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THE WEIGHTED SECOND MAIN THEOREM AND ALGEBRAIC DEPENDENCE OF MEROMORPHIC MAPPINGS SHARING MOVING HYPERPLANES

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Abstract. In this article, we establish some new second main theorems for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, in which the truncated counting functions have different weights. As for application, we deal with the algebraic dependence problem of meromorphic mappings sharing moving hyperplanes in general position.

1. Introduction

In 1933, H. Cartan [1] extended Nevanlinna theory to holomorphic mappings from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ intersecting hyperplanes, where $\mathbb{P}^n(\mathbb{C})$ is the n-dimensional complex projective space. Since that time, this problem has been studied intensively by many authors, see [7]. The core of this theory is to establish the second main theorem.

The second main theorem for meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$ with moving hyperplanes was first given by M. Ru, W. Stoll [8] and M. Ru, J. Wang [9] (the truancated case). But there is no sharp second main theorem with truncated counting functions for meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$ $(n \geq 2)$ intersecting moving hyperplanes. And then, S. D. Quang [2] considered the case where the truncated counting functions with different weights. To state some recent results, we recall the following.

Let $a_1, \ldots, a_q$ $(q \geq n+1)$ be q meromorphic mappings of $\mathbb{C}^m$ into the dual space $\mathbb{P}^n(\mathbb{C})^*$ with reduced representations $a_i = (a_{i0} : \cdots : a_{in})$ $(1 \leq i \leq q)$. Let $\mathcal{M}_m$ be the field of all meromorphic functions on $\mathbb{C}^m$, denote by $\mathcal{R}_{\{a_i\}_{i=1}^q} \subset \mathcal{M}_m$ the smallest subfield which contains $\mathbb{C}$ and all $\frac{a_{ik}}{a_{il}}$ with $a_{il} \not\equiv 0$ (for brevity we will write $\mathcal{R}$ if there is no confusion).

In 2019, S. D. Quang [3] proved a current best second main theorem for meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$ intersecting moving hyperplanes as follows.

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Theorem A ([3, Theorem 1.1])**.** *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q (q \geq 2n-k+2)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \not\equiv 0 (1 \leq i \leq q)$, where $k+1 = \mathrm{rank}_{\mathcal{R}}(f)$. Then the following assertions hold*:

(a)

$$\| \frac{q-(n-k)}{n+2} T_f(r) \leq \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r));$$

(b)

$$\| \frac{q-2(n-k)}{k(k+2)} T_f(r) \leq \sum_{i=1}^{q} N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

Here, by the notation "$\|P$" we mean the assertion P holds for all $r \in [0, \infty)$ outside a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

Recently, P. D. Thoan, N. H. Nam and N. V. An [11] generalized Theorem A(a) as follows.

Theorem B ([11, Theorem 1.1])**.** *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_j\}_{j=1}^q (q \geq 2n-k+2)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_j) \not\equiv 0 (1 \leq j \leq q)$, where $k+1 = \mathrm{rank}_{\mathcal{R}}(f)$. Let $\lambda_1, \ldots, \lambda_q$ be q positive numbers with $(2n-k+2) \max_{1\leq i\leq q} \lambda_i \leq \sum_{i=1}^q \lambda_i$. Then for every positive number $\eta \in [\max_{1\leq i\leq q} \lambda_i, \frac{\sum_{i=1}^q \lambda_i}{2n-k+2}]$, we have*

$$\| \frac{\sum_{j=1}^q \lambda_j - (n-k)\eta}{n+2} T_f(r) \leq \sum_{j=1}^{q} \lambda_j N^{[k]}_{(f,a_j)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

Our aim is to generalize Theorem A(b) and get the following result.

Theorem 1.1. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant meromorphic mapping. Let $\{a_i\}_{i=1}^q (q \geq 2n-k+2)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \not\equiv 0 (1 \leq i \leq q)$, where $k+1 = \mathrm{rank}_{\mathcal{R}}(f)$. Let $\lambda_1, \ldots, \lambda_q$ be q positive numbers with $2(n-k) \max_{1\leq i\leq q} \lambda_i \leq \sum_{i=1}^q \lambda_i$. Then*

$$\begin{aligned} &\| \frac{\sum_{i=1}^q \lambda_i - 2(n-k) \max_{1\leq i\leq q} \lambda_i}{k(k+2)} T_f(r) \\ \leq & \sum_{i=1}^{q} \lambda_i N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)). \end{aligned}$$

For the case which the number of moving hyperplanes is more than $(n-k)(k+1)+n+2$, S. D. Quang [3] proved the following second main theorem.

Theorem C ([3, Theorem 1.3]). *Under the assumptions of Theorem A and further assume $q \geq (n-k)(k+1)+n+2$. Then*

$$\|\frac{q}{k+2}T_f(r) \leq \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

We will prove the corresponding second main theorem with the weighted counting functions as follows.

Theorem 1.2. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant meromorphic mapping. Let $\{a_i\}_{i=1}^q (q \geq (n-k)(k+1)+n+2)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \not\equiv 0 (1 \leq i \leq q)$, where $k+1 = \operatorname{rank}_{\mathcal{R}}(f)$. Let $\lambda_1, \ldots, \lambda_q$ be q positive numbers satisfying $(n-k+1)(k+2)\max_{1\leq i\leq q}\lambda_i \leq \sum_{i=1}^q \lambda_i$. Then*

$$\|\frac{\sum_{i=1}^q \lambda_i}{k+2}T_f(r) \leq \sum_{i=1}^{q} \lambda_i N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

As an application of second main theorem, W. Stoll [10] and M. Ru [6] dealed with algebraic dependence problem. S. D. Quang [3] generalized W. Stoll's work and considered the case of the number l depending on the moving hyperplanes. It means that for each j, we suppose that there exists a positive number l_j such that $f_{i_1} \wedge \cdots \wedge f_{i_{l_j}} = 0$ on A_j for any l_j mappings. We will consider such algebraic dependence problem.

