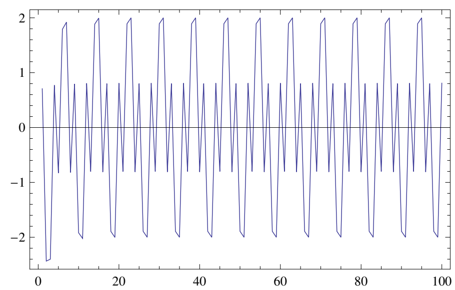

Figure 1. $x_{n+1} = \frac{x_n x_{n-2}}{\frac{1}{\sqrt{2}} x_n - \frac{1}{\sqrt{2}} x_{n-2}}$

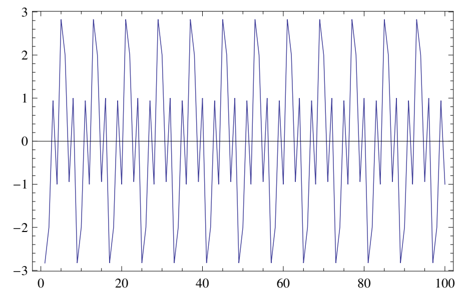

Figure 2. $x_{n+1} = \frac{x_n x_{n-2}}{\frac{1}{\sqrt{2}} x_n - \frac{1}{\sqrt{2}} x_{n-2}}$

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