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MODULAR INVARIANTS UNDER THE ACTIONS OF SOME REFLECTION GROUPS RELATED TO WEYL GROUPS

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Abstract. Some modular representations of reflection groups related to Weyl groups are considered. The rational cohomology of the classifying space of a compact connected Lie group G with a maximal torus T is expressed as the ring of invariants, $H^*(BG;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^{W(G)}$, which is a polynomial ring. If such Lie groups are locally isomorphic, the rational representations of their Weyl groups are equivalent. However, the integral representations need not be equivalent. Under the mod p reductions, we consider the structure of the rings, particularly for the Weyl group of symplectic groups $Sp(n)$ and for the alternating groups A_n as the subgroup of $W(SU(n))$. We will ask if such rings of invariants are polynomial rings, and if each of them can be realized as the mod p cohomology of a space. For $n = 3, 4$, the rings under a conjugate of $W(Sp(n))$ are shown to be polynomial, and for $n = 6, 8$, they are non–polynomial. The structures of $H^*(BT^{n-1};\mathbb{F}_p)^{A_n}$ will be also discussed for $n = 3, 4$.

The invariant theory of some finite groups will be discussed, [20] and [19]. For any prime p we note that $H^*(BT^n;\mathbb{F}_p) = \mathbb{F}_p[t_1, t_2, \ldots, t_n]$, a polynomial ring generated by n elements of degree 2. When p does not divide the order of a subgroup W of $GL(n, \mathbb{F}_p)$, it is well-known that the invariant ring $\mathbb{F}_p[t_1, t_2, \ldots, t_n]^W$ is a polynomial ring if and only if W is a pseudo-reflection group, [13, §20-2, §20-3]. This result can fail in a modular case. Namely, even if W is a pseudo-reflection group, the invariant ring need not be a polynomial ring for $|W| \equiv 0 \bmod p$. This paper concerns such uncertainty.

The Weyl group of $SU(n)$ is isomorphic to the symmetric group Σ_n. The reflection group $W(SU(n))$ is generated by the permutation matrices Σ_{n-1} and an $(n-1)\times(n-1)$ reflection, [14, Ch. 3]. For instance,

$$W(SU(3)) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

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In [15], Kudo considers the invariant ring $H^*(BT^{n-1};\mathbb{F}_p)^{W_{n,d}}$, where $W_{n,d} = \phi_d W(SU(n))\phi_d^{-1} = W(SU(n)/\mathbb{Z}_d)$ for the following matrix:

$$\phi_d = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \frac{1-d}{d} & \dots & \frac{1-d}{d} & \frac{1}{d} \end{pmatrix}.$$

In this paper, we consider some mod 2 invariant rings related to the symplectic groups, namely $H^*(BT^n;\mathbb{F}_2)^{\overline{W_n}}$ where $\overline{W_n}$ is the mod 2 reduction of $W_n = \phi_2 W(Sp(n))\phi_2^{-1}$. We note that the reflection group $W(Sp(n))$ is generated by the permutation matrices Σ_n and the $n \times n$ diagonal matrix $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$. We also note that $\phi_d W(Sp(n))\phi_d^{-1} \subset GL(n,\mathbb{Z})$ if and only if $d = 2$.

For a subgroup W of $GL(n,\mathbb{F}_2)$, we see [18, §8.1] that the Dickson algebra $\mathbb{F}_2[t_1,t_2,\dots,t_n]^{GL(n,\mathbb{F}_2)}$ is included in the invariant ring $\mathbb{F}_2[t_1,t_2,\dots,t_n]^W$, and $\mathbb{F}_2[t_1,t_2,\dots,t_n]^{GL(n,\mathbb{F}_2)} = \mathbb{F}_2[c_{n,n-1},c_{n,n-2},\dots,c_{n,0}]$. Let W^* denote the dual representation of a subgroup W of $GL(n,\mathbb{F}_p)$. Some comparisons between $H^*(BT^n;\mathbb{F}_p)^W$ and $H^*(BT^n;\mathbb{F}_p)^{W^*}$ are done in [9] and [10]. In the case of $n = d$, the representation $W_{n,n} = W(PSU(n))$ is equivalent to the dual representation $W(SU(n))^*$. It is known, [7], that for $p \geq 5$, the invariant ring $H^*(BT^{p-1};\mathbb{F}_p)^{W(SU(p))^*}$ is not polynomial. At $p = 2$ or 3, the invariant rings are polynomial. Kudo shows [15] that even $H^*(BT^3;\mathbb{F}_2)^{W_{4,4}}$ is also polynomial. However, for $n = 6, 8$, the rings $H^*(BT^{n-1};\mathbb{F}_2)^{W_{n,n}}$ are not polynomial, and the rings $H^*(BT^{n-1};\mathbb{F}_3)^{W_{n,n}}$ are not polynomial for $n = 6, 9$. Duan announced some related work at The 2nd Pan-Pacific International Conference on Topology and Applications, [4].

