Bull. Korean Math. Soc. **57** (2020), No. 1, pp. 167–177
https://doi.org/10.4134/BKMS.b190133
pISSN: 1015-8634 / eISSN: 2234-3016

COLOCALIZATION OF LOCAL HOMOLOGY MODULES

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Abstract. Let I be an ideal of Noetherian local ring $(R,\mathfrak{m})$ and M an artinian R-module. In this paper, we study colocalization of local homology modules. In fact we give Colocal-global Principle for the artinianness and minimaxness of local homology modules, which is a dual case of Local-global Principle for the finiteness of local cohomology modules. We define the representation dimension $r^I(M)$ of M and the artinianness dimension $a^I(M)$ of M relative to I by $r^I(M)=\inf\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not representable$\}$, and $a^I(M)=\inf\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not artinian$\}$ and we will prove that

i) $a^I(M)=r^I(M)=\inf\{r^{IR_\mathfrak{p}}({}_\mathfrak{p}M):\mathfrak{p}\in\mathrm{Spec}(R)\}\geq\inf\{a^{IR_\mathfrak{p}}({}_\mathfrak{p}M):\mathfrak{p}\in\mathrm{Spec}(R)\}$,

ii) $\inf\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not minimax$\}=\inf\{r^{IR_\mathfrak{p}}({}_\mathfrak{p}M):\mathfrak{p}\in\mathrm{Spec}(R)\setminus\{\mathfrak{m}\}\}$.

Also, we define the upper representation dimension $R^I(M)$ of M relative to I by $R^I(M)=\sup\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not representable$\}$, and we will show that

i) $\sup\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)\neq 0\}=\sup\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not artinian$\}=\sup\{R^{IR_\mathfrak{p}}({}_\mathfrak{p}M):\mathfrak{p}\in\mathrm{Spec}(R)\}$,

ii) $\sup\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not finitely generated$\}=\sup\{i\in\mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not minimax$\}=\sup\{R^{IR_\mathfrak{p}}({}_\mathfrak{p}M):\mathfrak{p}\in\mathrm{Spec}(R)\setminus\{\mathfrak{m}\}\}$.

1. Introduction

Throughout this paper assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring, I is an ideal of R and M is an R-module. Cuong and Nam in [2] defined the local homology modules $\mathrm{H}_i^I(M)$ with respect to I by

$$\mathrm{H}_i^I(M)=\varprojlim_n \mathrm{Tor}_i^R(R/I^n,M).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenless and May in [5] for an artinian R-module M. For basic results about local homology we refer the reader to [2], [4] and [13]; for local cohomology refer to [1].

Received February 2, 2019; Revised August 4, 2019; Accepted August 23, 2019.
2010 *Mathematics Subject Classification.* 13D45, 13E99.
Key words and phrases. Local homology, artinianness, colocalization.

An important problem in local cohomology is Faltings' Local-global Principle for finiteness of local cohomology modules which states that for a positive integer n and an ideal $\mathfrak{a}$ of R the $R_\mathfrak{p}$-module $\mathrm{H}^i_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p})$ is finitely generated for all $i < n$ and for all $\mathfrak{p} \in \mathrm{Spec}(R)$ if and only if $\mathrm{H}^i_\mathfrak{a}(M)$ is finitely generated R-module for all $i < n$. In the other word, in terms of the finiteness dimension $f_\mathfrak{a}(M)$ of M relative to $\mathfrak{a}$ we have

$$f_\mathfrak{a}(M) = \inf\{f_{\mathfrak{a}_{R_\mathfrak{p}}}(M_\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}(R)\},$$

where, $f_\mathfrak{a}(M) := \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_\mathfrak{a}(M) \text{ is not finitely generated}\}$.

In this paper, we investigate the dual of Faltings' Local-global Principle and we call it Colocal-global Principle for artinianness of local homology modules. At first, for a representable R-module M, we define the representation dimension $r^I(M)$ of M relative to I by

$$r^I(M) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^I_i(M) \text{ is not representable}\}$$

and artinianness dimension $a^I(M)$ of M relative to I by

$$a^I(M) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^I_i(M) \text{ is not artinian}\}$$

and we prove the following main result:

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let n be an integer. Then the following conditions are equivalent*:

i) $\mathrm{H}^I_i(M)$ *is an artinian R-module for all $i < n$,*
ii) $\mathrm{H}^I_i(M)$ *is a representable R-module for all $i < n$,*
iii) ${}_\mathfrak{p}(\mathrm{H}^I_i(M))$ *is a representable $R_\mathfrak{p}$-module for all $i < n$ and all $\mathfrak{p} \in \mathrm{Spec}(R)$,*
and we conclude that

$$\begin{aligned} a^I(M) = r^I(M) &= \inf\{r^{IR_\mathfrak{p}}({}_\mathfrak{p}M) : \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &\geq \inf\{a^{IR_\mathfrak{p}}({}_\mathfrak{p}M) : \mathfrak{p} \in \mathrm{Spec}(R)\}. \end{aligned}$$

