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TOTAL DOMINATION NUMBER OF CENTRAL TREES

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Abstract. Let $\gamma_t(G)$ and $\tau(G)$ denote the total domination number and vertex cover number of graph G, respectively. In this paper, we study the total domination number of the central tree $C(T)$ for a tree T. First, a relationship between the total domination number of $C(T)$ and the vertex cover number of tree T is discussed. We characterize the central trees with equal total domination number and independence number. Applying the first result, we improve the upper bound on the total domination number of $C(T)$ and solve one open problem posed by Kazemnejad et al..

1. Introduction

Graph theory terminology not presented here can be found in [2] and [3]. Let G be a simple and undirected graph. The *vertex set* and the *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively. The *degree, neighborhood* and *closed neighborhood* of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. If the graph G is clear from context, we simply write $d(v)$, $N(v)$ and $N[v]$, respectively. The *minimum degree* and *maximum degree* of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. The subgraph induced by $S \subseteq V$ is denoted by $G[S]$. For any two vertices u and v, let $d(u, v)$ denote the distance between vertex u and vertex v. The *diameter* of G, denoted by $diam(G)$, is the maximum distance among pairs of vertices in G. Let P_n and C_n denote the path and cycle with order n, respectively. Let $K_{1,n}$ denote the star with order $n + 1$.

A set $S \subseteq V(G)$ in a graph G is called a *total dominating set* if every vertex in G is adjacent to at least one vertex in S. The *total domination number* $\gamma_t(G)$ equals to the minimum cardinality of a total dominating set in G. Moreover, a total dominating set of G of cardinality $\gamma_t(G)$ is called a γ_t-set of G.

The concept of total domination in graphs was first introduced by Cockayne, Dawes and Hedetniemi [1] and has been studied extensively by many researchers

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in the last years. The literature on the subject has been surveyed and detailed in the recent book [4].

A *vertex cover* of the graph G is a set $D \subseteq V(G)$ such that every edge of G is incident to at least one element of D. The *vertex cover number* of G, denoted by $\tau(G)$, is the minimum cardinality of a vertex cover of G. An *independent set* of G is a subset of vertices of G, no two of which are adjacent. A *maximum independent set* is an independent set with the largest cardinality in G. This cardinality is called the *independence number* of G, and is denoted by $\alpha(G)$. A *matching* of G is a subset of edges of G such that no two of which have a common vertex. For a tree T, any vertex of degree one is called a *leaf* and the neighbour of a leaf is called a *support vertex* of G.

By doing an operation on a given graph G, Vernold et al. [6] obtained the central graph of G as follows.

Definition 1 ([6])**.** The central graph $C(G)$ of a graph G of order n and size m is a graph of order $n+m$ and size $\binom{n}{2}+m$ which is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.

We fix a notation for the vertex set and the edge set of the central graph $C(G)$ to work with throughout the paper. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We set $V(C(G)) = V(G) \cup \mathcal{C}$, where $\mathcal{C} = \{c_{ij} : v_iv_j \in E(G)\}$ and $E(C(G)) = \{v_ic_{ij}, v_jc_{ij} : v_iv_j \in E(G)\} \cup \{v_iv_j : v_iv_j \notin E(G)\}$. The central graph $C(T)$ of a tree T is called a *central tree*.

Kazemnejad et al. [5] gave the following results.

Proposition 1 ([5])**.** *For any connected graph G of order $n \geq 2$, $\gamma_t(C(G)) \geq \tau(G)$.*

Proposition 2 ([5])**.** *Let T be a tree of order $n \geq 3$ such that $\Delta(T) \geq n-3$. Then $\gamma_t(C(T)) = 3$.*

Proposition 3 ([5])**.** *Let T be a tree of order $n \geq 7$ such that $\Delta(T) \leq n-4$. Then $\gamma_t(C(T)) \leq \lfloor \frac{2n}{3} \rfloor$. Moreover, the upper bound is tight.*

Proposition 4 ([5])**.** *For any connected graph G of order $n \geq 3$ and size m, $\alpha(C(G)) = m$.*

Proposition 5 ([5])**.** *For any connected graph G of order $n \geq 3$ with $\Delta(G) \leq n-2$, $\gamma_t(C(T))) \geq 3$.*

Furthermore, they gave the following two open problems.