Although $|\overline{W_n}| = \frac{|W(Sp(n))|}{2}$, in a way, our results are similar to the ones in [15] as long as n is small. Namely, for $n = 3, 4$, we will show that all of the invariant rings are polynomial rings, though $H^*(BT^n;\mathbb{F}_2)^{\overline{W_n}} \ncong H^*(BT^n;\mathbb{F}_2)^{\overline{W_n}^*}$. And, except $H^*(BT^3;\mathbb{F}_2)^{\overline{W_3}^*} \cong H^*(BT^3;\mathbb{F}_2)^{W(SU(4))}$, the other three invariant rings are not isomorphic to the mod 2 cohomology of spaces. See [3], [17, §3], [18, Ch 10], and [1] for a detail of the realization problem.

Theorem 1. (a) *$H^*(BT^3;\mathbb{F}_2)^{\overline{W_3}} = \mathbb{F}_2[x_2,x_8,x_{12}]$, where $x_2 = t_3$, $x_8 = (t_1+t_2)^4 + (t_1t_2+t_1t_3+t_2t_3)^2 + t_1t_2t_3(t_1+t_2+t_3)$, and $x_{12} = \{t_1t_2(t_1+t_2) + t_1t_3(t_1+t_3) + t_2t_3(t_2+t_3)\}^2 + t_1t_2t_3(t_1^3+t_2^3+t_3^3) + t_1^2t_2^2t_3^2$.*

(b) *$H^*(BT^3;\mathbb{F}_2)^{\overline{W_3}^*} = \mathbb{F}_2[y_4,y_6,y_8]$, where $y_4 = t_1^2+t_2^2+t_1t_2$, $y_6 = t_1t_2(t_1+t_2)$, and $y_8 = c_{3,2} = (t_1+t_2+t_3)^4 + (t_1t_2+t_1t_3+t_2t_3)^2 + t_1t_2t_3(t_1+t_2+t_3)$.*

(c) *$H^*(BT^3;\mathbb{F}_2)^{\overline{W_3}}$ is not realizable.*

Theorem 2. (a) *$H^*(BT^4;\mathbb{F}_2)^{\overline{W_4}} = \mathbb{F}_2[x_2,x_4,c_{4,3},c_{4,2}]$, where $x_2 = t_4$, $x_4 = (t_1+t_2+t_3)(t_1+t_2+t_3+t_4)$.*

(b) $H^*(BT^4;\mathbb{F}_2)^{\overline{W_4}^*} = \mathbb{F}_2[z_4, z_6, z_8, c_{4,3}]$, *where* $H^*(BT^3;\mathbb{F}_2)^{W(SU(4))} = \mathbb{F}_2[z_4, z_6, z_8]$.

(c) *Both of these invariant rings are not realizable.*

When n gets larger, our results suggest that the ring $H^*(BT^n;\mathbb{F}_2)^{\overline{W_n}}$ would not be a polynomial ring. A direct application of a method of [15] implies that, for $n = 6, 8$, they are not polynomial rings, as shown in Proposition 3.1. While doing this work, the case of $n = 5$ had been remained open due to a heavy calculation involved. Now it can be seen [12] that the invariant rings are polynomial.

The mod p reduction of an integral reflection group is also a reflection group. The converse need not be true. We will see a few examples using the alternating groups A_n as the subgroup of $W(SU(n))$. For instance, one can see that A_3 is a pseudo–reflection group at p if and only if $p = 3$. The following shows the structure of invariant rings under A_n for small n.

Theorem 3. *For* $A_n \subset W(SU(n))$, *the following hold*:

(a) $H^*(BT^2;\mathbb{F}_2)^{A_3} = \mathbb{F}_2[x_4, x_6, y_6]/x_4^3 + x_6^2 + y_6^2 + x_6y_6 = 0$, *where* $x_4 = t_1^2 + t_1t_2 + t_2^2$, $x_6 = t_1^2t_2 + t_1t_2^2$, *and* $y_6 = t_1^3 + t_1^2t_2 + t_2^3$.

(b) $H^*(BT^2;\mathbb{F}_3)^{A_3} = \mathbb{F}_3[t_1 - t_2, t_1t_2(t_1 + t_2)]$ *and* $H^*(BT^2;\mathbb{F}_3)^{A_3^*} = \mathbb{F}_3[t_1 + t_2, t_1t_2(t_1 - t_2)]$. *Both of these are not realizable.*

(c) *For* $p \geq 5$, *we have* $H^*(BT^2;\mathbb{F}_p)^{A_3} = \mathbb{F}_p[x_4, x_6, z_6]/4x_4^3 = 27x_6^2 + z_6^2$, *where* $x_4 = t_1^2 + t_1t_2 + t_2^2$, $x_6 = t_1^2t_2 + t_1t_2^2$, *and* $z_6 = (t_1 - t_2)(t_1 + 2t_2)(2t_1 + t_2)$.