In the second main result we study Colocal-global Principle for minimax local homology modules by proving the following result:

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let n be an integer. Then the following conditions are equivalent*:

i) $\mathrm{H}^I_i(M)$ *is minimax R-module for all $i < n$,*
ii) ${}_\mathfrak{p}(\mathrm{H}^I_i(M))$ *is a representable $R_\mathfrak{p}$-module for all $i < n$ and all $\mathfrak{p} \in \mathrm{Spec}(R) \backslash \{\mathfrak{m}\}$,*

And so we obtain that $\inf\{i : \mathrm{H}^I_i(M) \text{ is not minimax}\} = \inf\{r^{IR_\mathfrak{p}}({}_\mathfrak{p}M) : \mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}\}$. Also, we define the upper representation dimension $R^I(M)$ of M relative to I by

$$R^I(M) = \sup\{i \in \mathbb{N}_0 : \mathrm{H}^I_i(M) \text{ is not representable}\},$$

and we will show that

i) $\sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M) \neq 0\} = \sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not artinian$\} = \sup\{R^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \mathrm{Spec}(R)\}$,

ii) $\sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not finitely generated$\} = \sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M)$ is not minimax$\} = \sup\{R^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}\}$.

2. The results

A Hausdorff linearly topologized R-module M is said to be linearly compact if M has the following property: if $\mathcal{F}$ is a family of closed cosets (i.e., the cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection. It is clear that artinian R-modules are linearly compact with the discrete topology. If $(R, \mathfrak{m})$ is a complete local ring, then finitely generated R-modules are also linearly compact and discrete. The local homology modules $\mathrm{H}_i^I(M)$ of a linearly compact R-module M are also linearly compact R-modules by [4, Proposition 3.3]. For more facts about linearly compact modules see [7] and [15].

We need the concept of Noetherian dimension of an R-module in some of our proofs. Let M be an artinian R-module. The Noetherian dimension of M, $\mathrm{Ndim}_R(M)$, is defined by induction. If $M = 0$, we put $\mathrm{Ndim}_R(M) = 1$. For any integer $t \geq 0$, if $\mathrm{Ndim}_R(M) < t$ is false and whenever $M_1 \subseteq M_2 \subseteq \cdots$ is an ascending chain of submodules of M then there exists an integer m_0 such that $\mathrm{Ndim}_R(M_{m+1}/M_m) < t$ for all $m \geq m_0$, then we put $\mathrm{Ndim}_R(M) = t$. In case M is an artinian module, $\mathrm{Ndim}_R(M) < \infty$ (see [6]).

The R-module M is said to be a minimax module if there is a finitely generated submodule N of M, such that M/N is artinian. The class of minimax modules includes all finitely generated and all artinian modules. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules.

A module is called cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. A prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \mathrm{Ann}_R T$ [14]. The set of coassociated primes of M is denoted by $\mathrm{Coass}_R(M)$.

Let $S \subseteq R$ be a multiplicative set. The R_S-module $\mathrm{Hom}_R(R_S, M)$ is called the co-localization of M with respect to S and denoted by ${}_SM$. When M is an artinian R-module, it is known that ${}_SM$ is almost never an artinian R_S-module (see [9]), while by [9, Theorem 3.2] ${}_SM$ is a representable R_S-module. Thus the functor co-localization is not closed on the category artinian modules. In the case M is linearly compact R-module, ${}_SM$ is a linearly compact R-module.

The cosupport of M is defined by $\mathrm{Cos}_R M = \{\mathfrak{p} \in \mathrm{Spec}\, R : {}_{\mathfrak{p}}M \neq 0\}$ (see [9]). Next, Yassemi [14] defined the co-support of an R-module M, denoted by $\mathrm{Cosupp}_R(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\mathrm{Ann}(L) \subseteq \mathfrak{p}$. It is well known that in case

M is an artinian R-module or M is a linearly compact R-module the equality $\mathrm{Cos}_R(M) = \mathrm{Cosupp}_R(M)$ is true.

Definition 2.1. Let M be a representable R-module. We define the representation dimension $r^I(M)$ of M relative to I by

$$r^I(M) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M) \text{ is not representable}\},$$

artinianness dimension $a^I(M)$ of M relative to I by

$$a^I(M) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M) \text{ is not artinian}\}$$

and the upper representation dimension $R^I(M)$ of M relative to I by

$$R^I(M) = \sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M) \text{ is not representable}\}.$$

Note that, if M is an artinian R-module, then in [12, Theorem 2.3] we have proved that

$$\inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M) \text{ is not artinian}\} = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M) \text{ is not representable}\}$$

which implies that $r^I(M) = a^I(M)$.