Problem 1 ([5])**.** Find some families graphs G of order n and size m where $m \geq n \geq 5$ with $\gamma_t(C(G)) = \alpha(C(G))$.

Problem 2 ([5])**.** Characterize the trees T satisfying $\gamma_t(C(T)) = \lfloor \frac{2n}{3} \rfloor$.

In this paper, we study the total domination number of the central tree $C(T)$ for a tree T. First, a relationship between the total domination number of $C(T)$ and the vertex cover number of tree T is discussed. We characterize the central trees with equal total domination number and independence number. Applying the first result, we improve the upper bound on the total domination number of $C(T)$ and solve one open problem posed by Kazemnejad et al..

2. Main results

Let T be a tree, $L(T) = \{v : d(v) = 1\}$ and $I(T) = V(T) \setminus L(T)$. Let $S(T) = \{v : v \text{ is a support vertex}\}$ and $N_2(T) = \{v : d(v) = 2\}$. Let $S_{1,n,n}$ be the tree obtained from star $K_{1,n}$ by replacing every edge by a path of length 2. Let $S_{1,2}$ be the tree obtained from vertex v and P_4 by adding an edge between vertex v and one support vertex of P_4. Let $\mathcal{T} = \{P_5, K_{1,4}, S_{1,2}, S_{1,3,3}\}$.

Lemma 1. *Let T be a tree. If $diam(T) \geq 5$, then $\gamma_t(C(T)) = \tau(T)$.*

Proof. Suppose that $diam(T) = d$. Then $d \geq 5$. Let $v_0v_1 \cdots v_d$ be a longest path in T. Then both v_0 and v_d are leaves. Let $L_i(T) = \{v : d(v_0, v) = i\}$ for $i = 0, 1, 2, \ldots, d$. It is clearly $\bigcup_{i=0}^{d} L_i(T) = V(T)$. Let D be a minimal vertex cover of T such that $\tau(T) = |D|$. Since $v_0v_1 \in E(T)$ and $v_{d-1}v_d \in E(T)$, it follows that $D \cap \{v_0, v_1\} \neq \emptyset$ and $D \cap \{v_{d-1}, v_d\} \neq \emptyset$. If $v_1 \notin D$, then $v_0 \in D$. Then $(D \setminus \{v_0\}) \cup \{v_1\}$ is a minimal vertex cover of T with cardinality $\tau(T)$. Without loss of generality, we can assume that $\{v_1, v_{d-1}\} \subseteq D$. For any vertex $c_{ij} \in \mathcal{C}$, since $v_iv_j \in E(T)$, it follows that $D \cap \{v_i, v_j\} \neq \emptyset$. Hence vertex c_{ij} is dominated by D. It is obvious that every vertex in $\bigcup_{i=3}^{d} L_i(T)$ is dominated by vertex v_1 in $C(T)$ and every vertex in $\bigcup_{i=0}^{2} L_i(T)$ is dominated by vertex v_{d-1} in $C(T)$. So D is a total dominating set of $C(T)$. Hence, $\gamma_t(C(T)) \leq |D| = \tau(T)$. By Proposition 1, $\gamma_t(C(T)) = \tau(T)$. □

Lemma 2. *Let T be a tree with $diam(T) = 4$. If $I(T) = S(T)$ and $I(T) \cap N_2(T) \neq \emptyset$, then $\gamma_t(C(T)) = \tau(T)$.*

Proof. Let $v_0v_1 \cdots v_4$ be a longest path in T. Then both v_0 and v_4 are leaves. Let $L_i(T) = \{v : d(v_0, v) = i\}$ for $i = 0, 1, 2, \ldots, 4$. It is clearly $\bigcup_{i=0}^{4} L_i(T) = V(T)$. Let D be a minimal vertex cover of T such that $|D| = \tau(T)$.