(d) $H^*(BT^3;\mathbb{F}_2)^{A_4} = \mathbb{F}_2[\alpha_4, \alpha_6, \beta_6, x_8]/\alpha_4^3 + \alpha_6^2 + \beta_6^2 + \alpha_6\beta_6 = 0$, *where* $\alpha_4 = (t_1 + t_2)^2 + (t_1 + t_2)(t_1 + t_3) + (t_1 + t_3)^2$, $\alpha_6 = (t_1 + t_2)(t_1 + t_3)(t_2 + t_3)$, $\beta_6 = (t_1 + t_2)^3 + (t_1 + t_2)^2(t_1 + t_3) + (t_1 + t_3)^3$, *and* $x_8 = t_1t_2t_3(t_1 + t_2 + t_3)$.

In §1, we will show some basic results. It includes the matrix presentations, the order of $\overline{W_n}$, systems of parameters and the Dickson algebras. In §2 both Theorem 1 and Theorem 2 will be proved. For the non–realizability, our proof uses the classification theorem of 2–compact groups, [2]. Finally in §3, using Poincaré series and others, non–polynomial cases are discussed.

Major results in this work were announced at a fall meeting of the Japan Math. Soc., [11] together with The 2nd Pan-Pacific International Conference on Topology and Applications, Busan Korea [21].

1. Basic results

The integral representation of the Weyl group $W(Sp(n)) = (\mathbb{Z}/2)^n \rtimes \Sigma_n$ can be given by $\left\langle \Sigma_n, \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right\rangle$ as reflection groups, [14, Ch. 3]. Let $\phi = \phi_2^{-1}$

so that $\phi = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & & 1 & 0 \\ 1 & \dots & & 1 & 2 \end{pmatrix} = \left(\begin{array}{c|cc} E_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0\dots 0 & 1 & 0 \\ 1\dots 1 & 1 & 2 \end{array}\right)$.

We will show that $\phi^{-1}W(Sp(n))\phi \subset GL(n,\mathbb{Z})$.

Proposition 1.1. *The group $W_n = \phi^{-1}W(Sp(n))\phi$ is included in $GL(n,\mathbb{Z})$.*

Proof. For each σ of the reflections which generate $W(Sp(n))$, it's enough to show that $\phi^{-1}\sigma\phi \in GL(n,\mathbb{Z})$. First recall that $\Sigma_{n-1} \subset W(SU(n))$, and that $\phi^{-1}W(SU(n))\phi \subset GL(n-1,\mathbb{Z})$. Since Σ_n is generated by Σ_{n-1} together with the transposition σ_{n-1} switching $n-1$ and n, we see $\phi^{-1}\Sigma_n\phi$ is included in $GL(n,\mathbb{Z})$ by the following:

$$\phi^{-1}\sigma_{n-1}\phi = \frac{1}{2}\left(\begin{array}{c|cc} 2E_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0\ \dots\ 0 & 2 & 0 \\ -1\dots-1 & -1 & 1 \end{array}\right)\left(\begin{array}{c|cc} E_{n-2} & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 0 & 1 \\ & 1 & 0 \end{array}\right)\left(\begin{array}{c|cc} E_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0\dots0 & 1 & 0 \\ 1\dots1 & 1 & 2 \end{array}\right)$$

$$= \left(\begin{array}{c|cc} E_{n-2} & \multicolumn{2}{c}{0} \\ \hline 1\ \dots\ 1 & 1 & 2 \\ -1\dots-1 & 0 & -1 \end{array}\right).$$

Moreover we see that

$$\phi^{-1}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\phi$$

$$= \frac{1}{2}\left(\begin{array}{c|cc} 2E_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0\ \dots\ 0 & 2 & 0 \\ -1\dots-1 & -1 & 1 \end{array}\right)\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\left(\begin{array}{c|cc} E_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0\dots0 & 1 & 0 \\ 1\dots1 & 1 & 2 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cc} -1 & & & & & \\ & 1 & & & & \\ & & \ddots & & \multicolumn{2}{c}{0} \\ & & & 1 & & \\ \hline 0 & & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 & 1 \end{array}\right).$$

This completes the proof. □

Next consider the mod 2 reduction $\overline{W_n}$ of W_n. Since the scalar matrix $-E_n \in W(Sp(n))$, it is also contained in W_n. Hence the kernel of the projection $W_n \longrightarrow \overline{W_n}$ contains $\mathbb{Z}/2$. It turns out $|\overline{W_n}| = \frac{|W_n|}{2} = 2^{n-1}\cdot n!$ for $n \geq 3$.