We need the following lemma in the proof of some of our results.

Lemma 2.2. *Let R be a Noetherian ring, I an ideal of R and M a linearly compact R-module. Then*

i) *IM and M/IM are linearly compact R-modules,*
ii) *${}_{\mathfrak{p}}(IM) \simeq I_{\mathfrak{p}}({}_{\mathfrak{p}}M)$ for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$,*
iii) *$\mathrm{Coass}_R(IM) \subseteq \mathrm{Coass}_R M$.*

Proof. i) See [7, Lemma 3.14] and [7, Lemma 3.5].

ii) Let $\mathfrak{p} \in \mathrm{Spec}(R)$. By (i), $0 \to IM \to M \to M/IM \to 0$ is a short exact sequence of linearly compact R-modules. Now [10, Lemma 3.2] implies that

$$0 \to {}_{\mathfrak{p}}(IM) \to {}_{\mathfrak{p}}M \to {}_{\mathfrak{p}}(M/IM) \to 0$$

is also exact. But, by using [10, Lemma 3.4] we have

$${}_{\mathfrak{p}}(M/IM) \simeq {}_{\mathfrak{p}}(R/I \otimes_R M) \simeq R_{\mathfrak{p}}/IR_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} {}_{\mathfrak{p}}M \simeq {}_{\mathfrak{p}}M/I_{\mathfrak{p}}({}_{\mathfrak{p}}M).$$

Now, by using the above short exact sequence and the exact sequence

$$0 \to I_{\mathfrak{p}}({}_{\mathfrak{p}}M) \to {}_{\mathfrak{p}}M \to {}_{\mathfrak{p}}M/I_{\mathfrak{p}}({}_{\mathfrak{p}}M) \to 0,$$

we obtain the result.

iii) Since IM is a homomorphic image of $I \otimes_R M$ we have $\mathrm{Coass}_R(IM) \subseteq \mathrm{Coass}_R(I \otimes_R M)$. But by [14, Theorem 1.21] $\mathrm{Coass}_R(I \otimes_R M) \subseteq \mathrm{Coass}_R M$. Thus we conclude that $\mathrm{Coass}_R(IM) \subseteq \mathrm{Coass}_R M$. □

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let n be an integer. Then the following conditions are equivalent:*

i) *$\mathrm{H}_i^I(M)$ is an artinian R-module for all $i < n$,*
ii) *$\mathrm{H}_i^I(M)$ is a representable R-module for all $i < n$,*

iii) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ *is a representable* $R_{\mathfrak{p}}$*-module for all* $i < n$ *and all* $\mathfrak{p} \in \mathrm{Spec}(R)$*,*

Proof. i) $\Leftrightarrow$ ii) By [12, Theorem 2.3].

i) $\Rightarrow$ iii) By [9, Theorem 3.2], colocalization of any artinian R-module is representable.

iii) $\Rightarrow$ i) By induction on n. Let $n = 1$. There exists an epimorphism $M \to M/I^tM \to 0$ for all $t > 0$ and so we have an epimorphism $M \to \mathrm{H}_0^I(M) \to 0$. Hence $\mathrm{H}_0^I(M)$ is an artinian R-module. Suppose that $n > 1$. By the inductive hypothesis, $\mathrm{H}_i^I(M)$ is an artinian R-module for all $i < n-1$. We show that $\mathrm{H}_{n-1}^I(M)$ is artinian. By [3, Theorem 3.6] ${}_{\mathfrak{p}}(\mathrm{H}_{n-1}^I(M)) \simeq \mathrm{H}_{n-1}^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M)$. Thus, by assumption, $\mathrm{H}_{n-1}^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M)$ is a representable $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$. By [12, Corollary 2.2] for each $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists an integer $n_{\mathfrak{p}}$ such that $(IR_{\mathfrak{p}})^{n_{\mathfrak{p}}}\,\mathrm{H}_{n-1}^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) = 0$. On the other hand, by [3, Theorem 4.5] $\mathrm{Coass}_R(\mathrm{H}_{n-1}^I(M))$ is finite. Let $k = \max\{n_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{Coass}_R(\mathrm{H}_{n-1}^I(M))\}$. Thus $(IR_{\mathfrak{p}})^k\,\mathrm{H}_{n-1}^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) = 0$ for all $\mathfrak{p} \in \mathrm{Coass}_R(\mathrm{H}_{n-1}^I(M))$. Since by [4, Proposition 3.3] $\mathrm{H}_{n-1}^I(M)$ is linearly compact R-module, Lemma 2.2(ii) implies that ${}_{\mathfrak{p}}(I^k\,\mathrm{H}_{n-1}^I(M)) = 0$ for all $\mathfrak{p} \in \mathrm{Coass}_R(\mathrm{H}_{n-1}^I(M))$. But, by Lemma 2.2(iii) we get that ${}_{\mathfrak{p}}(I^k\,\mathrm{H}_{n-1}^I(M)) = 0$ for all $\mathfrak{p} \in \mathrm{Coass}_R(I^k\,\mathrm{H}_{n-1}^I(M))$. By Lemma 2.2(i) $I^k\,\mathrm{H}_{n-1}^I(M)$ is linearly compact R-module and so by [3, Theorem 4.2], it follows that $I^k\,\mathrm{H}_{n-1}^I(M) = 0$. Now by using [2, Proposition 4.7] we conclude that $\mathrm{H}_{n-1}^I(M)$ is artinian R-module and the proof is complete. □

