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AN EXAMPLE OF LARGE GROUPS

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Abstract. The fundamental idea of this article is to present an effective way to obtain the large groups in terms of the split extension obtained by a finite cyclic group and a free abelian group rank 2. The proof of the main result on largeness property of this specific split extension groups will be given by using the connection of large groups with the groups having deficiency one presentations.

1. Introduction and preliminaries

Let G be a group and it has a finite generating set. Now, if G has a subgroup H with a finite index such that it has an epimorphism onto a non-abelian free group, then G is called *large* ([3]). In fact in the same reference one may find a good survey about known results on large groups as well as the connection between large and defficient groups. More detailed, in [3, Theorem 3.6] it has been showed that if G accepts a deficient presentation such that one of its relator is actually commutator, in this case G is isomorphic to one of the groups $\mathbb{Z} \times \mathbb{Z}$, residually abelianised (which is non-abelian) or large. The main tool of our approximation is actually hidden in that result. In fact, by considering a presentation $\mathcal{P}$ for the semi-direct product (equivalently, split extension) of a finite cyclic group with a free abelian rank 2 group, we will first prove that $\mathcal{P}$ is a deficiency one presentation with one of its relators is commutator. After that, by using [3, Theorem 3.6], we will obtain our group with the above presentation $\mathcal{P}$ is large, and so will satisfy all related consequences of this algebraic property. One can summarize some of the consequences of largeness for groups as having the non-abelian free subgroup and having the solution of the word problem generically in linear time.

Assume that G is a finitely presented group and $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ is its presentation. As defined in [5–7], $\mathcal{P}$ is *deficient* (or, equivalently efficient) if the equality $def(\mathcal{P}) = -|\mathbf{x}| + |\mathbf{r}|$ holds. According to the same references, we also have another bound for the group G which is $\delta(G) = -rk_{\mathbb{Z}}(H_1(G)) + d(H_2(G))$ such

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that $rk(\cdot)$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $d(\cdot)$ means the number of generators. It is a well known fact that the inequality $def(\mathcal{P}) \geq \delta(G)$ always satisfies. On the other hand the deficiency of G is the maximal deficiencies among all such presentations. Moreover, if $def(G) = \delta(G)$ with $def(\mathcal{P}) = \delta(G)$, then G is called an *efficient* group with an *efficient presentation* $\mathcal{P}$.

There exists a geometric way to show the efficiency for group (and actually for monoids), namely *spherical pictures* ([2, 5–7, 13, 17]) over the presentation $\mathcal{P}$. In fact these geometric figures are representing the elements of $\pi_2(\mathcal{P})$ (the second homotopy group such that a left $\mathbb{Z}G$-module) on $\mathcal{P}$, and then determining the *p-Cockcroft* property (for a prime p) over presentations. Until now, there are so many studies have been done on spherical pictures and p-Cockcroft property. Therefore we may cite references, for instance, [5–7, 13, 17] for some fundamental definitions, properties and results on them. Among these results, the following theorem (cf. [17]) is quite important which points out the connection between efficiency and p-Cockcroft property.

Theorem 1.1. *For a group presentation $\mathcal{P}$, the necessary and sufficient condition on $\mathcal{P}$ to be efficient is it must satisfy p-Cockcroft property for any prime p.*

This fact will be used in the proof of the main result of this paper.

Let us take $A = \mathbb{Z}_t$ (finite cyclic group of order t) and $D = F_2$ (the free abelian group having rank 2), and let $\mathcal{P}_A = \langle a \,;\, a^t \rangle$ and $\mathcal{P}_D = \langle s, c \,;\, sc = cs \rangle$ be their presentations, respectively. It is a well known fact that if we want to obtain a semi-direct product $G = D \times_\theta A$, then we need to define a regular homomorphism θ from A to the automorphism group of D. Now if we regard the elements $[c^m d^n]_D$ of D as 1×2 matrices $[m \ n]$, then we can represent automorphisms of D by 2×2 matrices with integer entries. In other words we can represent automorphisms $\theta_{[a]}$ of D by the matrix