Let $f_i : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})(1 \leq i \leq \lambda)$ be meromorphic mappings with reduced representations $f_i := (f_{i_0} : \cdots : f_{i_n})$. Assume that $(f_t, a_i) := \sum_{j=0}^n f_{t_j} a_{i_j} \neq 0$ for each $1 \leq t \leq \lambda$, $1 \leq i \leq q$, and $(f_1, a_i)^{-1}\{0\} = \cdots = (f_\lambda, a_i)^{-1}\{0\}$. Put $A_i = (f_1, a_i)^{-1}\{0\}$, and assume that every analytic set A_i has the irreducible decomposition as follows: $A_i = \bigcup_{j=1}^{t_i} A_{ij}$. Set $A = \bigcup_{A_{ij}\not\equiv A_{kl}}\{A_{ij} \cap A_{kl}\}$ with $1 \leq j \leq t_i$, $1 \leq l \leq t_k$, $1 \leq i, k \leq q$.

Denote by $T[n+1, q]$ the set of all injective maps from $\{1, \ldots, n+1\}$ to $\{1, \ldots, q\}$. For each $z \in \mathbb{C}^m \backslash \{\bigcup_{\beta\in T[n+1,q]}\{z | a_{\beta(1)}(z) \wedge \cdots \wedge a_{\beta(n+1)}(z) = 0\} \cup A \cup \bigcup_{i=1}^{\lambda} I(f_i)\}$, we define $\rho(z) = \sharp\{j \in A_j\}$, then $\rho(z) \leq n$.

For any positive number $r > 0$, define $\rho(r) = \sup\{\rho(z) || z| \leq r\}$, where the supremum is taken over all $z \in \mathbb{C}^m \backslash \{\bigcup_{\beta\in T[n+1,q]} z | a_{\beta_{(1)}}(z) \wedge \cdots \wedge a_{\beta_{(n+1)}}(z) = 0\} \cup A \cup \bigcup_{i=1}^{\lambda} I(f_i)\}$. Then $\rho(r)$ is a decreasing function. Let

$$d := \lim_{r\to\infty} \rho(r),$$

then $d < n$, and $d = 1$ when $\dim\{A_i \cap A_j\} \leq m-2 (i \neq j)$.

In 2017, L. N. Quynh [5] gave the following algebraic dependence theorem for meromorphic mappings sharing moving hyperplanes.

Theorem D ([5, Theorem 1.1]). *Let $f_1, \ldots, f_\lambda : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be nonconstant meromorphic mappings. Let $\{a_j\}_{j=1}^q$ be moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general* *position satisfying* $T(r,a_j)=o(\max_{1\le i\le\lambda} T(r,f_i))(1\le j\le q)$. *Let* $k_j(1\le j\le q)$ *be positive integers or* $+\infty$. *Assume that* $(f_i,a_j)\not\equiv 0$ *for* $1\le i\le\lambda$, $1\le j\le q$. *Put* $A_j := Supp\nu^0_{(f_1,a_j),\le k_j}=\cdots=Supp\nu^0_{(f_\lambda,a_j),\le k_j}$ *for each* $1\le j\le q$, *and* $A=\bigcup_{j=1}^q A_j$. *Let* l $(\le l\le\lambda)$ *be an integer such that for any* $1\le i_1<\cdots<i_l\le\lambda$, $z\in A$, $f_{i_1}(z)\wedge\cdots\wedge f_{i_l}(z)=0$. *If*

$$\sum_{j=1}^q\frac{1}{k_j}<\frac{q}{n(n+2)}-\frac{d\lambda}{\lambda-l+1},$$

then $f_1,\dots,f_\lambda$ *are algebraically dependent over* $\mathbb{C}$, *i.e.,* $f_1\wedge\cdots\wedge f_\lambda\equiv 0$.

For the case of the number l depending on the moving hyperplanes, we will apply Theorem 1.1 and show the following result.

Theorem 1.3. *Let* $f_1,\dots,f_\lambda:\mathbb{C}^m\to\mathbb{P}^n(\mathbb{C})$ *be nonconstant meromorphic mappings. Let* $\{a_j\}_{j=1}^q(q\ge(2n-1)(\lambda-1)+1)$ *be moving hyperplanes of* $\mathbb{P}^n(\mathbb{C})$ *in general position satisfying* $T(r,a_j)=o(\max_{1\le i\le\lambda}T(r,f_i))(1\le j\le q)$. *Let* $k_j(1\le j\le q)$ *be positive integers or* $+\infty$. *Assume that* $(f_i,a_j)\not\equiv 0$ *and* $A_j := Supp\nu^0_{(f_1,a_j),\le k_j}=\cdots=Supp\nu^0_{(f_\lambda,a_j),\le k_j}$ *for each* $1\le j\le q$. *Denote* $A=\bigcup_{j=1}^q A_j$. *Let* $l_1,\dots,l_q$ *be* q *positive integers with* $2\le l_i\le\lambda$. *Further assume any* $1\le i_1<\cdots<i_{l_j}<q$, $f_{i_1}(z)\wedge\cdots\wedge f_{i_{l_j}}(z)=0$ *for* $z\in A$. *If* $k+1=\max_{1\le t\le\lambda}\{\mathrm{rank}_{\mathcal{R}}(f_t),1\le j\le q\}$,

$$\sum_{j=1}^q\frac{1}{k_j}<\frac{q(\lambda+1)-\sum_{j=1}^q l_j-2(n-k)(\lambda-1)-k(k+2)d\lambda}{k(k+2)(\lambda-1)},$$

then $f_1,\dots,f_\lambda$ *are algebraically dependent over* $\mathbb{C}$.

Remark. From Theorem 1.3, we have the following results.

(1) Letting $k_j=+\infty(1\le j\le q)$, we have

$$q>\frac{1}{\lambda+1}\left(d\lambda k(k+2)+\sum_{j=1}^q l_j+2(n-k)(\lambda-1)\right).$$

(2) Letting $l_1=\cdots=l_q=l$, $k_j=+\infty(1\le j\le q)$, we have

$$q>\frac{1}{\lambda-l+1}\left(d\lambda k(k+2)+2(n-k)(\lambda-1)\right).$$

(3) Letting $\lambda=l_j=2$ and $k_j=+\infty$, it implies the uniqueness theorem: if $q>2k^2+2n+2k$, then $f_1=f_2$.