Since there exists a matching M of T such that every edge of M is incident to one leaf and one support vertex, $|D| \geq |S(T)|$. Since $I(T) = S(T)$, $|D| \geq |I(T)|$. So $\tau(T) \geq |I(T)|$. It is obvious that $I(T)$ is a vertex cover of T and $\tau(T) \leq |I(T)|$. Hence $\tau(T) = |I(T)|$. Since $I(T) \cap N_2(T) \neq \emptyset$, it follows that $N_2(T) \subseteq (L_1(T) \cup L_3(T))$. Without loss of generality, we can assume that $v_3 \in I(T) \cap N_2(T)$. Let $I'(T) = (I(T) \setminus \{v_3\}) \cup \{v_4\}$. It is obvious that $I'(T)$ is a vertex cover of T with cardinality $\tau(T)$. In the following, we will prove that $I'(T)$ is a total dominating set of $C(T)$.

For any vertex $c_{ij} \in \mathcal{C}$, since $v_iv_j \in E(T)$, it follows that $I'(T) \cap \{v_i, v_j\} \neq \emptyset$. Hence vertex c_{ij} is dominated by $I'(T)$. It is obvious that every vertex in $\bigcup_{i=0}^{2} L_i(T)$ is dominated by vertex v_4 in $C(T)$ and every vertex in $\bigcup_{i=3}^{4} L_i(T)$

is dominated by vertex v_1 in $C(T)$. So $I'(T)$ is a total dominating set of $C(T)$. Hence, $\gamma_t(C(T)) \leq |I'(T)| = \tau(T)$. By Proposition 1, $\gamma_t(C(T)) = \tau(T)$. □

Lemma 3. *Let T be a tree with $diam(T) = 4$. If $I(T) \neq S(T)$ or $I(T) \cap N_2(T) = \emptyset$, then $\gamma_t(C(T)) = \tau(T) + 1$.*

Proof. Let $v_0 v_1 \cdots v_4$ be a longest path in T. Then both v_0 and v_4 are leaves. Let $L_i(T) = \{v : d(v_0, v) = i\}$ for $i = 0, 1, 2, \ldots, 4$. It is clearly $\bigcup_{i=0}^{4} L_i(T) = V(T)$. We will discuss it from the following cases.

Case 1. $I(T) = S(T)$ and $I(T) \cap N_2(T) = \emptyset$. Let D be a minimal vertex cover of T such that $|D| = \tau(T)$. Since there exists a matching M of T such that every edge of M is incident to one leaf and one support vertex, $|D| \geq |S(T)|$. Since $I(T) = S(T)$, $|D| \geq |I(T)|$. So $\tau(T) \geq |I(T)|$. It is obvious that $I(T)$ is a vertex cover of T and $\tau(T) \leq |I(T)|$. Hence $\tau(T) = |I(T)|$. Furthermore, if v_2 is adjacent to at least two leaves, then $I(T)$ is the unique minimum vertex cover of T. If v_2 is adjacent to exactly one leaf, say u, then $I(T)$ or $(I(T) \setminus \{v_2\}) \cup \{u\}$ is the unique two minimum vertex covers of T.

By Proposition 1, $\gamma_t(C(T)) \geq \tau(T)$. Assume that $\gamma_t(C(T)) = \tau(T)$.

Let S be a minimal total dominating set of $C(T)$ such that $\gamma_t(C(T)) = |S|$. Then $S \cap V(T)$ is a vertex cover of T. So, $\tau(T) \leq |S \cap V(T)| \leq |S| = \gamma_t(C(T))$. Since $\gamma_t(C(T)) = \tau(T)$, it follows that $S \cap V(T) = S$. Then S is a minimum vertex cover of T. Hence, if v_2 is adjacent to at least two leaves, then $S = I(T)$. If v_2 is adjacent to exactly one leaf, say u, then $S = I(T)$ or $S = (I(T) \setminus \{v_2\}) \cup \{u\}$. For any case, vertex v_2 is not dominated by S in $C(T)$, which is a contradiction. Hence, $\gamma_t(C(T)) \geq \tau(T) + 1$. Since $I(T) \cup \{c_{12}\}$ is a total dominating set of $C(T)$, it follows that $\gamma_t(C(T)) \leq |I(T)| + 1 = \tau(T) + 1$. So $\gamma_t(C(T)) = \tau(T) + 1$.