$$\mathcal{M} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

For simplicity, let us label $\mathcal{M}$ as the form $\left[\begin{smallmatrix} U_1 & V_1 \\ W_1 & Z_1 \end{smallmatrix}\right]$, and then let us multiply it by itself. Now by relabelling the matrix $\mathcal{M}^2$ as $\left[\begin{smallmatrix} U_2 & V_2 \\ W_2 & Z_2 \end{smallmatrix}\right]$ and iterating this procedure, we finally have

$$\mathcal{M}^t = \begin{bmatrix} U_{t-1}\alpha_{11} + V_{t-1}\alpha_{21} & U_{t-1}\alpha_{12} + V_{t-1}\alpha_{22} \\ W_{t-1}\alpha_{11} + Z_{t-1}\alpha_{21} & W_{t-1}\alpha_{12} + Z_{t-1}\alpha_{22} \end{bmatrix}, \text{ say } \begin{bmatrix} U_t & V_t \\ W_t & Z_t \end{bmatrix}.$$

In fact this t^{th} power of $\mathcal{M}$ will be needed for the following lemma.

In general, if we have any two groups G_1 and G_2 with the generating sets $\mathbf{x}$ and $\mathbf{y}$, respectively, then for each element$x$ in $\mathbf{x}$ and y in $\mathbf{y}$, and for a given homomorphism θ, we are allowed to choose a word $y\theta_x$ on $\mathbf{y}$ with $[y\theta_x]_{G_2} = [y]_{G_2}\theta_{[x]_{G_1}}$ (see, for instance, [7]). In our case, we will restrict ourselves only to the choice

$$s\theta_a = s^{\alpha_{11}}c^{\alpha_{12}} \quad \text{and} \quad c\theta_a = s^{\alpha_{21}}c^{\alpha_{22}}.$$

Hence, for the function $\theta : A \to \mathrm{Aut}(D)$ to be a well-defined homomorphism, we must require $\theta_{[a^t]} = \theta_{[1]}$ or equivalently $\mathcal{M}^t$ is equal to identity matrix. So we have:

Lemma 1.2. *The function $\theta : A \to \mathrm{Aut}(D)$, $[a] \mapsto \theta_{[a]}$ defines a group homomorphism (well-defined) if and only if*

$$U_t = 1\,, \quad V_t = 0\,, \quad W_t = 0 \quad \textit{and} \quad Z_t = 1.$$

Proof. This is clear from the equality of $\mathcal{M}^t = I_{2\times 2}$. □

By this lemma, we definitely have a homomorphism and so have a semi-direct product $G = D \times_\theta A$ with a presentation

$$\mathcal{P}_G = \big\langle a, s, c\,;\ a^t,\ [s,c],\ T_{sa},\ T_{ca} \big\rangle \tag{1.1}$$

(see [11]), where

$$T_{sa} : sa = as^{\alpha_{11}}c^{\alpha_{12}} \quad \text{and} \quad T_{ca} : ca = as^{\alpha_{21}}c^{\alpha_{22}},$$

respectively.

Therefore the main theorem of this paper can be given as follows.

Theorem 1.3 (Main Result)**.** *The group G with a presentation $\mathcal{P}_G$ as in* (1.1) *is large.*

Example 1.4. By Lemma 1.2, a group G having one of the following presentations

i) $\mathcal{P}_1 = \big\langle a, s, c\,;\ a^2,\ [s,c],\ sa = as^k c^{1-k},\ ca = as^{1+k}c^{-k} \big\rangle$,
ii) $\mathcal{P}_2 = \big\langle a, s, c\,;\ a^2,\ [s,c],\ sa = as^{-1},\ ca = as^k c \big\rangle$, where $k = 2n \in \mathbb{Z}$,
iii) $\mathcal{P}_3 = \big\langle a, s, c\,;\ a^3,\ [s,c],\ sa = asc,\ ca = as^{-3}c^{-2} \big\rangle$,
iv) $\mathcal{P}_4 = \big\langle a, s, c\,;\ a^3,\ [s,c],\ sa = ac,\ ca = as^{-1}c^{-1} \big\rangle$,

defines a semi-direct product. Also each of $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ has deficiency 1. Hence G with these presentations can be an example of Theorem 1.3.