When $d=1$ and $q>(n-k)(k+1)+n+2$, we will obtain the following algebraic dependence theorem.

Theorem 1.4. *Let* $f_1,\dots,f_\lambda:\mathbb{C}^m\to\mathbb{P}^n(\mathbb{C})$ *be nonconstant meromorphic mappings. Let* $a_i:\mathbb{C}^m\to P^n(\mathbb{C})^*$ *be meromorphic mappings in general position* *satisfying* $T(r, a_j) = o(\max_{1\le i\le\lambda} T(r, f_i))$, $(1 \le i \le \lambda,\ 1 \le j \le q)$. *Let* $k_j (1 \le j \le q)$ *be positive integers or* $+\infty$. *Assume that* $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$ *and the following conditions are satisfied.*

(a) $\min\{1, \nu^0_{(f_1,a_j),\le k_j}\} = \cdots = \min\{1, \nu^0_{(f_\lambda,a_j),\le k_j}\}$ *for each* $1 \le j \le q$;

(b) $\dim(f_1, a_{j_1})^{-1}\{0\} \cap (f_1, a_{j_2})^{-1}\{0\} \le m-2$ *for each* $1 \le j_1 < j_2 \le q$;

(c) *Let* $l_1, \ldots, l_q$ *be* q *positive integers with* $2 \le l_i \le \lambda$, *such that for any increasing sequence* $1 \le i_1 < \cdots < i_{l_j} \le \lambda$, $f_{i_1}(z) \wedge \cdots \wedge f_{i_{l_j}}(z) = 0, z \in \cup_{j=1}^q (f_1, a_j)^{-1}\{0\}$.

Put $\max\{\mathrm{rank}_{\mathcal{R}}(f_i), 1 \le i \le \lambda\} = k+1$, *where* k *is a positive integer. Further assume* $q \ge (n-k+1)(k+2)(\lambda-1)$ *for* $k \le \frac{n-1}{2}$, *or* $q \ge \frac{(n+3)^2}{4}(\lambda-1)$ *for* $k > \frac{n-1}{2}$. *If*

$$\sum_{j=1}^{q} \frac{1}{k_j+1-k} < \frac{q^2(\lambda+1) - q\sum_{j=1}^q l_j + q\lambda(k-1) - q\lambda k(k+2)}{k(k+2)(q(\lambda-1)+\lambda(k-1))},$$

then $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$.

Remark. (1) When $\mathrm{rank}_{\mathcal{R}}(f_i) = k+1$ and $k_j = +\infty$ $(1 \le i \le \lambda, 1 \le j \le q)$, we see that the above inequality becomes

$$q \ge \max\{(n-k+1)(k+2)(\lambda-1), \frac{1}{\lambda+1}(\sum_{j=1}^q l_j + \lambda k(k+2) - \lambda(k-1))\}.$$

(2) Let $\lambda = l = 2$ when $k_j = +\infty$, according to the Lemma 1 in [4], we may show that $\mathrm{rank}_{\mathcal{R}}(f_1) = \mathrm{rank}_{\mathcal{R}}(f_2)$. So if

$$q > \max\{(n-k+1)(k+2), 2k^2+2k+2\},$$

then $f_1 = f_2$.

2. Basic nations and auxiliary results from Nevanlinna theory

2.1. We set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and define $B(r) := \{z \in \mathbb{C}^n : \|z\| < r\}, S(r) := \{z \in \mathbb{C}^n : \|z\| = r\} (0 < r < \infty)$. Define

$$\nu_{n-1}(z) := (dd^c\|z\|^2)^{n-1}, \sigma_n(z) := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{n-1}$$

on $\mathbb{C}^n \setminus \{0\}$.

2.2. For a divisor ν on $\mathbb{C}$, we denote by $N(r, \nu)$ the counting function of the divisor ν as usual in Nevanlinna theory. For a positive integer M or $M = \infty$, we define the truncated divisor of ν by

$$\nu^{[M]}(z) = \min\{M, \nu(z)\},$$

$$\nu^{[M]}_{\le k}(z) := \begin{cases} \nu^{[M]}(z) & \text{if } \nu^{[M]}(z) \le k, \\ 0 & \text{if } \nu^{[M]}(z) > k. \end{cases}$$

Similarly, we define $\nu^{[M]}_{>k}$. We will write $N^{[M]}(r,\nu), N^{[M]}_{\leq k}(r,\nu), N^{[M]}_{>k}(r,\nu)$ for $N(r,\nu^{[M]}), N(r,\nu^{[M]}_{\leq k}), N(r,\nu^{[M]}_{>k})$ as respectively. We will omit the character $[M]$ if $M=\infty$.

Let V be a complex vector space of dimension $n>1$. The vectors $\{v_1,\dots,v_k\}$ are said to be in general position if for each selection of integers $1\leq i_1<\cdots<i_p\leq k$ with $p\leq n$, then $v_{i_1}\wedge\cdots\wedge v_{i_p}\neq 0$. The vectors $\{v_1,\dots,v_k\}$ are said to be in a special position if they not in general position. Take $1\leq p\leq k$, then $\{v_1,\dots,v_k\}$ are said to be in p-special position if for each selection of integers $1\leq i_1<\cdots<i_p\leq k$, the vectors $v_{i_1},\dots,v_{i_p}$ are in a special position.

Definition. A subset $\mathcal{L}$ of $\mathcal{M}$ (or $\mathcal{M}^{n+1}$) is said to be minimal over the field $\mathcal{R}$ if it is linearly dependent over $\mathcal{R}$ and each proper subset of $\mathcal{L}$ is linearly independent over $\mathcal{R}$.