Case 2. $I(T) \neq S(T)$. Then $v_2 \notin S(T)$ and $I(T) = S(T) \cup \{v_2\}$. Let D be a minimal vertex cover of T such that $|D| = \tau(T)$. If a leaf v belongs to D, then $(D \setminus \{v\}) \cup \{w\}$ is a minimum vertex cover of T, where $w \in N(v) \cap S(T)$. Without loss of generality, we can assume that $D \cap L(T) = \emptyset$. That is $S(T) \subseteq D$. If $v_2 \in D$, then $D \setminus \{v_2\}$ is a vertex cover of T with cardinality less than $|D|$, which is a contradiction. So $v_2 \notin D$. Hence, $\tau(T) = |D| \geq |I(T)| - 1$. It is obvious that $I(T) \setminus \{v_2\}$ is a vertex cover of T and $\tau(T) \leq |I(T)| - 1$. Hence $\tau(T) = |I(T)| - 1$. Furthermore, $I(T) \setminus \{v_2\}$ is the unique minimum vertex cover of T.

By Proposition 1, $\gamma_t(C(T)) \geq \tau(T)$. Assume that $\gamma_t(C(T)) = \tau(T)$.

Let S be a minimal total dominating set of $C(T)$ such that $\gamma_t(C(T)) = |S|$. Then $S \cap V(T)$ is a vertex cover of T. So, $\tau(T) \leq |S \cap V(T)| \leq |S| = \gamma_t(C(T))$. Since $\gamma_t(C(T)) = \tau(T)$, it follows that $S \cap V(T) = S$. Then S is a minimum vertex cover of T. Hence, $S = I(T) \setminus \{v_2\}$. Then vertex v_2 is not dominated by S in $C(T)$, which is a contradiction. Hence, $\gamma_t(C(T)) \geq \tau(T) + 1$. Since $(I(T) \setminus \{v_2\}) \cup \{c_{12}\}$ is a total dominating set of $C(T)$, it follows that $\gamma_t(C(T)) \leq |(I(T) \setminus \{v_2\}) \cup \{c_{12}\}| = |I(T)| = \tau(T) + 1$. So $\gamma_t(C(T)) = \tau(T) + 1$. □

Lemma 4. *Let T be a tree. If $diam(T) = 3$, then $\gamma_t(C(T)) = \tau(T) + 1$. If $diam(T) = 2$, then $\gamma_t(C(T)) = \tau(T) + 2$.*

Proof. If $diam(T) = 3$, then it is obvious that $\gamma_t(C(T)) = 3$ and $\tau(T) = 2$ by Proposition 5. So $\gamma_t(C(T)) = \tau(T)+1$. If $diam(T) = 2$, then it is obvious that $\gamma_t(C(T)) = 3$ and $\tau(T) = 1$ by Proposition 2. So $\gamma_t(C(T)) = \tau(T) + 2$. □

By Lemma 1, Lemma 2, Lemma 3 and Lemma 4, we have the following.

Theorem 1. *Let T be a tree with order $n \geq 3$. Then*

$$\gamma_t(C(T)) = \begin{cases} \tau(T), & \text{if } diam(T) \geq 5, \\ \tau(T), & \text{if } diam(T)=4,\ I(T)=S(T) \text{ and } I(T)\cap N_2(T)\neq\emptyset, \\ \tau(T)+2, & \text{if } diam(T)=2, \\ \tau(T)+1, & \text{otherwise.} \end{cases}$$

For any tree T of order $n \geq 3$, it is well known that $\tau(T) \leq \lfloor \frac{n}{2} \rfloor$. By Theorem 1, an improved upper bound on total domination number of $C(T)$ is as follows.