Corollary 2.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* $I \subseteq \mathfrak{m}$ *be an ideal of* R *and* M *an artinian* R*-module. Then*

$$r^I(M) = \inf\{r^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

Proof. For any integer i and any $\mathfrak{p} \in \mathrm{Spec}(R)$ we have ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) \simeq \mathrm{H}_i^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M)$. Thus, the result follows by Theorem 2.3. □

Corollary 2.5. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* $I \subseteq \mathfrak{m}$ *be an ideal of* R *and* M *an artinian* R*-module. Let* n *be an integer. Then, if* ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ *is artinian* $R_{\mathfrak{p}}$*-module for all* $i < n$ *and all* $\mathfrak{p} \in \mathrm{Spec}(R)$*, then* $\mathrm{H}_i^I(M)$ *is artinian* R*-module for all* $i < n$*. Thus*

$$a^I(M) \geq \inf\{a^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

Proof. Since any artinian $R_{\mathfrak{p}}$-module is a representable $R_{\mathfrak{p}}$-module, the result follows by Theorem 2.3 iii)$\Rightarrow$ i). □

Recall that, module M is called coatomic, if every proper submodule of M is contained in a maximal submodule of M. Over Noetherian rings, the coatomic modules are closed under taking quotients, submodules and extensions. It is clear that every finitely generated R-module is coatomic and that every coatomic, artinian module has finite length. Also, it is well known that M is coatomic if and only if $\operatorname{Coass}_R(M)$ consists only of maximal ideals. For more details of coatomic modules, we refer the reader to [16] and [17].

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let n be an integer. Then the following conditions are equivalent*:

i) $\mathrm{H}_i^I(M)$ *is of finite length R-module for all $i < n$,*

ii) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ *is a coatomic and representable $R_{\mathfrak{p}}$-module for all $i < n$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. i) $\Rightarrow$ ii) By assumption $\mathrm{H}_i^I(M)$ is an artinian R-module for all $i < n$ and so Theorem 2.3 implies that ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ is a representable $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$. On the other hand, by [9, Proposition 7.4] we have $\operatorname{Cos}_R(\mathrm{H}_i^I(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Thus ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ and all $i < n$. It is sufficient to show that ${}_{\mathfrak{m}}(\mathrm{H}_i^I(M))$ is coatomic $R_{\mathfrak{m}}$-module for all $i < n$. By [9, Theorem 3.2] $\operatorname{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M))) = \{\mathfrak{q}R_{\mathfrak{m}} : \mathfrak{q} \in \operatorname{Att}_R(\mathrm{H}_i^I(M)) \text{ and } \mathfrak{q} \subseteq \mathfrak{m}\}$. But by assumption (i) and [9, Proposition 7.4] we have $\operatorname{Att}_R(\mathrm{H}_i^I(M)) \subseteq \{\mathfrak{m}\}$. Hence it follows that $\operatorname{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M))) \subseteq \{\mathfrak{m}R_{\mathfrak{m}}\}$. Since ${}_{\mathfrak{m}}(\mathrm{H}_i^I(M))$ is representable by [14, Theorem 1.14] $\operatorname{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M))) = \operatorname{Coass}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M)))$. Hence, $\operatorname{Coass}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M))) \subseteq \{\mathfrak{m}R_{\mathfrak{m}}\}$ and so ${}_{\mathfrak{m}}(\mathrm{H}_i^I(M))$ is coatomic $R_{\mathfrak{m}}$-module for all $i < n$, as required.