Theorem 2.1 (The First Main Theorem for general position, [10], p. 326)**.** *Let $f_j:\mathbb{C}^m\to\mathbb{P}^n(\mathbb{C}), 1\leq j\leq k$, be meromorphic mappings located in general position. Assume that $1\leq k\leq n$, then*

$$N(r,\mu_{f_1\wedge\cdots\wedge f_\lambda})+m(r,f_1\wedge\cdots\wedge f_\lambda)\leq\sum_{1\leq i\leq\lambda}T(r,f_i)+O(1).$$

Theorem 2.2 (The Second Main Theorem for general position, [10], Theorem 2.1, p. 320)**.** *Let M be a connected complex manifold of dimension m. Let A be a pure $(m-1)$-dimensional analytic subset of M and V be a complex vector space of dimension $n+1>1$. Let p and k be integers with $1\leq p\leq k\leq n+1$. Let $f_j:M\to P(V), 1\leq j\leq k$, be meromorphic mappings. Assume that $f_1,\dots,f_k$ are in general position, and $f_1,\dots,f_k$ are in p-special position on A. Then, we have*

$$\mu_{f_1\wedge\cdots\wedge f_k}\geq(k-p+1)\nu_A.$$

Here, by ν_A, we denote the reduced divisor whose support is the set A.

3. Proof of Theorems 1.1 and 1.2

Lemma 3.1 ([3, Lemma 3.1])**.** *Let $f:\mathbb{C}^m\to\mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be q meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position with* $\operatorname{rank}\{(f,\widetilde{a}_i);1\leq i\leq q\}=\operatorname{rank}_{\mathcal{R}}(f)$. *Assume that there exists a partition $\{1,\dots,q\}=I_1\cup I_2\cup\cdots\cup I_l$ satisfying*:

(i) *$\{(f,\widetilde{a}_i)\}_{i\in I_1}$ is minimal over $\mathcal{R}$, $\{(f,\widetilde{a}_i)\}_{i\in I_t}$ is linearly independent over $\mathcal{R}(2\leq t\leq l)$;*

(ii) *For any $2\leq t\leq l$, $i\in I_t$, there exists meromorphic function $c_i\in\mathcal{R}\backslash\{0\}$ such that*

$$\sum_{i\in I_t}c_i(f,\widetilde{a}_i)\in\left(\bigcup_{j=1}^{t-1}\bigcup_{i\in I_j}(f,\widetilde{a}_i)\right)_{\mathcal{R}}.$$

Then we have

$$\|T_f(r) \le \sum_{i=1}^{t}\sum_{j\in I_i}^{q} N^{[n_i]}_{(f,a_j)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)),$$

where $n_1 = \sharp I_1 - 2$ *and* $n_t = \sharp I_t - 1$ *for* $t = 2, \dots, l$.

3.1. Proof Theorem 1.1

We denote by $\mathcal{I}$ the set of all permutations of q-tuple $(1, \dots, q)$. For each element $I = (i_1, \dots, i_q) \in \mathcal{I}$, we set

$$M_I = \{r \in \mathbb{R}^+; N^{[1]}_{(f,a_{i_1})}(r) \le \cdots \le N^{[1]}_{(f,a_{i_q})}(r)\}.$$

We now consider an element $I = (i_1, \dots, i_q)$ of $\mathcal{I}$, and construct subsets I_t of the set $A_1 = \{1, \dots, 2n-k+2\}$. The proof of the first half is totally similar to the case of S. D. Quang [3]. By repeating the same lines of the proof, we also get the family of subsets $I_1, \dots, I_l$ satisfying the assumption Lemma 3.1. Thus

$$\begin{aligned} T_f(r) &\le \sum_{s=1}^{l}\sum_{j\in I_s} N^{[n_s]}_{(f,a_j)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)) \\ &\le \sum_{j\in J} N^{[n_s]}_{(f,a_j)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)) \end{aligned}$$

for all $r \in N_I$ (maybe outside a finite Borel measure subset of $\mathbb{R}^+$).

Put $(n_1+2)+(n_2+1)+\dots+(n_l+1) = d+2$, $n_1+1+n_2+\dots+n_l = k+1$. Since $\sharp J = d+2 \le n+2$, where $J = I_1\cup\dots\cup I_l$, we note that $J \subset \{1, 2, \dots, 2n-k+2\}$ and $\sharp J \le \min\{n+k+1, 2n-k+2\}$. Put $J_1 = \{1, 2, \dots, 2n-k+2\}\backslash J$, and $\sharp J_1 = (2n-k+2)-(d+2) = 2n-k-d$. We see that

$$n_s = k - \sum_{i=1, i\ne s}^{l} n_i \le k-(l-1).$$

Set $\eta \in [\max_{1\le i\le q}\lambda_i, \frac{\sum_{i=1}^q \lambda_i}{2(n-k)}]$, then $\sum_{j\in J_1}\lambda_{i_j} - \sharp J_1\eta \le 0$. We obtain

$$\begin{aligned} &\|(\sum_{i=1}^{q}\lambda_i - \sharp J_1\eta)T_f(r) \\ &\le (\sum_{i=1}^{q}\lambda_i - \sharp J_1\eta)\sum_{j\in J} N^{[n_s]}_{(f,a_{i_j})}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)) \\ &\le \sum_{j\notin J_1}\lambda_{i_j}\sum_{j\in J} N^{[n_s]}_{(f,a_{i_j})}(r) + (\sum_{j\in J_1}\lambda_{i_j} - \sharp J_1\eta)\sum_{j\in J} N^{[n_s]}_{(f,a_{i_j})}(r) \\ &\quad + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)) \end{aligned}$$

$$\leq \sharp J \sum_{j\in J} \frac{\sum_{j\notin J_1} \lambda_{i_j}}{\sharp J} N^{[n_s]}_{(f,a_{i_j})}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r))$$

$$\leq \sharp J \sum_{j\in J} \lambda_{i_j} N^{[n_s]}_{(f,a_{i_j})}(r) + \sharp J \sum_{j\in J} (\frac{\sum_{j\notin J_1} \lambda_{i_j}}{\sharp J} - \lambda_{i_j}) N^{[n_s]}_{(f,a_{i_j})}(r)$$
$$+ o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r))$$

$$\leq \sharp J \sum_{j=1}^{q} \lambda_{i_j} N^{[n_s]}_{(f,a_{i_j})}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