Corollary 1. *Let T be a tree with order $n \geq 3$. Then*

$$\gamma_t(C(T)) \leq \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } diam(T) \geq 5, \\ \lfloor \frac{n}{2} \rfloor, & \text{if } diam(T)=4,\ I(T)=S(T) \text{ and } I(T)\cap N_2(T)\neq\emptyset, \\ \lfloor \frac{n}{2} \rfloor+2, & \text{if } diam(T)=2, \\ \lfloor \frac{n}{2} \rfloor+1, & \text{otherwise.} \end{cases}$$

Theorem 2. *Let T be a tree with order $n \geq 3$. Then $\gamma_t(C(T)) = \alpha(C(T))$ if and only if $T = P_4$ or $T = K_{1,3}$.*

Proof. It is obvious that if $T = P_4$ or $T = K_{1,3}$, then $\gamma_t(C(T)) = \alpha(C(T))$ by Proposition 4 and Theorem 1. The sufficiency follows. Now we only prove the necessity. Suppose that $\gamma_t(C(T)) = \alpha(C(T))$. By Proposition 4, $\alpha(C(T)) = m(T) = n - 1$. Hence, $\gamma_t(C(T)) = n - 1$.

Suppose that $diam(T) \geq 4$. Then $n \geq 5$. By Corollary 1, $\gamma_t(C(T)) \leq \lfloor \frac{n}{2} \rfloor + 1 < n - 1$, which is a contradiction. So $diam(T) \leq 3$.

If $diam(T) = 3$, then $\gamma_t(C(T)) = \tau(T)+1 = 3$. So $n = 4$ and T is isomorphic to P_4. If $diam(T) = 2$, then $\gamma_t(C(T)) = \tau(T) + 2 = 3$. So $n = 4$ and T is isomorphic to $K_{1,3}$. □

Theorem 3. *Let T be a tree with order $n \geq 3$. Then $\gamma_t(C(T)) = \lfloor \frac{2n}{3} \rfloor$ if and only if $T \in \mathcal{T}$.*

Proof. It is obvious that if $T \in \mathcal{T}$, then $\gamma_t(C(T)) = \lfloor \frac{2n}{3} \rfloor$ by Theorem 1. The sufficiency follows. Now we only prove the necessity. Suppose that $\gamma_t(C(T)) = \lfloor \frac{2n}{3} \rfloor$.

Suppose that $diam(T) \geq 5$ or $diam(T) = 4$, $I(T) = S(T)$ and $I(T) \cap N_2(T) \neq \emptyset$. Then $n \geq 6$. By Corollary 1, $\gamma_t(C(T)) \leq \lfloor \frac{n}{2} \rfloor < \lfloor \frac{2n}{3} \rfloor$, which is a contradiction. Suppose that $diam(T) = 4$, $I(T) = S(T)$ and $I(T) \cap N_2(T) = \emptyset$. Then $n \geq 6$. By the proof of Case 1 in Lemma 3, $|I(T)| = \tau(T)$ and $\gamma_t(C(T)) = \tau(T) + 1$. Hence, $\gamma_t(C(T)) = |S(T)| + 1$. Since $I(T) = S(T)$ and

$I(T) \cap N_2(T) = \emptyset$, it follows that $|S(T)|+1 \leq \lfloor \frac{n}{2} \rfloor$. So $\gamma_t(C(T)) \leq \lfloor \frac{n}{2} \rfloor < \lfloor \frac{2n}{3} \rfloor$, which is a contradiction.

Hence, T is a tree with $diam(T) \leq 4$. Furthermore, if $diam(T) = 4$, then $I(T) \neq S(T)$.

If $diam(T) = 4$ and $I(T) \neq S(T)$, then $\gamma_t(C(T)) = \tau(T) + 1 = |I(T)|$ by the proof of Case 2 in Lemma 3. So, $|I(T)| = \lfloor \frac{2n}{3} \rfloor$. Hence, $|L(T)| = n - |I(T)| = \lceil \frac{n}{3} \rceil$. Since every vertex in $I(T) \setminus \{v_2\}$ is adjacent to at least one leaf, it follows that $\lceil \frac{n}{3} \rceil \geq \lfloor \frac{2n}{3} \rfloor - 1$. Hence $n = 5$ or $n = 7$. If $n = 5$, then $T = P_5$. Suppose that $n = 7$. Then $|I(T)| = 4$. So $T = S_{1,3,3}$.

If $diam(T) = 3$, then $\gamma_t(C(T)) = 3$. So $n = 5$ and $T = S_{1,2}$. If $diam(T) = 2$, then $\gamma_t(C(T)) = 3$. So $n = 5$ and $T = K_{1,4}$. Hence, in all cases, $T \in \mathcal{T}$. □

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