ii) $\Rightarrow$ i) By Theorem 2.3 $\mathrm{H}_i^I(M)$ is an artinian R-module for all $i < n$. By [9, Proposition 7.4] it is sufficient to show that $\operatorname{Cos}_R(\mathrm{H}_i^I(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Let $i < n$ be an integer and $\mathfrak{p} \in \operatorname{Cos}_R(\mathrm{H}_i^I(M))$. Then ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) \neq 0$ and so $\phi \neq \operatorname{Att}_{R_{\mathfrak{p}}}({}_{\mathfrak{p}}(\mathrm{H}_i^I(M))) = \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \operatorname{Att}_R(\mathrm{H}_i^I(M)) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$. Thus there exists $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \operatorname{Att}_R(\mathrm{H}_i^I(M))$. It follows that $\mathfrak{q}R_{\mathfrak{m}} \in \operatorname{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M)))$. But by assumption ${}_{\mathfrak{m}}(\mathrm{H}_i^I(M))$ is a representable and coatomic $R_{\mathfrak{m}}$-module and so $\operatorname{Att}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M))) = \operatorname{Coass}_{R_{\mathfrak{m}}}({}_{\mathfrak{m}}(\mathrm{H}_i^I(M))) \subseteq \{\mathfrak{m}R_{\mathfrak{m}}\}$. Thus we conclude that $\mathfrak{q} = \mathfrak{m}$ and so $\mathfrak{p} = \mathfrak{m}$. Therefore $\operatorname{Cos}_R(\mathrm{H}_i^I(M)) \subseteq \{\mathfrak{m}\}$. □

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let n and t be two arbitrary integers. If $\mathrm{H}_i^I(M)$ is minimax for all $i < n$, then $\mathrm{H}_n^I(M)/I^t\,\mathrm{H}_n^I(M)$ is minimax.*

Proof. i) We use induction on n. When $n = 0$, for all positive integers k the canonical epi-morphisms $M \to M/I^kM$ induces an epi-morphism $M \to \Lambda_I(M)$ where $\Lambda_I(M) = \varprojlim_k M/I^kM$ denotes the I-adic completion of M. Thus $\Lambda_I(M)$ is homomorphic image of an artinian module and so is minimax. Since $\mathrm{H}_0^I(M) \cong \Lambda_I(M)$, the result follows in this case. Now suppose, inductively that $n > 0$ and the result is true for $n - 1$. By [2, Corollary 4.5], we can replace M by $\cap_{n>0} I^nM$. But $\cap_{n>0} I^nM = I^kM$ for some $k \in \mathbb{N}$ and so we may assume that $IM = M$. Thus there exists $x \in I$ such that $xM = M$ by [8, 2.8] and so

$x^t M = M$. From the exact sequence

$$0 \to (0 :_M x^t) \to M \xrightarrow{x^t} M \to 0$$

and using [2, Corollary 4.2], we obtain the following long exact sequence

$$\cdots \to \mathrm{H}_n^I(M) \xrightarrow{x^t} \mathrm{H}_n^I(M) \xrightarrow{f} \mathrm{H}_{n-1}^I(0 :_M x^t) \xrightarrow{g} \mathrm{H}_{n-1}^I(M) \to \mathrm{H}_{n-1}^I(M) \to \cdots .$$

From the above exact sequence we get

$$0 \to \operatorname{Im} f \to \mathrm{H}_{n-1}^I(0 :_M x^t) \to \operatorname{Im} g \to 0,$$

$$\cdots \to \mathrm{H}_n^I(M) \xrightarrow{x^t} \mathrm{H}_n^I(M) \to \operatorname{Im} f \to 0.$$

Thus we have the following exact sequences:

$$\cdots \to \operatorname{Tor}_1^R(R/I^t, \operatorname{Im} g) \to \operatorname{Im} f/I^t \operatorname{Im} f \to \mathrm{H}_{n-1}^I(0 :_M x^t)/I^t\, \mathrm{H}_{n-1}^I(0 :_M x^t) \to \cdots ,$$

$$\cdots \to \mathrm{H}_n^I(M)/I^t\, \mathrm{H}_n^I(M) \xrightarrow{x^t} \mathrm{H}_n^I(M)/I^t\, \mathrm{H}_n^I(M) \to \operatorname{Im} f/I^t \operatorname{Im} f \to 0.$$

Induction hypothesis implies that $\mathrm{H}_{n-1}^I(0 :_M x^t)/I^t\, \mathrm{H}_{n-1}^I(0 :_M x^t)$ is minimax. But the class of minimax modules is a Serre subcategory of the category of R-modules and so it is easy to see that since $\operatorname{Im} g$ is minimax $\operatorname{Tor}_1^R(R/I^t, \operatorname{Im} g)$ is minimax. Hence $\operatorname{Im} f/I^t \operatorname{Im} f$ is minimax. Since $x^t \in I^t$, the first map in the above exact sequence is zero, and therefore $\mathrm{H}_n^I(M)/I^t\, \mathrm{H}_n^I(M) \simeq \operatorname{Im} f/I^t \operatorname{Im} f$. Hence, $\mathrm{H}_n^I(M)/I^t\, \mathrm{H}_n^I(M)$ is minimax, as desired. □