Remaining part of this paper will be divided in two sections and each of them will be a part of the proof of Theorem 1.3. As we expressed previously, to obtain the largeness property for our group G, we will first consider the related presentation $\mathcal{P}_G$, given in (1.1), and show that it is a deficiency one presentation (see Section 2). After that, by the result [3, Theorem 3.6], we will say that G is large since it has a deficient (or an efficient) presentation with one of its relators is commutator (see Section 3).

2. Deficiency part of Theorem 1.3

In this section we will first obtain a generating set (i.e., the generating pictures) of $\pi_2(\mathcal{P}_G)$, where $\mathcal{P}_G$ as in (1.1). After that, by considering this set, we will state and prove a result for $\mathcal{P}_G$ to be p-Cockcroft (and so, by Theorem 1.1, to be efficient) for some prime p or 0. Then we will pick one of the presentation in Example 1.4 and will show that it is efficient (more precisely, it is actually a deficient one presentation for G).

2.1. The generating set of the second homotopy module of presentation

Let us consider the group $G = D \times_\theta A$ with the presentation $\mathcal{P}_G$ in (1.1), where A and D are presented by $\mathcal{P}_A = \langle a\ ;\ a^t \rangle$ and $\mathcal{P}_D = \langle s, c\ ;\ sc = cs \rangle$, respectively. Recall that T_{sa} and T_{ca} denote the relators $sa = a(s\theta_a)$ and $ca = a(c\theta_a)$, respectively, where

$$s\theta_a = s^{\alpha_{11}} c^{\alpha_{12}} \ \text{ and } \ c\theta_a = s^{\alpha_{21}} c^{\alpha_{22}}.$$

For the relator a^t ($t \in \mathbb{Z}^+$) and for an element $y \in \{s, c\}$, let us denote the word $(\cdots((y\theta_a)\theta_a)\theta_a \cdots)\theta_a)$ by $y\theta_{a^t}$. This can be figured by a picture $\mathbb{A}_{a^t,y}$, as drawn in Figure 1.

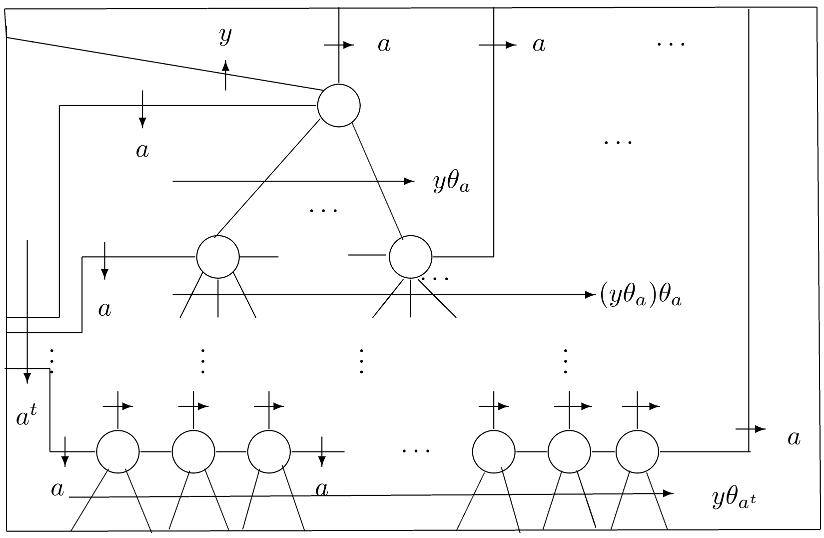

FIGURE 1. $\mathbb{A}_{a^t,y}$ ($y \in \{s, c\}$)

Moreover, if $W = s^{\varepsilon_1} c^{\varepsilon_2} s^{\varepsilon_3} c^{\varepsilon_4} \cdots s^{\varepsilon_{m-1}} c^{\varepsilon_m}$ is a word on the set $\{s, c\}$, then for the generator a, we denote the word

$$(s^{\varepsilon_1}\theta_a)(c^{\varepsilon_2}\theta_a) \cdots (s^{\varepsilon_{m-1}}\theta_a)(s^{\varepsilon_m}\theta_a)$$

by $W\theta_a$.