Proposition 1.2. *For $n \geq 3$, the subgroup $\overline{W_n}$ of $GL(n,\mathbb{F}_2)$ is isomorphic to $W_n/\mathbb{Z}/2$.*

Proof. Note that $W_n \cong (\mathbb{Z}/2)^n \rtimes \Sigma_n$. A result of Minkowski tells us, [18, Lemma 10.7.1], that the kernel of the projection $GL(n,\mathbb{Z})\longrightarrow GL(n,\mathbb{F}_2)$ is an elementary 2–abelian. For $n \geq 3$, the homomorphism $\Sigma_n \longrightarrow GL(n, \mathbb{F}_2)$ should be injective. Thus the kernel of $W_n \longrightarrow \overline{W_n}$ has to come from diagonal matrices of $W(Sp(n))$. The desired result is obtained from the following observation:

$$\phi^{-1} \begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_n \end{pmatrix} \phi = \left(\begin{array}{cccc|c} \varepsilon_1 & & & & \\ & \ddots & & & 0 \\ & & & \varepsilon_{n-1} & \\ \hline \frac{\varepsilon_n - \varepsilon_1}{2} & & \dots & \frac{\varepsilon_n - \varepsilon_{n-1}}{2} & \varepsilon_n \end{array}\right). \qquad \square$$

We recall how to see if a ring of invariants $H^*(BT^n; \mathbb{F}_p)^W$ is polynomial for a prime p (see [8, 14, 16, 18]). An element of $H^*(BT^n; \mathbb{F}_p)$ is considered as a function of n variables, $t_1, t_2, \dots, t_n$. A set of n elements $x_1, x_2, \dots, x_n \in H^*(BT^n; \mathbb{F}_p)^W$ is said to be a system of parameters if the solution of the following system of equations

$$\begin{cases} x_1(t_1, t_2, \dots, t_n) = 0, \\ x_2(t_1, t_2, \dots, t_n) = 0, \\ \qquad \vdots \\ x_n(t_1, t_2, \dots, t_n) = 0 \end{cases}$$

is trivial. Namely $t_1 = t_2 = \cdots = t_n = 0$. As before, we write $H^*(BT^n; \mathbb{F}_p) = \mathbb{F}_p[t_1, t_2, \dots, t_n]$. Let $d(x)$ denote $\frac{1}{2}\deg(x)$ so that $d(t_i) = 1$ for $1 \leq i \leq n$. Usually $d(x)$ is said to be the algebraic degree of x, while $\deg(x)$ is the topological degree. According to [18, Proposition 5.5.5], for a finite group W, if we can find a system of parameters $\{x_1, x_2, \dots, x_n\}$ with $\prod_{i=1}^{n} d(x_i) = |W|$, then $H^*(BT^n; \mathbb{F}_p)^W = \mathbb{F}_p[x_1, x_2, \dots, x_n]$.

Next we recall some basic things about generators of the Dickson algebra $\mathbb{F}_p[t_1, t_2, \dots, t_n]^{GL(n, \mathbb{F}_p)}$, [18, §8.1]. Let

$$V = \mathbb{F}_p\langle t_1 \rangle \oplus \mathbb{F}_p\langle t_2 \rangle \oplus \cdots \oplus \mathbb{F}_p\langle t_n \rangle,$$

the vector space over $\mathbb{F}_p$ with basis $t_1, t_2, \dots, t_n$. Consider the polynomial $f(X) = \prod_{v \in V}(X - v)$. Then $f(X) = X^{p^n} + \sum_{i=0}^{n-1}(-1)^{n-i} c_{n,i} X^{p^i}$ and $\mathbb{F}_p[t_1, t_2, \dots, t_n]^{GL(n, \mathbb{F}_p)} = \mathbb{F}_p[c_{n,n-1}, c_{n,n-2}, \dots, c_{n,0}]$ with $d(c_{n,i}) = p^n - p^i$. For instance, if $p = 2$ and $n = 2$, then $f(X) = X^4 + c_{2,1}X^2 + c_{2,0}X$ with $c_{2,1} = t_1^2 + t_1 t_2 + t_2^2$ and $c_{2,0} = t_1^2 t_2 + t_1 t_2^2$. The Dickson invariants $c_{n,i}$ $(0 \leq i \leq n-1)$ can be expressed using determinants. Consider the following polynomial:

$$\Delta_n(X) = \begin{vmatrix} t_1 & \dots & t_n & X \\ t_1^p & \dots & t_n^p & X^p \\ \vdots & & \vdots & \vdots \\ t_1^{p^n} & \dots & t_n^{p^n} & X^{p^n} \end{vmatrix}.$$