Proposition 2.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and L a linearly compact R-module. If L is a minimax R-module, then ${}_{\mathfrak{p}}L$ is a representable $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$,*

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$. By assumption there exists a finitely generated submodule N of L such that L/N is an artinian R-module. By [9, Theorem 3.2] ${}_{\mathfrak{p}}(L/N)$ is a representable $R_{\mathfrak{p}}$-module. On the other hand, since L/N is a linearly compact R-module with the discrete topology, N is an open submodule L. Thus, N is open, hence N is closed in L. Thus by [4, Lemma 2.3] N is a linearly compact submodule of L. Now by [10, Lemma 3.2], the short exact sequence $0 \to N \to L \to L/N \to 0$ of linearly compact R-modules induces an exact sequence $0 \to {}_{\mathfrak{p}}N \to {}_{\mathfrak{p}}L \to {}_{\mathfrak{p}}(L/N) \to 0$. It follows that ${}_{\mathfrak{p}}L/{}_{\mathfrak{p}}N \simeq {}_{\mathfrak{p}}(L/N)$ and so ${}_{\mathfrak{p}}L/{}_{\mathfrak{p}}N$ is a representable $R_{\mathfrak{p}}$-module. Since N is finitely generated and $\mathfrak{p} \in \operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$ we have ${}_{\mathfrak{p}}N = 0$ by [14, Theorem 2.10]. Therefore ${}_{\mathfrak{p}}L$ is a representable $R_{\mathfrak{p}}$-module and the proof is complete. □

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let i be an integer. If $\mathrm{H}_i^I(M)$ is a minimax R-module, then ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ is a representable $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$,*

Proof. By [4, Proposition 3.3] $\mathrm{H}_i^I(M)$ is a linearly compact R-module. Therefore, the result follows by Proposition 2.8. □

Theorem 2.10. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let n be an integer. Then the following conditions are equivalent*:

i) $\mathrm{H}_i^I(M)$ *is a minimax R-module for all $i < n$.*
ii) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ *is a representable $R_{\mathfrak{p}}$-module for all $i < n$ and all $\mathfrak{p} \in \mathrm{Spec}(R) \backslash \{\mathfrak{m}\}$.*

Proof. i)$\Rightarrow$ ii) By Theorem 2.9.

ii) $\Rightarrow$ i) We use induction on n. Let $n = 1$. There exists an epimorphism $M \to M/I^tM \to 0$ for all $t > 0$ and so we have an epimorphism $M \to \mathrm{H}_0^I(M) \to 0$. Hence $\mathrm{H}_0^I(M)$ is an artinian R-module and so $\mathrm{H}_0^I(M)$ is a minimax R-module. Suppose that $n > 1$. By the inductive hypothesis, $\mathrm{H}_i^I(M)$ is a minimax R-module for all $i < n-1$. It is sufficient to show that $\mathrm{H}_{n-1}^I(M)$ is a minimax R-module. At first, we prove that there exists an integer t such that $\mathrm{Cosupp}_R(I^t\,\mathrm{H}_{n-1}^I(M)) \subseteq \{\mathfrak{m}\}$. By [11, Theorem 3.2] $\mathrm{Coass}_R(\mathrm{H}_{n-1}^I(M))$ is finite. Let $\mathrm{Coass}_R(\mathrm{H}_{n-1}^I(M))\backslash\{\mathfrak{m}\} = \{\mathfrak{p}_1,\dots,\mathfrak{p}_k\}$. By assumption ${}_{\mathfrak{p}_i}(\mathrm{H}_{n-1}^I(M))$ is a representable $R_{\mathfrak{p}_i}$-module for all $1 \le i \le k$. Since ${}_{\mathfrak{p}_i}(\mathrm{H}_{n-1}^I(M)) \simeq \mathrm{H}_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M)$ for all $1 \le i \le k$, it follows by [12, Theorem 2.3] that for any integer $1 \le i \le k$ there exists an integer t_i such that $(IR_{\mathfrak{p}_i})^{t_i}\,\mathrm{H}_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M) = 0$. Let $t = \mathrm{Max}\{t_1,\dots,t_k\}$. Thus $(IR_{\mathfrak{p}_i})^{t}\,\mathrm{H}_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M)$ $= 0$ for all $1 \le i \le k$. By Lemma 2.2(ii), $(IR_{\mathfrak{p}_i})^{t}\,\mathrm{H}_{n-1}^{IR_{\mathfrak{p}_i}}({}_{\mathfrak{p}_i}M) \simeq {}_{\mathfrak{p}_i}(I^t\,\mathrm{H}_{n-1}^I(M))$ for all $1 \le i \le k$ and so ${}_{\mathfrak{p}_i}(I^t\,\mathrm{H}_{n-1}^I(M)) = 0$ for all $1 \le i \le k$. Hence $\mathfrak{p}_i \notin \mathrm{Cosupp}_R(I^t\,\mathrm{H}_{n-1}^I(M))$ for all $1 \le i \le k$ and so

$$\{\mathfrak{p}_1,\dots,\mathfrak{p}_k\} \cap \mathrm{Coass}_R(I^t\,\mathrm{H}_{n-1}^I(M)) = \phi.$$