Let X_A and X_D be a generating set of $\pi_2(\mathcal{P}_A)$ and $\pi_2(\mathcal{P}_D)$, respectively. By [2], each of X_A and X_D contains a single generating picture $\mathbb{P}_A$ and $\mathbb{P}_D$, respectively as drawn in Figure 2.

For simplicity, let us denote the commutator relator $[s, c]$ by $\mathcal{R}$.

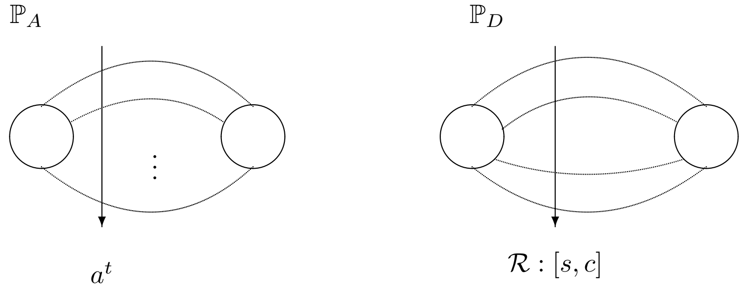

FIGURE 2.

Since $[\mathcal{R}\theta_a]_{\mathcal{P}_D} = [1\theta_a]_{\mathcal{P}_D}$, there exists a non-spherical picture $\mathbb{B}_{s,c}$ over the presentation $\mathcal{P}_D$ with the boundary label

$$\mathcal{R}\theta_a = s^{\alpha_{11}} c^{\alpha_{12}} s^{\alpha_{21}} c^{\alpha_{22}} (s^{\alpha_{21}} c^{\alpha_{22}} s^{\alpha_{11}} c^{\alpha_{12}})^{-1}.$$

Since one may choose different type of homomorphisms other than θ_a, these choices will be effected the choice of matrix $\mathcal{M}$ and so obtain different pictures $\mathbb{B}_{s,c}$.

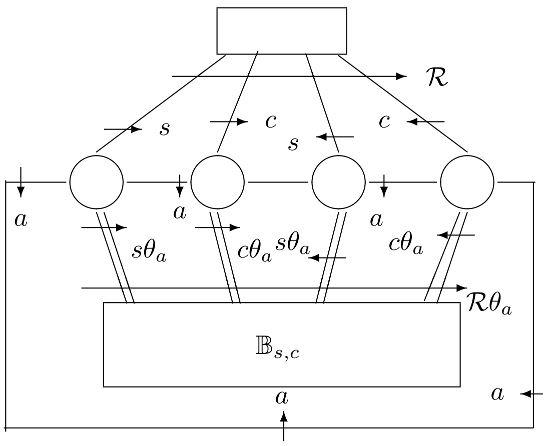

FIGURE 3. $\mathbb{P}_{sc}$

Let us consider the relator a^t and the set of generators $\{s, c\}$ for the presentation $\mathcal{P}_D$. We know that there exists a non-spherical picture $\mathbb{A}_{a^t,y}$ for each element $y \in \{s, c\}$, as depicted in Figure 1. Clearly the pictures $\mathbb{A}_{a^t,y}$ consist of only T_{ya} ($y \in \{s, c\}$) discs. In addition to above non-spherical pictures, since $[y\theta_{a^t}]_{\mathcal{P}_D} = [y\theta_1]_{\mathcal{P}_D}$, for each $y \in \{s, c\}$, there also exits a non-spherical picture $\mathbb{B}_{y,a^t}$ having boundary label $y\theta_{a^t}$.

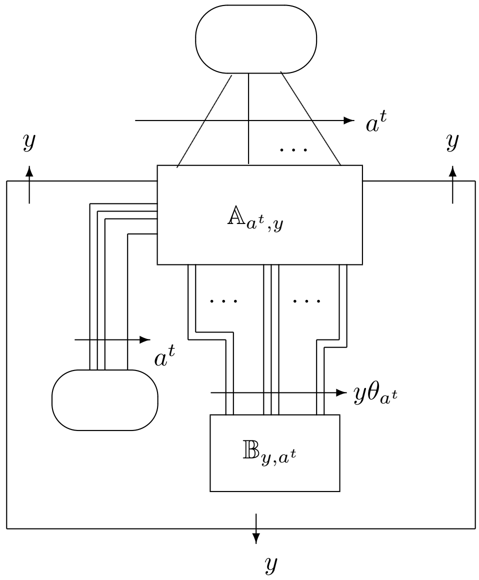

FIGURE 4. $\mathbb{P}_{ya} \quad (y \in \{s, c\})$

We aim now to construct generating spherical pictures via these above non-spherical pictures.