Then $\Delta_n(X) = c_n f(X)$ where $c_n = \Delta_{n-1}(t_n)$. From this observation, for $p = 2$, we see the following:

$$c_{3,2} = \frac{\begin{vmatrix} t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^8 & t_2^8 & t_3^8 \end{vmatrix}}{\begin{vmatrix} t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^4 & t_2^4 & t_3^4 \end{vmatrix}}$$

and

$$c_{4,3} = \frac{\begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ t_1^{16} & t_2^{16} & t_3^{16} & t_4^{16} \end{vmatrix}}{\begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ t_1^8 & t_2^8 & t_3^8 & t_4^8 \end{vmatrix}}.$$

2. Polynomial rings and non–realizability

Some invariant elements can be found using orbit sums or orbit polynomials. A typical example is $\prod_{i=1}^n (X + t_i)$ for the permutation representation of Σ_n. Another way of finding elements of unstable algebras $H^*(BT^n; \mathbb{F}_p)^W$ can be given by use of cohomology operations, [18, Ch. 10]. In fact, in the part (a) of Theorem 1 we see $x_{12} = Sq^4(x_8)$.

Proof of Theorem 1. (a) Suppose $x_2 = t_3$, $x_8 = (t_1+t_2)^4+(t_1t_2+t_1t_3+t_2t_3)^2+t_1t_2t_3(t_1+t_2+t_3)$, $x_{12} = \{t_1t_2(t_1+t_2)+t_1t_3(t_1+t_3)+t_2t_3(t_2+t_3)\}^2+t_1t_2t_3(t_1^3+t_2^3+t_3^3)+t_1^2t_2^2t_3^2$. We notice that $\overline{W_3}$ is generated by the 3 reflections:

$$\overline{W_3} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle$$

It is easy to check that the elements x_2,x_8,x_{12} are $\overline{W_3}$–invariant. For instance, the middle matrix sends t_1 to $t_1+t_2+t_3$, and fixes both t_2 and t_3. Consequently x_8 is sent to $(t_1+t_3)^4+\{(t_1+t_2+t_3)(t_2+t_3)+t_2t_3\}^2+(t_1+t_2+t_3)t_2t_3t_1 = x_8$. The solution of the following system of equations

$$\begin{cases} t_3 = 0, \\ (t_1+t_2)^4 + (t_1t_2+t_1t_3+t_2t_3)^2 + t_1t_2t_3(t_1+t_2+t_3) = 0, \\ \{t_1t_2(t_1+t_2) + t_1t_3(t_1+t_3) + t_2t_3(t_2+t_3)\}^2 \\ +t_1t_2t_3(t_1^3+t_2^3+t_3^3) + t_1^2t_2^2t_3^2 = 0 \end{cases}$$

is trivial. Hence $\{x_2, x_8, x_{12}\}$ is a system of parameters. Consequently we see that $H^*(BT^3; \mathbb{F}_2)^{\overline{W_3}}$ is the polynomial ring generated by these elements, since $|\overline{W_3}| = 2^2 \cdot 3! = d(x_2) \cdot d(x_8) \cdot d(x_{12})$.

(b) Notice that $\{y_4, y_6, y_8\}$ is a system of parameters. Furthermore $|\overline{W_3}^*| = 2^2 \cdot 3! = d(y_4) \cdot d(y_6) \cdot d(y_8)$. A similar argument shows the desired result.

(c) If the unstable algebra $H^*(BT^3; \mathbb{F}_2)^{\overline{W_3}}$ is realizable, there is a 2–compact group X, [6] such that $H^*(BT^3; \mathbb{F}_2)^{\overline{W_3}} \cong H^*(BX; \mathbb{F}_2)$. Since the polynomial algebra is generated by even-degree elements, the classifying space BX is 2–torsion free. So the 2–adic cohomology is also a polynomial algebra generated by elements of the same degree. We can find, [2], a compact connected Lie group G such that $H^*(BX; \mathbb{Z}_2^\wedge) \cong H^*(BG; \mathbb{Z}_2^\wedge)$. However, any Lie group G does not satisfy the condition that $H^*(BG; \mathbb{F}_2) = \mathbb{F}_2[x_2, x_8, x_{12}]$, since this cohomology does not contain a generator of degree 4. Thus, $H^*(BT^3; \mathbb{F}_2)^{\overline{W_3}}$ is not realizable. This completes the proof. □

The representation of $\Sigma_n = W(SU(n))$ is generated by the permutation matrices together with the following $(n-1) \times (n-1)$ matrix:

$$\begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & \vdots \\ & & & -1 \end{pmatrix}.$$

We will prove Theorem 2.

Proof of Theorem 2. (a) We notice that $\overline{W_4}$ is generated by the 4 reflections:

$$\overline{W_4} = \left\langle \underbrace{\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}}_{\Sigma_3}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1 \end{pmatrix} \right\rangle.$$

The argument is analogous to the previous ones. Notice that $\{x_2, x_4, c_{4,3}, c_{4,2}\}$ is a system of parameters. Furthermore $|\overline{W_4}| = 2^6 \cdot 3 = d(x_2) \cdot d(x_4) \cdot d(c_{4,3}) \cdot d(c_{4,2})$. Thus $H^*(BT^4; \mathbb{F}_2)^{\overline{W_4}} = \mathbb{F}_2[x_2, x_4, c_{4,3}, c_{4,2}]$.