On the other hand, by Lemma 2.2(iii)

$$\mathrm{Coass}_R(I^t\,\mathrm{H}_{n-1}^I(M)) \setminus \{\mathfrak{m}\} \subseteq \mathrm{Coass}_R(\mathrm{H}_{n-1}^I(M)) \setminus \{\mathfrak{m}\} = \{\mathfrak{p}_1,\dots,\mathfrak{p}_k\}.$$

It follows that $\mathrm{Coass}_R(I^t\,\mathrm{H}_{n-1}^I(M)) \subseteq \{\mathfrak{m}\}$. Now by [18, satz 2.4], it follows that there exists an integer s such that $\mathfrak{m}^sI^t\,\mathrm{H}_{n-1}^I(M)$ is finitely generated. Thus $I^{s+t}\,\mathrm{H}_{n-1}^I(M)$ is finitely generated and so is minimax. On the other hand, by Theorem 2.7 $\mathrm{H}_{n-1}^I(M)/I^{s+t}\,\mathrm{H}_{n-1}^I(M)$ is minimax. Since the class of minimax modules is a Serre subcategory of the category of R-modules, it follows that $\mathrm{H}_{n-1}^I(M)$ is minimax and the proof is complete. □

Corollary 2.11. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Then*

$$\inf\{i : \mathrm{H}_i^I(M) \textit{ is not minimax}\} = \inf\{r^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) \mid \mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}\}.$$

Proof. It follows by Theorem 2.10. □

Theorem 2.12. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I \subseteq \mathfrak{m}$ be an ideal of R and M an artinian R-module. Let n be an integer. Then for any $\mathfrak{p} \in \mathrm{Spec}(R)$ the following conditions are equivalent*:

i) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ *is a representable* $R_{\mathfrak{p}}$*-module for all* $i > n$.
ii) $IR_{\mathfrak{p}} \subseteq \sqrt{(0 :_{R_{\mathfrak{p}}} {}_{\mathfrak{p}}(\mathrm{H}_i^I(M)))}$ *for all* $i > n$.
iii) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) = 0$ *for all* $i > n$.

Proof. i) $\Rightarrow$ ii) Since ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) \simeq \mathrm{H}_i^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M)$, it follows by [12, Corollary 2.2].

ii) $\Rightarrow$ iii) We use induction on $u := \mathrm{Ndim}_R M$. Let $u = 0$. Since $\mathrm{H}_i^I(M) = 0$ for all $i > 0$, by [2, Proposition 4.8], the result follows in this case. Now suppose, inductively that $u > 0$ and the result is true for $u - 1$. By Lemma [2, Corollary 4.5], we can replace M by $\cap_{n>0} I^n M$. But $\cap_{n>0} I^n M = I^k M$ for some $k \in \mathbb{N}$ and so we may assume that $IM = M$. Since M is artinian, $xM = M$ for some $x \in I$ by [8, 2.8]. Now for all $i > n$, from the exact sequence

$$0 \to (0 :_M x) \to M \xrightarrow{x} M \to 0$$

we obtain the following long exact sequence of linearly compact R-modules

$$\cdots \to \mathrm{H}_{i+1}^I(M) \to \mathrm{H}_i^I(0 :_M x) \to \mathrm{H}_i^I(M) \xrightarrow{x} \mathrm{H}_i^I(M) \to \cdots .$$

By [10, Lemma 3.2] we have the following long exact sequence

$$\cdots \to {}_{\mathfrak{p}}(\mathrm{H}_{i+1}^I(M)) \to {}_{\mathfrak{p}}(\mathrm{H}_i^I(0 :_M x)) \to {}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) \xrightarrow{x/1} {}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) \to \cdots .$$

By [4, Lemma 4.7] $\mathrm{Ndim}_R(0 :_M x) \leq u - 1$ and so the induction hypothesis implies that ${}_{\mathfrak{p}}(\mathrm{H}_i^I(0 :_M x)) = 0$ for all $i > n$. Thus for any $i > n$ we have an injection $0 \to {}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) \xrightarrow{x/1} {}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$. Suppose that ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) \neq 0$ for some $i > n$. Since $\frac{x}{1} \in IR_{\mathfrak{p}}$, by assumption (ii) there exists a positive integer k such that $(x/1)^k {}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) = 0$. Now from the above injection we get a contradiction. Hence ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) = 0$ for all $i > n$ and the proof is complete.