Let us take a single $\mathbb{B}_{s,c}$ picture and process its boundary with a single a-arc. Then for each fixed $y \in \{s, c\}$, we get one positive and one negative T_{ya}-discs. Therefore, for the same T_{ya}-dics, we have two discs with opposite sign and so these give us that we have one $\mathcal{R}$-disc. Hence we get a new picture (non spherical) containing the unique $\mathbb{B}_{s,c}$ and also two different types of T_{ya}-discs (such that each of has one positive and one negative disc), and finally one $\mathcal{R}$-disc. The boundary label of this new picture is $a^{-1}a$. Clearly to obtain a spherical picture $\mathbb{P}_{sc}$ in terms of this last picture (non-spherical), we should combine elements a and a^{-1} by an arc (see Figure 3). Thus let us define the set of pictures $\{\mathbb{P}_{sc}\}$ by X_{sc}.

Now let us consider one of the non-spherical picture $\mathbb{A}_{a^t,y}$ with the boundary label

$$ya^t y^{-1}(y\theta_a)^{-1}a^{-t}.$$

To obtain a spherical picture, we first need to fixed two a^t-discs which one of them is positive and the other is negative. After that we can combine y and y^{-1} by an arc. So we finally need to fix the subpicture $(\mathbb{B}_{y,a^t})^{-1}$ for the part of the boundary $(y\theta_{a^t})^{-1}$. Thus, for each element $y \in \{s, c\}$, we get a spherical picture $\mathbb{P}_{ya}$ as given in Figure 4. Therefore let $X_{sca} = \{\mathbb{P}_{sa}, \mathbb{P}_{ca}\}$.

Although the monoid version of the following proposition can be found in [21], the group version can be either proved directly by the result in [2].

Lemma 2.1. *Suppose that $G = D \times_\theta A$ is a semi-direct product with a related presentation $\mathcal{P}_G$, as in* (1.1). *Therefore a generating set of the second homotopy module $\pi_2(\mathcal{P}_G)$ is*

$$X_A \cup X_D \cup X_{sc} \cup X_{sca}.$$

We should note that, by applying completely the same progress, the above proposition could be constructed for the semi-direct product of any two groups G_1 and G_2 having presentations $\mathcal{P}_{G_1} = \langle \mathbf{x}; \mathbf{r} \rangle$ and $\mathcal{P}_{G_2} = \langle \mathbf{y}; \mathbf{s} \rangle$.

2.2. The deficiency result and its proof

By concerning the generating pictures defined in Proposition 2.1, we get the following result which is the first piece of Theorem 1.3.

Theorem 2.2. *The presentation $\mathcal{P}_G$ in* (1.1) *is p-Cockcroft* (*for a prime p or* 0) *if and only if the whole following conditions hold*:

(i) $\det \mathcal{M} \equiv 1 \pmod p$,

(ii) $\displaystyle\sum_{i=1}^{t-1} U_i \equiv 1 \pmod p, \qquad \sum_{i=1}^{t-1} V_i \equiv 0 \pmod p,$

$\displaystyle\sum_{i=1}^{t-1} W_i \equiv 0 \pmod p, \qquad \sum_{i=1}^{t-1} Z_i \equiv 1 \pmod p,$

(iii) *For $y \in \{s, c\}$*, $\exp_S(\mathbb{B}_{y,a^t}) \equiv 0 \pmod p$.

Proof. In here, we will basically calculate the p-Cockcroft property by counting the number of discs in each of spherical pictures $\mathbb{P}_A$, $\mathbb{P}_D$, $\mathbb{P}_{sc}$ and $\mathbb{P}_{ya}$, where $y \in \{s, c\}$. By Figure 2, it is quite clear that $\mathbb{P}_A$ and $\mathbb{P}_D$ are Cockcroft, and so p-Cockcroft.