Therefore,

$$\begin{aligned} \|(\sum_{i=1}^{q} \lambda_i - \sharp J_1 \eta) T_f(r) &= (\sum_{i=1}^{q} \lambda_i - (2n-k-d)\eta) T_f(r) \\ &= (\sum_{i=1}^{q} \lambda_i - (2(n-k) - l + 1)) T_f(r) \\ &\leq \sharp J \sum_{j=1}^{q} \lambda_{i_j} N^{[n_s]}_{(f,a_{i_j})}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)) \qquad (1) \\ &= (d+2) \sum_{j=1}^{q} \lambda_{i_j} N^{[n_s]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{i=1}^{q} \lambda_i - \sharp J_1 \eta &= \sum_{i=1}^{q} \lambda_i - (2n-k+2-(d+2))\eta \\ &= \sum_{i=1}^{q} \lambda_i - (2(n-k) - l + 1)\eta \qquad (2) \\ &\geq \sum_{i=1}^{q} \lambda_i - 2(n-k)\eta > 0. \end{aligned}$$

From the inequalities (1) and (2), we get

$$\| \frac{\sum_{i=1}^{q} \lambda_i - (2n-k-d)\eta}{d+2} T_f(r)$$
$$\leq \sum_{j=1}^{q} \lambda_{i_j} N^{[n_s]}_{(f,a_{i_j})}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r)).$$

Hence for any $\eta \in [\max_{1\leq i\leq q} \lambda_i, \frac{\sum_{i=1}^{q} \lambda_i}{2(n-k)}]$,

$$\|(\sum_{i=1}^{q} \lambda_i - (2(n-k))\eta) T_f(r)$$

$$
\begin{aligned}
&\le (\sum_{i=1}^{q} \lambda_i - (2n-k-d)\eta) T_f(r) \\
&\le (d+2) n_s \sum_{i=1}^{q} \lambda_i N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)) \\
&\le (k+1+l)(k+1-l) \sum_{i=1}^{q} \lambda_i N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) \\
&\quad + O(\max_{1\le i\le q} T_{a_i}(r)) \\
&\le k(k+2) \sum_{i=1}^{q} \lambda_i N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)).
\end{aligned}
$$

Therefore,

$$\| \frac{\sum_{i=1}^{q} \lambda_i - 2(n-k)\eta}{k(k+2)} T_f(r) \le \sum_{i=1}^{q} \lambda_i N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)).$$

Since

$$\frac{\sum_{i=1}^{q} \lambda_i - 2(n-k)\eta}{k(k+2)} \le \frac{\sum_{i=1}^{q} \lambda_i - 2(n-k) \max\limits_{1\le i\le q} \lambda_i}{k(k+2)},$$

we get

$$
\begin{aligned}
&\| \frac{\sum_{i=1}^{q} \lambda_i - 2(n-k) \max\limits_{1\le i\le q} \lambda_i}{k(k+2)} T_f(r) \\
&\le \sum_{i=1}^{q} \lambda_i N^{[1]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)).
\end{aligned}
$$

We see that $\bigcup_{I\in\mathcal{I}} N_I = \mathbb{R}^+$ and the above inequality holds for every $r \in N_I, I \in \mathcal{I}$. Theorem 1.1 is proved.

Proof Theorem 1.2. We denote by $\mathcal{I}$ the set of all permutations of q-tuple $(1, \ldots, q)$. For each element $I = (i_1, \ldots, i_q) \in \mathcal{I}$, we set

$$N_I = \{r \in \mathbb{R}^+; N^{[k]}_{(f,a_{i_1})}(r) \le \cdots \le N^{[k]}_{(f,a_{i_q})}(r)\}.$$

We now consider an element $I = (i_1, \ldots, i_q)$ of $\mathcal{I}$. By repeating the same lines of the proof, we also get a similar result of the family of subsets I satisfying the assumption Lemma 3.1. Then for all $r \in N_I$, we have

$$
\begin{aligned}
\|T_f(r) &\le \sum_{j=1}^{k+l} N^{[k]}_{(f,a_j)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)) \\
&\le N^{[k]}_{(f,a_1)}(r) + \sum_{i=n-k+2}^{n+l} N^{[k]}_{(f,a_i)}(r) + N^{[k]}_{(f,a_{(n-k+1)(k+1)+1})}(r) + o(T_f(r))
\end{aligned}
$$

$$+ O(\max_{1\le i\le q} T_{a_i}(r))$$
$$\le \frac{1}{n-k+1}\left(\sum_{i=1}^{(n-k+1)(k+1)+1} N^{[k]}_{(f,a_i)}(r)\right) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)).$$

In the same way, we have

$$\| \sum_{i=1}^{q} \lambda_i T_f(r)$$
$$\le \frac{\sum_{i=1}^{q}\lambda_i}{n-k+1}\sum_{j=1}^{(n-k+1)(k+2)} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r))$$
$$= \frac{(n-k+1)(k+2)}{n-k+1}\sum_{j=1}^{(n-k+1)(k+2)}\left(\frac{\sum_{i=1}^{q}\lambda_i}{(n-k+1)(k+2)} N^{[k]}_{(f,a_i)}(r)\right)$$
$$+ o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r))$$
$$\le (k+2)\sum_{j=1}^{(n-k+1)(k+2)}\left(\frac{\sum_{i=1}^{q}\lambda_i}{(n-k+1)(k+2)} - \lambda_{i_j}\right) N^{[k]}_{(f,a_i)}(r)$$
$$+ (k+2)(\sum_{j=1}^{(n-k+1)(k+2)} \lambda_{i_j} N^{[k]}_{(f,a_i)}(r)) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r))$$
$$\le (k+2)\sum_{j=1}^{q} \lambda_{i_j} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)).$$

Then we have

$$\| \frac{\sum_{i=1}^{q}\lambda_i}{k+2} T_f(r) \le \sum_{i=1}^{q} \lambda_i N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r)).$$

We see that above inequality holds for every $r \in N_I$, $I \in \mathcal{I}$ outside a finite Borel measure subset. Therefore, Theorem 1.2 is proved. □