(b) Recall that $H^*(BSU(n); \mathbb{F}_2) = H^*(BT^{n-1}; \mathbb{F}_2)^{W(SU(n))}$ for $n \geq 3$ and $H^*(BSU(n); \mathbb{F}_2) \cong H^*(BU(n); \mathbb{F}_2)/(c_1)$. Here notice that

$$\overline{W_4}^* = \left\{ \left(\begin{array}{c|c} & b_1 \\ A & b_2 \\ & b_3 \\ \hline 0\,0\,0 & 1 \end{array} \right) \;\middle|\; A \in W(SU(4)),\ \ b_i \in \mathbb{F}_2 \ \text{ for } i = 1, 2, 3 \right\}.$$

Consequently we can see that, for $H^*(BT^3;\mathbb{F}_2)^{W(SU(4))} = \mathbb{F}_2[z_4, z_6, z_8]$, the set $\{z_4, z_6, z_8, c_{4,3}\}$ is a system of parameters. And the product of their algebraic degree is equal to the order of $\overline{W_4}^*$. So the desired result follows.

(c) Again the argument is analogous to the part (c) of Theorem 1. Using the classification of Lie groups, we can show that $H^*(BG;\mathbb{Z}_2^\wedge)$ is isomorphic to neither $H^*(BT^4;\mathbb{F}_2)^{\overline{W_4}}$ nor $H^*(BT^4;\mathbb{F}_2)^{\overline{W_4}^*}$ for any compact connected Lie group G. This completes the proof. □

Remark 1. The following is the case of $n = 2$. The group $W_2 = \phi^{-1}W(Sp(2))\phi \cong D_8$ is generated by the two reflections $\left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. The mod 2 reduction is $\overline{W_2} = \mathbb{Z}/2\,\langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \rangle$ and $H^*(BT^2;\mathbb{F}_2)^{\overline{W_2}} = \mathbb{F}_2[t_2, t_1(t_1+t_2)]$.

Remark 2. Another set of generators for the polynomial ring $H^*(BT^4;\mathbb{F}_2)^{\overline{W_4}}$ is obtained as follows. The higher dimensional elements are less computational. Let $V = \mathbb{F}_2\langle t_1\rangle \oplus \mathbb{F}_2\langle t_2\rangle \oplus \mathbb{F}_2\langle t_3\rangle \oplus \mathbb{F}_4\langle t_4\rangle$. The $\overline{W_4}$–action divides V into four invariant subsets. They are $\{0\}$, $\{t_4\}$, $\{t_1+t_2+t_3, t_1+t_2+t_3+t_4\}$ and the rest of 12 vectors. For $A = t_1(t_1+t_4)(t_2+t_3)(t_2+t_3+t_4)$, $B = t_2(t_1+t_3)(t_2+t_4)(t_1+t_3+t_4)$ and $C = t_3(t_1+t_2)(t_3+t_4)(t_1+t_2+t_4)$, we can see that $\overline{W_4}$ permutes these three elements and $A+B+C=0$. The invariant ring contains $y_{16} = A^2+AB+B^2$ and $y_{24} = AB(A+B)$. Since $\{x_2, x_4, y_{16}, y_{24}\}$ is a system of parameters, it follows that $H^*(BT^4;\mathbb{F}_2)^{\overline{W_4}} = \mathbb{F}_2[x_2, x_4, y_{16}, y_{24}]$. The following is the orbit polynomial for the set U of the 12 vectors.

$$\begin{aligned} f(X) = &\prod_{u\in U}(X+u) \\ = &X^{12} + (x_2^2+x_4)X^{10} + x_2x_4X^9 + (x_2^4+x_4^2)X^8 + (x_2^6+x_2^4x_4+x_4^3)X^6 \\ &+ x_2x_4(x_2^4+x_4^2)X^5 + \{x_2^2x_4^2(x_2^2+x_4)+y_{16}\}X^4 + x_2^3x_4^3X^3 \\ &+ (x_2^2+x_4)y_{16}X^2 + x_2x_4y_{16}X + y_{24}. \end{aligned}$$

3. Structure of invariant rings

We will see examples of invariant rings that are not polynomial in this section. We use a result of Dwyer–Wilkerson [7, Theorem 1.4]. Suppose that V is a finite dimensional vector space over the field $\mathbb{F}_p$, and that W is a subgroup of $\mathrm{Aut}(V)$. Note that the symmmetric algebra $S(V)$ is isomorphic to $H^*(BT^n;\mathbb{F}_p)$ if $\dim V = n$. Let U be a subset of V, and W_U the subgroup of W consisting of elements which fix U pointwise. Then if $S(V)^{W^*}$ is a polynomial ring over $\mathbb{F}_p$, then W_U must be a pseudo–reflection group and $S(V)^{W_U}$ is also a polynomial ring.