Now consider the picture $\mathbb{P}_{sc}$ in Figure 3. It contains a unique negative $\mathcal{R}$-disc, a unique $\mathbb{B}_{s,c}$ picture and balanced (one positive and one negative) number of T_{sa} and T_{ca}-discs. Actually the boundary of $\mathbb{B}_{s,c}$ is equal to the $\mathcal{R}\theta_a$, more clearly,

$$s^{\alpha_{11}} c^{\alpha_{12}} s^{\alpha_{21}} c^{\alpha_{22}} (s^{\alpha_{21}} c^{\alpha_{22}} s^{\alpha_{11}} c^{\alpha_{12}})^{-1}.$$

That means, inside of $\mathbb{B}_{s,c}$, we get $\alpha_{11}\alpha_{22}$-times and $\alpha_{12}\alpha_{21}$-times positive and negative $\mathcal{R}$-discs, respectively, i.e.,

$$\exp_{\mathcal{R}}(\mathbb{B}_{s,c}) = \det \mathcal{M} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}.$$

So to balanced the single negative $\mathcal{R}$-disc in $\mathbb{P}_{sc}$, there must be $det\mathcal{M} \equiv 1 \pmod p$, as required. This gives the condition (i).

For a fixed $y \in \{s, c\}$, let us consider a picture $\mathbb{P}_{ya}$ (see Figure 4). It contains one positive and one negative a^t-discs and two subpictures $\mathbb{A}_{a^t,y}$ (as in Figure 1) and $\mathbb{B}_{y,a^t}$, where $y \in \{s, c\}$. Clearly $\exp_{a^t}(\mathbb{P}_{ya}) = 1 - 1 = 0$, and so there is nothing to do. Now let us consider the matrices $\mathcal{M}$, $\mathcal{M}^2, \ldots, \mathcal{M}^{t-1}$ to use in the calculation of exponent sums in the subpicture $\mathbb{A}_{a^t,y}$. We know that the each of the subpicture $\mathbb{A}_{a^t,y}$ consists of only T_{ya}-discs ($y \in \{s,c\}$). By using the morphism $\theta_{[a]}$ of D defined by $[s] \mapsto [s^{\alpha_{11}}c^{\alpha_{12}}]$ and $[c] \mapsto [s^{\alpha_{21}}c^{\alpha_{22}}]$, an easy determination implies that the sum of first row and first column for all matrices $\mathcal{M}^j$ $(1 \leq j \leq t-1)$ gives the exponent sum of T_{sa}-discs in $\mathbb{A}_{a^t,s}$, the sum of first row and second column elements gives the exponent sum of T_{ca}-discs in $\mathbb{A}_{a^t,c}$, etc. In other words

$$U_1 + \cdots + U_{t-1} = \exp_{T_{sa}}(\mathbb{A}_{a^t,s}), \quad V_1 + \cdots + V_{t-1} = \exp_{T_{ca}}(\mathbb{A}_{a^t,s}),$$
$$W_1 + \cdots + W_{t-1} = \exp_{T_{sa}}(\mathbb{A}_{a^t,c}), \quad Z_1 + \cdots + Z_{t-1} = \exp_{T_{ca}}(\mathbb{A}_{a^t,c}).$$

Therefore to obtain the p-Cockcroft property, there must be

$$\sum_{i=1}^{t-1} U_i \equiv 1 \pmod p, \qquad \sum_{i=1}^{t-1} V_i \equiv 0 \pmod p,$$
$$\sum_{i=1}^{t-1} W_i \equiv 0 \pmod p, \qquad \sum_{i=1}^{t-1} Z_i \equiv 1 \pmod p,$$

as required. This gives the condition (ii).

In picture $\mathbb{P}_{ya}$, we also have a subpicture $\mathbb{B}_{y,a^t}$ having boundary label $y\theta_{a^t}$. (We note that the boundary word $y\theta_{a^t}$ is actually a piece of the boundary label $a^{-t}da^td^{-1}(y\theta_{a^t})^{-1}$ of the subpicture $\mathbb{A}_{a^t,y}$.) In fact the word $y\theta_{a^t}$ contains a finite number of only "s" and "c" letters, and so the subpicture $\mathbb{B}_{y,a^t}$ contains only commutator $\mathcal{R}$-discs. Therefore the exponent sum of $\mathcal{R}$-discs in the picture $\mathbb{B}_{y,a^t}$ has to congruent zero by modulo p, as required.