4. Proof of Theorems 1.3 and 1.4

Lemma 4.1 ([5, Claim 3.1])**.** *For every $1 \le i \le \lambda$, $1 \le j \le q$ and $1 \le k \le n$, we have*

$$N^{[k]}_{\le k_j}(r, \nu^0_{(f_i,a_j)}) \ge \frac{k_j+1}{k_j+1-k} N^{[k]}(r, \nu^0_{(f_i,a_j)}) - \frac{k}{k_j+1-k} T(r, f_i).$$

Lemma 4.2 ([2, Claim 4.2])**.** *For every $1 \le t \le \lambda$, we have*

$$\sum_{j=1}^{q} (\lambda - l_j + 1)\min\{1, \nu_{(f_t,a_j)}(z)\} \le d\mu_{f_1\wedge\cdots\wedge f_\lambda}(z) + q(\lambda-1)\sum_{\beta} \mu_{a_{\beta(1)}\wedge\cdots\wedge a_{\beta(n+1)}}(z)$$

for each $z \notin A \cup \bigcup_{i=1}^{\lambda} I(f_i)$*, where the sum is taken over all injective map* $\beta : \{1, 2, \ldots, n+1\} \to \{1, 2, \ldots, q\}$.

Proof Theorem 1.3. Since $\nu^{[1]}_{(f_i,a_j),\le k_j}(z) \le \nu^{[1]}_{(f_i,a_j)}(z)$ and Lemma 4.2, it yields that

$$\begin{aligned}
&\sum_{j=1}^{q}(\lambda - l_j + 1)N^{[1]}_{(f_t,a_j),\le k_j}(r)\\
&\le \sum_{j=1}^{q}(\lambda - l_j + 1)N^{[1]}_{(f_t,a_j)}(r)\\
&\le dN_{\mu f_1\wedge\cdots\wedge f_\lambda}(r) + q(\lambda-1)\sum_{\beta\in T[n+1,q]} N_{\mu a_{\beta(1)}\wedge\cdots\wedge a_{\beta(n+1)}}(r)\\
&\le d\sum_{i=1}^{\lambda}T_{f_i}(r) + q(\lambda-1)\sum_{\beta\in T[n+1,q]}\sum_{i=1}^{n+1}T_{a_{\beta(i)}}(r)\\
&= d\sum_{i=1}^{\lambda}T_{f_i}(r) + o(\max_{1\le i\le q}T_{f_i}(r)).
\end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned}
d\sum_{i=1}^{\lambda}T_{f_i}(r) &\ge \sum_{j=1}^{q}(\lambda - l_j + 1)N^{[1]}(r, \nu_{(f_t,a_j),\le k_j})\\
&\ge \sum_{j=1}^{q}(\lambda - l_j + 1)\left(\frac{k_j+1}{k_j}N^{[1]}_{(f_t,a_j)}(r) - \frac{1}{k_j}T_{f_i}(r)\right)\\
&\ge \sum_{j=1}^{q}(\lambda - l_j + 1)N^{[1]}(r, \nu_{(f_t,a_j)}) - \sum_{j=1}^{q}\frac{\lambda - l_j + 1}{k_j}T_{f_i}(r).
\end{aligned}$$

Hence,

$$d\lambda\sum_{i=1}^{\lambda}T_{f_i}(r) + \sum_{j=1}^{q}\sum_{i=1}^{\lambda}\frac{\lambda - l_j + 1}{k_j}T_{f_i}(r) \ge \sum_{j=1}^{q}\sum_{i=1}^{\lambda}(\lambda - l_j + 1)N^{[1]}(r, \nu_{(f_t,a_j)}).$$

Put $\lambda_i = \lambda - l_i + 1$, $\lambda_{i_0} = \max\{\lambda_i, 1 \le i \le \lambda\}$ and $k_t + 1 = \operatorname{rank}_{\mathcal{R}}(f_t)$, $k = \max\{k_t : 1 \le t \le \lambda\}$, $(1 \le i \le q,\ 1 \le t \le \lambda)$. We easily see that

$$\frac{\sum_{j=1}^{q}\lambda_j}{\lambda_{i_0}} \ge 1 + \frac{q-1}{\lambda-1} \ge 2n > 2(n - k_t).$$

From Theorem 1.1, we obtain

$$\| \frac{q(\lambda+1) - \sum_{j=1}^{q} l_j - 2(n-k_t)\max\limits_{1\le j\le q}(\lambda - l_j + 1)}{k_t(k_t+2)}T_{f_i}(r)$$

$$\leq \sum_{j=1}^{q}(\lambda - l_j + 1)N^{[1]}(r, \nu_{(f_t,a_j)}) + o(\max_{1\leq i\leq q} T_{a_i}(r)).$$

On the other hand,

$$\frac{q(\lambda+1) - \sum_{j=1}^{q} l_j - 2n(\lambda-1) + 2k(\lambda-1)}{k(k+2)}$$
$$\leq \frac{q(\lambda+1) - \sum_{j=1}^{q} l_j - 2n(\lambda-1) + 2k_t(\lambda-1)}{k_t(k_t+2)}.$$

Hence

$$\|(d\lambda + \sum_{j=1}^{q} \frac{\lambda - l_j + 1}{k_j})T(r)$$
$$\geq \sum_{i=1}^{\lambda} \frac{q(\lambda+1) - \sum_{j=1}^{q} l_j - 2(n-k)(\lambda-1)}{k(k+2)} T_{f_i}(r) + o(\max_{1\leq i\leq q} T_{a_i}(r)).$$

Letting $r \to +\infty$, we get

$$\sum_{j=1}^{q} \frac{\lambda-1}{k_j} \geq \sum_{j=1}^{q} \frac{\lambda - l_j + 1}{k_j}$$
$$\geq \frac{q(\lambda+1) - \sum_{j=1}^{q} l_j - 2(n-k)(\lambda-1) - k(k+2)d\lambda}{k(k+2)}.$$