Proposition 3.1. *Let $n = 6, 8$. Then $H^*(BT^n;\mathbb{F}_2)^{\overline{W_n}}$ is not a polynomial ring.*

Proof. According to the result of Dwyer–Wilkerson, we need to find a subset U such that the subgroup W_U is not generated by pseudo–reflections. Our method is an immediate consequence of that of Kudo, [15].

The dual representation $\overline{W_n}^*$ is expressed as follows:

$$\overline{W_n}^* = \left\{ \left(\begin{array}{c|c} & b_1 \\ A & \vdots \\ & b_{n-1} \\ \hline 0 \ \dots \ 0 & 1 \end{array} \right) \middle| \begin{array}{l} A \in W(SU(n)), \\ b_i \in \mathbb{F}_2 \ \text{ for } 1 \le i \le n-1 \end{array} \right\}.$$

First we consider the case of $n = 6$. Let $U = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ for

$$\boldsymbol{x} = {}^t(1,1,1,0,0,0), \quad \boldsymbol{y} = {}^t(1,1,0,1,1,0), \quad \boldsymbol{z} = {}^t(0,0,0,0,0,1).$$

Recall that any element of $W(SU(6))$ is a 5×5 matrix such that each column is one of the set of the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5\}$ and the vector $\boldsymbol{b} = {}^t(1,1,1,1,1)$ at $p = 2$. As in [15, proof of Theorem 3], it follows that

$$W_U = \left\{ \begin{array}{c} e,\ (1,2),\ (4,5),\ (1,2)(4,5),\ (1,4)(2,5)(3,6), \\ (1,5,2,4)(3,6),\ (1,4,2,5)(3,6),\ (1,5)(2,4)(3,6) \end{array} \right\}.$$

Since W_U is not a pseudo–reflection group, we see that $H^*(BT^6; \mathbb{F}_2)^{\overline{W_6}}$ is not a polynomial ring by [7].

The case of $n = 8$ is analogous. Let $U = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}\}$ for

$$\boldsymbol{x} = {}^t(1,1,1,1,0,0,0,0), \quad \boldsymbol{y} = {}^t(1,1,0,0,1,1,0,0),$$
$$\boldsymbol{z} = {}^t(1,0,1,0,1,0,1,0), \quad \boldsymbol{w} = {}^t(0,0,0,0,0,0,0,1).$$

Again, the group W_U is not a pseudo–reflection group, hence $H^*(BT^8; \mathbb{F}_2)^{\overline{W_8}}$ is not a polynomial ring. □

The concept of the Poincaré series can be useful to find the structure of invariant rings, [18] and [14]. For a graded vector space $M = \oplus_{i=0}^{\infty} M_{2i}$ over a field $\mathbb{F}$, we define the Poincaré series by $P_{\mathbb{F}}(M, t) = \sum_{i=0}^{\infty} (\dim_{\mathbb{F}} M_{2i}) \cdot t^i$. If $M = \mathbb{F}[f_1, \dots, f_m]/(h_1, \dots, h_k)$, where $\{f_1, \dots, f_m\}$ are generators and $\{h_1, \dots, h_k\}$ are relations, then the following holds:

$$P_{\mathbb{F}}(M, t) = \frac{\prod_{i=1}^{k} (1 - t^{\mathrm{d}(h_i)})}{\prod_{j=1}^{m} (1 - t^{\mathrm{d}(f_j)})}.$$

Proof of Theorem 3. (a) The alternating group A_3 as a subgroup of $W(SU(3))$ is generated by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. In a non-modular case, the Poincaré series can be calculated by Molien's theorem:

$$P_{\mathbb{F}_2}(H^*(BT^2; \mathbb{F}_2)^{A_3}, t) = \frac{1}{|A_3|} \sum_{w \in A_3} \frac{1}{\det(E_2 - tw)} = \frac{(1 - t^6)}{(1 - t^2)(1 - t^3)^2}.$$

The three elements $\{x_4, x_6, y_6\}$ are A_3–invariant with $x_4^3 + x_6^2 + y_6^2 + x_6 y_6 = 0$. So we obtain the desired result.

(b) It is easy to show that both $\{t_1 - t_2, t_1 t_2 (t_1 + t_2)\}$ and $\{t_1 + t_2, t_1 t_2 (t_1 - t_2)\}$ are systems of parameters. Clearly the product of their algebraic degrees is equal to the order of A_3. Thus the two invariant rings have to be polynomial rings. The nonrealizability of each invariant ring is based on a result of [5]. If a polynomial ring $H^*(BT^n;\mathbb{F}_p)^W$ is realizable for an odd prime p, the modular representation $W\longrightarrow GL(n,\mathbb{F}_p)$ should lift to a p–adic representation as a pseudo–reflection group. This is impossible in each case.