Conversely, let us suppose that three conditions of the theorem satisfy. Then, by considering the generating set of $\pi_2(\mathcal{P}_G)$, it will be easy to obtain the presentation $\mathcal{P}_G$ is p-Cockcroft (p is a prime as usual or 0).

These complete the proof. $\square$

After completed this above proof, we can easily say that $\mathcal{P}_G$ is efficient (by Theorem 1.1). Since the number of relators is precisely one more than number of generators, $\mathcal{P}_G$ is actually a deficiency one presentation.

Let us consider the presentation $\mathcal{P}_1$ in Example 1.4. Clearly it presents a semi-direct product since the square of matrix $\left[\begin{smallmatrix} k & 1-k \\ 1+k & -k \end{smallmatrix}\right]$ is equal to the identity (by Lemma 1.2). Assume $k = 1$ in $\mathcal{P}_1$. By considering Figures 1, 2, 3 and 4, one can easily draw the generating pictures for $\pi_2(\mathcal{P}_1)$ while $k = 1$. In this case, the subpicture $\mathbb{B}_{s,c}$ contains only a single positive $\mathcal{R}$-disc that balanced one negative $\mathcal{R}$-disc in $\mathbb{P}_{sc}$. Thus all discs in the spherical picture $\mathbb{P}_{sc}$ are balanced. Also, for the picture $\mathbb{P}_{sa}$, there is no subpicture $\mathbb{B}_{s,a^2}$. In $\mathbb{P}_{sa}$, we actually have one positive and one negative a^2-discs, and again one positive and one negative T_{sa}-discs. So, as in $\mathbb{P}_{sc}$, all discs in $\mathbb{P}_{sa}$ are balanced as well. Finally, for the subpicture $\mathbb{A}_{a^2,c}$ of $\mathbb{P}_{ca}$, we have one positive and one negative T_{ca}-discs, and two positive T_{sa}-discs. In other words, $\exp_{T_{ca}}(\mathbb{A}_{a^2,c}) = 1-1 = 0$

and $\exp_{T_{sa}}(\mathbb{A}_{a^2,c}) = 2$. Additionally, in the subpicture $\mathbb{A}_{a^2,c}$ of $\mathbb{P}_{ca}$, we have two positive $\mathcal{R}$-discs. Therefore the presentation

$$\mathcal{P}_1^1 = \langle a, s, c\ ;\ a^2,\ [s,c],\ sa = as,\ ca = as^2c^{-1} \rangle$$

is 2-Cockcroft and so efficient (by Theorem 1.1). More precisely, $\mathcal{P}_1^1$ is a deficiency 1 presentation.

In fact, the deficiencies of other presentations $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ in Example 1.4 can be seen quite similar as in $\mathcal{P}_1$ case. In detailed, while $\mathcal{P}_2$ is 2-Cockcroft, $\mathcal{P}_3$ and $\mathcal{P}_4$ are 0-Cockcroft and so p-Cockcroft.

3. Largeness part of Theorem 1.3

Although the largeness property of groups has been studied widely in [9, 10, 16], some other significant papers have been published in this area. Specially, in [19], Stöhr proved that a group having a deficiency 1 presentation such that one of its relator is a proper power is actually large, and then, in [9], Edjvet showed same result for a group with a deficiency 0 presentation. On the other hand, by concerning the deficiency 1 presentations, it has been presented the following lemma as a main result in a key paper written by Button (cf. [3]). Before stating this result, let us remind the following well known definition.

Definition. Assume that the group G is finitely generated. Then

1) if the intersection of all the subgroups having finite index of G is trivial, then it is called *residually finite* ($\mathcal{R}F$),
2) if the intersection of all the subgroups having finite index of G is equal to the commutator subgroup of it, then G is called *residually abelianised* ($\mathcal{R}A$).

Further, if G is non-abelian while it is $\mathcal{R}A$, then it is called $NARA$ (non-abelian residually abelianised).