This implies that

$$\sum_{j=1}^{q} \frac{1}{k_j} \geq \frac{q(\lambda+1) - \sum_{j=1}^{q} l_j - 2(n-k)(\lambda-1) - k(k+2)d\lambda}{k(k+2)(\lambda-1)}.$$

It is a contradiction. The proof of Theorem 1.3 is finished. □

Lemma 4.3 ([5, Claim 3.3])**.** *Let $h_i : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})(1 \leq i \leq p \leq n+1)$ be meromorphic mappings with reduced representations $h_i := (h_{i0} : \cdots : h_{in})$, $a_i := (a_{i0} : \cdots : a_{in})$. Put $\widetilde{h} := ((h_i, a_1) : \cdots : (h_i, a_{n+1}))$, and assume that $a_1, \ldots, a_{n+1}$ are located in general position such that $(h_i, a_j) \not\equiv 0 (1 \leq i \leq p, 1 \leq j \leq n+1)$. Let S be a pure $(m-1)$dimensional analytic subset of $\mathbb{C}^m$ such that $S \not\subset (a_1 \wedge \cdots \wedge a_{n+1})^{-1}\{0\}$. Then $h_1 \wedge \cdots \wedge h_p = 0$ on S if and only if $\widetilde{h}_1 \wedge \cdots \wedge \widetilde{h}_p = 0$ on S.*

Put $J = \{j_1, \ldots, j_\lambda\}$, $J^c = \{1, \ldots, q\}\backslash J$ and

$$B_J = \begin{pmatrix} (f_1, a_{j_1}) & \cdots & (f_\lambda, a_{j_1}) \\ (f_1, a_{j_2}) & \cdots & (f_\lambda, a_{j_2}) \\ \vdots & \vdots & \vdots \\ (f_1, a_{j_\lambda}) & \cdots & (f_\lambda, a_{j_\lambda}) \end{pmatrix}.$$

Lemma 4.4 ([5, Claim 3.4]). *If B_J is nondegenerate, i.e.,* $\det B_J \not\equiv 0$, *then*

$$\nu_{\widetilde{f}_1\wedge\cdots\wedge\widetilde{f}_\lambda} \geq \sum_{j\in J}\big(\min_{1\leq i\leq\lambda}\{\nu^0_{(f_i,a_j),\leq k_j}\} - \min\{1,\nu^0_{(f_i,a_j),\leq k_j}\}\big) + \sum_{j=1}^{q}(\lambda - l_j + 1)\min\{1,\nu^0_{(f_i,a_j),\leq k_j}\}$$

on the set $\mathbb{C}^n\backslash(A\cup\bigcup_{i=1}^{\lambda} I(f_i)\cup(a_{j_1\wedge\cdots\wedge a_{j_\lambda}})^{-1}(0))$, *where* $\widetilde{f}_i := ((f_i,a_{j_1}) : \cdots : ((f_i,a_{j_\lambda})$ *and* $A = \bigcup_{1\leq i<j\leq q} Z_{(f_1,a_i)}\cap Z_{(f_1,a_j)}$.

Proof Theorem 1.4. For each j, $1\leq j\leq q$, we set

$$N_j(r) = \sum_{i=1}^{\lambda} N^{[k]}_{\leq k_j}(r,\nu^0_{(f_i,a_j)}) - ((\lambda-1)k+1)N^{[1]}_{\leq k_j}(r,\nu^0_{(f_1,a_j)})$$

and for each permutation $I = (j_1,\ldots,j_q)$ of $(1,\ldots,q)$, put

$$T_I = \{r\in[1,+\infty); N_{j_1}(r)\geq\cdots\geq N_{j_q}(r)\}.$$

It is clear that $\bigcup_I T_I = [1,+\infty)$. Therefore, there exists a permutation, for instance, it is $I_1 = (1,\ldots,q)$ such that $\int_{T_{I_1}} dr = +\infty$. Then, we have

$$N_1(r)\geq N_{j_1}(r)\geq N_{j_2}(r)\geq\cdots\geq N_{j_\lambda}(r)\geq N_q(r)$$

for each $r\in T_{I_1}$.

We see that $\min\limits_{1\leq i\leq\lambda} b_i \geq \sum_{i=1}^{\lambda}\min\{k,b_i\} - (\lambda-1)k$, for every λ non-negative integers $b_1,\ldots,b_\lambda$. Lemma 4.4 implies that

$$\sum_{j\in J}\Big(\sum_{i=1}^{\lambda}\min\{k,\nu^0_{(f_i,a_j),\leq k_j}\} - ((\lambda-1)k+1)\min\{1,\nu^0_{(f_i,a_j),\leq k_j}\}\Big) + \sum_{j=1}^{q}(\lambda-l_j+1)\min\{1,\nu^0_{(f_i,a_j),\leq k_j}\} \leq \mu_{\widetilde{f}_1\wedge\cdots\wedge\widetilde{f}_\lambda}$$

on the set $\mathbb{C}^n\backslash(A\cup\bigcup_{i=1}^{\lambda} I(f_i)\cup(a_{j_1}\wedge\cdots\wedge a_{j_\lambda})^{-1}(0))$. Integrating both sides of the inequality, we have

$$\sum_{j\in J}\Big(\sum_{i=1}^{\lambda} N^{[k]}_{\leq k_j}(r,\nu^0_{(f_i,a_j)}) - ((\lambda-1)k+1)N^{[1]}_{\leq k_j}(r,\nu^0_{(f_1,a_j)})\Big) + \sum_{j=1}^{q}(\lambda-l_j+1)N^{[1]}_{\leq k_j}(r,\nu^0_{(f_1,a_j)}) \leq N_{\widetilde{f}_1\wedge\cdots\wedge\widetilde{f}_\lambda}(r) = N_{\det B_J}(r). \tag{3}$$

By Jensen's formula, we obtain

$$N_{\det B_J}(r) \leq \int_{S(r)}\log|\det B_J|\sigma_n + O(1) \leq \sum_{i=1}^{\lambda} T(r,f_i) + o(\max_{1\leq i\leq\lambda} T(r,f_i)). \tag{4}$$