(c) For $p \geq 5$, we see the following:

$$P_{\mathbb{F}_p}(H^*(BT^2;\mathbb{F}_p)^{A_3},t) = \frac{(1-t^6)}{(1-t^2)(1-t^3)^2}.$$

The three elements $\{x_4, x_6, z_6\}$ are A_3–invariant with $4x_4^3 = 27x_6^2 + z_6^2$, and the desired result follows.

(d) Recall that $A_4 = (\mathbb{Z}/2\langle a\rangle \times \mathbb{Z}/2\langle b\rangle) \rtimes \mathbb{Z}/3\langle c\rangle$, where $a = (12)(34)$, $b = (13)(24)$, $c = (123)$ with $c^{-1}ac = ab$ and $c^{-1}bc = a$. Each of the integral matrix presentations is as follows:

$$a = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Under the mod 2 reduction, we can show that $\mathbb{Z}/2\langle a\rangle \times \mathbb{Z}/2\langle b\rangle$ is a reflection group and $H^*(BT^3;\mathbb{F}_2)^{\mathbb{Z}/2\langle a\rangle\times\mathbb{Z}/2\langle b\rangle} = \mathbb{F}_2[x_2, y_2, x_8]$, where $x_2 = t_1 + t_2$, $y_2 = t_1 + t_3$, $x_8 = t_1t_2t_3(t_1 + t_2 + t_3)$. The group $\mathbb{Z}/3\langle c\rangle$ acts on $\mathbb{Z}/2\langle a\rangle \times \mathbb{Z}/2\langle b\rangle$ as A_3, and fixes x_4. Thus the Poincaré series of $\mathbb{F}_2[x_1, y_1]^{\mathbb{Z}/3\langle c\rangle}$ is given by the following:

$$P_{\mathbb{F}_2}(\mathbb{F}_2[x_1, y_1]^{\mathbb{Z}/3\langle c\rangle}, t) = \frac{(1-t^6)}{(1-t^2)(1-t^3)^2}.$$

Since $H^*(BT^3;\mathbb{F}_2)^{A_4} = \big(H^*(BT^3;\mathbb{F}_2)^{\mathbb{Z}/2\langle a\rangle\times\mathbb{Z}/2\langle b\rangle}\big)^{\mathbb{Z}/3\langle c\rangle}$, we obtain the desired result. □

Remark 3. As mentioned before, we will see that A_3 is a pseudo–reflection group at p if and only if $p = 3$. For $p \neq 3$ (non–modular case), it follows from the invariant ring not being polynomial. For $p = 3$, the rank of the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is 1 and the desired result follows.

Remark 4. We consider the rational invariant rings for the groups in Theorem 3 whose mod p reductions are pseudo–reflection groups. First it is straightforward to see the following:

$$H^*(BT^2;\mathbb{Q})^{A_3} = \mathbb{Q}[x_4, x_6, z_6]/4x_4^3 = 27x_6^2 + z_6^2,$$

where $x_4 = t_1^2 + t_1t_2 + t_2^2$, $x_6 = t_1^2t_2 + t_1t_2^2$, and $z_6 = (t_1 - t_2)(t_1 + 2t_2)(2t_1 + t_2)$.

Next we consider $H^*(BT^3;\mathbb{Q})^{\mathbb{Z}/2\langle a\rangle\times\mathbb{Z}/2\langle b\rangle}$. The Poincaré series is the following:

$$P_{\mathbb{Q}}(H^*(BT^3;\mathbb{Q})^{\mathbb{Z}/2\langle a\rangle\times\mathbb{Z}/2\langle b\rangle}, t) = \frac{(1-t^6)}{(1-t^2)^3(1-t^3)}.$$

Let $x_4 = t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_1t_3 + t_2t_3$, $y_4 = (t_1 - t_2)(t_1 + t_2 + 2t_3)$, $z_4 = (t_1 - t_3)(t_1 + 2t_2 + t_3)$ and $x_6 = t_1t_2t_3 - (t_1t_2 + t_1t_3 + t_2t_3)(t_1 + t_2 + t_3)$. Then

$$H^*(BT^3; \mathbb{Q})^{\mathbb{Z}/2\langle a\rangle \times \mathbb{Z}/2\langle b\rangle} = \mathbb{Q}[x_4, y_4, z_4, x_6]/\sim,$$

where $27x_6^2 = 8x_4^3 + 2y_4^3 + 2z_4^3 - 6x_4y_4^2 - 6x_4z_4^2 + 6x_4y_4z_4 - 3y_4^2z_4 - 3y_4z_4^2$ must be satisfied.

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