Lemma 3.1 ([3, Theorem 3.6]). *Assume that the group G has a deficiency 1 presentation such that one of its relator is commutator. In that case one of the following satisfies*:

i) *G is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*
ii) *G is a $NARA$ group such that the abelianization is $\mathbb{Z} \times \mathbb{Z}$.*
iii) *G is large.*

Specifically, if there exists a finite index subgroup H of G such that $\overline{H} \neq \mathbb{Z} \times \mathbb{Z}$, then G is large.

We recall that the presentation $\mathcal{P}_G$, given in (1.1), for the group $G = D \times_\theta A$ is actually a deficiency 1 presentation (by Theorem 2.2) and one of its relators is commutator, namely $\mathcal{R} : [s,c]$. Thus this group G satisfies the third condition of Lemma 3.1. Besides that, in [3, Section 5], it has been also defined largeness conditions on special semi-direct products. In fact, by considering an automorphism α from a free group F to form a mapping torus, it is obtained a semi-direct product $F \times_\alpha \mathbb{Z}$ and since all free-by-$\mathbb{Z}$ group can be written as in this form, the truthfulness of following result is clearly seen.

Lemma 3.2 ([3, Theorem 5.1])**.** *If the finitely generated group G is free-by-$\mathbb{Z}$, then G is large if the free group F is infinitely generated, or if $\mathbb{Z} \times \mathbb{Z} \leq G$ and F has rank at least* 2.

Therefore this above lemma can also be used to prove the largeness of our group G since $\mathbb{Z} \times \mathbb{Z} \leq G$, and since our free group in the product is F_2 and so rank 2.

Additionally, it is known that a semi-direct product is $\mathcal{R}F$ if both factors in this product are $\mathcal{R}F$ while the first factor is finitely generated. Hence the group G with presentation (1.1) is clearly $\mathcal{R}F$. Therefore, by the definition of residually finiteness, it cannot be $\mathcal{R}A$ and so cannot be $NARA$. (Actually this was a clear fact because if a group is large then it cannot be $NARA$.)

4. Conclusions and open problems

This paper mainly deals with the largeness property over (infinite) group examples in terms of the deficient one presentation defined on the group. The usage of spherical group pictures in the theory will imply some new approximations to solve such these group theoretical problems. We should also note that the consequences of largeness property (for instance, solvability of the word problem (cf. [12])) are also held for the presentations $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ given in Example 1.4.

We can also present the following open problems related to this topic for future studies:

• In [15], Lustig developed a test to investigate the *minimality* of a group presentation. In fact this test has been widely used to show the minimality while the presentation is inefficient (see, for instance, [2, 4]). Therefore no one can prove that this group (presented by this minimal but inefficient presentation) is actually efficient. Lustig test basically works on the Fox ideals obtained from the generating pictures of the second homotopy modules. In our semi-direct product case, by considering the presentation $\mathcal{P}_G$ in (1.1) and then using this test, we could not get a minimal but inefficient presentation as an example. (For instance, by taking into account $k \neq 2n$ for any integer n in the presentation $\mathcal{P}_2$ defined in Example 1.4, although it is an inefficient presentation, it cannot be showed that it is minimal according to the Lustig test.) Therefore obtaining a relationship (if any) between largeness property and inefficient but minimal group presentations can be studied in the future projects.

• The monoid version of the p-Cockcroft property and minimality while having inefficiency of the semi-direct product have been defined and examined in detail in [5–8]. In fact it is not hard to find monoids having deficiencies 1 presentations. We suspect but cannot prove that the result presented in ([3, Theorem 3.6]) whether still holds for deficiency one monoids.

In addition, we would also like to suggest the following problem for a future study:

• It is known that not every finite groups are efficient (for example finite metabelian groups [20]). Therefore it would be worth to study the connection between the efficiency (by using group generating pictures as in this paper) and permutability of all maximal subgroups of the Sylow subgroups of the generalized Fitting subgroup of some normal subgroup of a finite group G (we may refer [1, 14] for such group classifications) with a similar idea as in [18]. In fact this problem would not be easy since it is not clear which types of efficiency should be considered on such subgroups. To do that, we need some results having similar ideas as in [3, Theorem 3.6].

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