Set $T(r)=\sum_{i=1}^{\lambda}T(r,f_i)$. Combining (3) and (4), we get for all $r\in I_1$,

$$\begin{aligned}\|T(r) &\geq \sum_{i=1}^{\lambda}N_{j_i}(r)+\sum_{j=1}^{q}(\lambda-l_j+1)N^{[1]}_{\leq k_j}(r,\nu^0_{(f_1,a_j)})+o(\max_{1\leq i\leq\lambda}T(r,f_i))\\ &\geq \frac{\lambda}{q}\sum_{j=1}^{q}N_j(r)+\sum_{j=1}^{q}(\lambda-l_j+1)N^{[1]}_{\leq k_j}(r,\nu^0_{(f_1,a_j)})+o(\max_{1\leq i\leq\lambda}T(r,f_i))\\ &= \sum_{j=1}^{q}(\lambda-l_j+1-\frac{\lambda((\lambda-1)k+1)}{q})N^{[1]}_{\leq k_j}(r,\nu^0_{(f_1,a_j)})\\ &\quad+\sum_{j=1}^{q}\sum_{i=1}^{\lambda}\frac{\lambda}{q}N^{[k]}_{\leq k_j}(r,\nu^0_{(f_i,a_j)})+o(\max_{1\leq i\leq\lambda}T(r,f_i))\\ &\geq \sum_{j=1}^{q}\sum_{i=1}^{\lambda}\left(\frac{\lambda-l_j+1}{\lambda k}-\frac{(\lambda-1)k+1}{qk}+\frac{\lambda}{q}\right)N^{[k]}_{\leq k_j}(r,\nu^0(f_i,a_j))\\ &\quad+o(\max_{1\leq i\leq\lambda}T(r,f_i)).\end{aligned}$$

From Lemma 4.1 and the above inequality, we get

$$\begin{aligned}\|T(r)\geq &\sum_{j=1}^{q}\sum_{i=1}^{\lambda}\left(\frac{q(\lambda-l_j+1)+\lambda(k-1)}{q\lambda k}\right)N^{[k]}(r,\nu^0_{(f_i,a_j)})\\ &+\sum_{j=1}^{q}\left(\frac{q(\lambda-l_j+1)+\lambda(k-1)}{q\lambda k}\right)\frac{k}{k_j+1-k}T(r)+o(\max_{1\leq i\leq\lambda}T(r,f_i)).\end{aligned}$$

It implies that

$$\begin{aligned}(5)\qquad &\|q\lambda k\left(1+\sum_{j=1}^{q}\frac{k}{k_j+1-k}(\frac{\lambda-l_j+1}{\lambda k}-\frac{(\lambda-1)k+1}{qk})+\frac{\lambda}{q}\right)T(r)\\ &\geq \sum_{j=1}^{q}\sum_{i=1}^{\lambda}(q(\lambda-l_j+1)+\lambda(k-1))N^{[k]}(r,\nu^0_{(f_i,a_j)})+o(\max_{1\leq i\leq\lambda}T(r,f_i)).\end{aligned}$$

For each $1\leq j\leq q$, put $\lambda_j=q(\lambda-l_j+1)+\lambda(k-1)$, we see that

$$\frac{\sum_{i=1}^{q}\lambda_i}{\max\lambda_i}\geq\frac{q^2+q\lambda(k-1)}{q(\lambda-1)+\lambda(k-1)}\geq\frac{q}{\lambda-1}.$$

Case a. If $k\leq\frac{n-1}{2}$, $q\geq(n-k+1)(k+2)(\lambda-1)$, we get

$$\frac{\sum_{i=1}^{q}\lambda_i}{\max\lambda_i}\geq(n-k+1)(k+2)\geq(n-k_t+1)(k_t+2).$$

Applying Theorem 1.2,

$$\begin{aligned}(6)\quad &\| \sum_{j=1}^{q}(q(\lambda-l_j+1)+\lambda(k-1))N^{[k]}(r,\nu^0_{(f_t,a_j)})\\ &\geq \frac{\sum_{j=1}^{q}(q(\lambda-l_j+1)+\lambda(k-1))}{k+2}T_{f_t}(r)+o(\max_{1\leq i\leq\lambda}T_{f_i}(r)).\end{aligned}$$

Combining inequalities (5) and (6), we have

$$\begin{aligned}&\|q\lambda k\left(1+\sum_{j=1}^{q}\frac{k}{k_j+1-k}(\frac{\lambda-l_j+1}{\lambda k}-\frac{(\lambda-1)k+1}{qk}+\frac{\lambda}{q})\right)T(r)\\ &\geq \sum_{t=1}^{\lambda}\frac{\sum_{j=1}^{q}((q(\lambda-l_j+1)+\lambda(k-1))}{k+2}T_{f_t}(r)+o(\max_{1\leq i\leq\lambda}T_{f_i}(r)).\end{aligned}$$

Letting $r\to+\infty$,

$$\begin{aligned}&q^2(\lambda+1)-q\sum_{j=1}^{q}l_j+q\lambda(k-1)\\ &\leq q\lambda k(k+2)+q\lambda k(k+2)\sum_{j=1}^{q}\frac{k}{k_j+1-k}\frac{q(\lambda-l_j+1)+\lambda(k-1)}{q\lambda k}.\end{aligned}$$

Thus

$$\sum_{j=1}^{q}\frac{1}{k_j+1-k}\geq\frac{q^2(\lambda+1)-q\sum_{j=1}^{q}l_j+q\lambda(k-1)-q\lambda k(k+2)}{k(k+2)(q(\lambda-1)+\lambda(k-1))}.$$

This is a contradiction. Hence, we have $f_1\wedge\cdots\wedge f_\lambda\equiv 0$.

Case b. If $k>\frac{n-1}{2}$, $q\geq\frac{(n+3)^2}{4}(\lambda-1)$, then

$$\frac{\sum_{i=1}^{q}\lambda_i}{\max\lambda_i}\geq\frac{(n+3)^2}{4}\geq(n-k_t+1)(k_t+2).$$

By Case (a), we also have $f_1\wedge\cdots\wedge f_\lambda\equiv 0$. □

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