iii) $\Rightarrow$i) It is clear. □

Theorem 2.13. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* $I \subseteq \mathfrak{m}$ *be an ideal of* R *and* M *an artinian* R*-module. Let* n *be an integer. Then the following conditions are equivalent*:

i) $\mathrm{H}_i^I(M)$ *is an artinian* R*-module for all* $i > n$.
ii) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ *is a representable* $R_{\mathfrak{p}}$*-module for all* $i > n$ *and all* $\mathfrak{p} \in \mathrm{Spec}(R)$.
iii) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) = 0$ *for all* $i > n$ *and all* $\mathfrak{p} \in \mathrm{Spec}(R)$.
iv) $\mathrm{H}_i^I(M) = 0$ *for all* $i > n$.

Proof. i)$\Rightarrow$ ii) By [9, Theorem 3.2], colocalization of any artinian R-module is representable.

ii) $\Rightarrow$ iii) By Theorem 2.12 i) $\Rightarrow$ iii).

iii) $\Rightarrow$ iv) Assumption implies that $\mathrm{Cos}_R(\mathrm{H}_i^I(M)) = \phi$ for all $i > n$. Thus $\mathrm{H}_i^I(M) = 0$ for all $i > n$ by [3, 4.3].

iv) $\Rightarrow$ i) It is clear. □

Theorem 2.14. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring,* $I \subseteq \mathfrak{m}$ *be an ideal of* R *and* M *an artinian* R*-module. Let* n *be an integer. Then the following conditions are equivalent*:

i) $\mathrm{H}_i^I(M)$ *is a finitely generated* R*-module for all* $i > n$.
ii) $\mathrm{H}_i^I(M)$ *is a minimax* R*-module for all* $i > n$.
iii) ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M))$ *is a representable* $R_{\mathfrak{p}}$*-module for all* $i > n$ *and all* $\mathfrak{p} \in \mathrm{Spec}(R)\backslash\{\mathfrak{m}\}$.

Proof. i) $\Rightarrow$ ii): Any finitely generated R-module is minimax.

ii) $\Rightarrow$ iii): By Theorem 2.9.

iii) $\Rightarrow$ i): By Theorem 2.12, ${}_{\mathfrak{p}}(\mathrm{H}_i^I(M)) = 0$ for all $i > n$ and all $\mathfrak{p} \in \mathrm{Spec}(R)\backslash\{\mathfrak{m}\}$. Thus $\mathrm{Cos}_R((\mathrm{H}_i^I(M))) \subseteq \{\mathfrak{m}\}$ for all $i > n$ and so

$$\mathrm{Coass}_R((\mathrm{H}_i^I(M))) \subseteq \{\mathfrak{m}\}$$

for all $i > n$. Now by [18, satz 2.4], for any $i > n$ there exists a positive integer t_i such that $I^{t_i}\,\mathrm{H}_i^I(M)$ is finitely generated. Since $\mathrm{H}_i^I(M) = 0$ for all $i > \mathrm{Ndim}_R M$ by [2, Proposition 4.8], we can find an integer t such that $I^t\,\mathrm{H}_i^I(M)$ is finitely generated for all $i > n$. Now we use induction on $u := \mathrm{Ndim}_R M$. Let $u = 0$. Since $\mathrm{H}_i^I(M) = 0$ for all $i > 0$, by [2, Proposition 4.8], the result follows in this case. Now suppose, inductively that $u > 0$ and the result is true for $u-1$. By an argument analogue to that used in the proof of Theorem 2.12, we may assume that $xM = M$ for some $x \in I$ and so $x^tM = M$. Thus for all $i > n$, the exact sequence

$$0 \to (0 :_M x^t) \to M \xrightarrow{x^t} M \to 0$$

implies that

$$\cdots \to \mathrm{H}_i^I(0 :_M x^t) \xrightarrow{\varphi_i} \mathrm{H}_i^I(M) \xrightarrow{x^t} \mathrm{H}_i^I(M) \to \cdots .$$

Since $\mathrm{Ndim}_R(0 :_M x^t) \leq u-1$, induction hypothesis implies that $\mathrm{H}_i^I(0 :_M x^t)$ is finitely generated for all $i > n$ and so we have the exact sequence $0 \to \mathrm{Im}\,\varphi_i \to \mathrm{H}_i^I(M) \to x^t\,\mathrm{H}_i^I(M) \to 0$ for all $i > n$. Since $\mathrm{Im}\,\varphi_i$ and $x^t\,\mathrm{H}_i^I(M)$ are finitely generated we conclude that $\mathrm{H}_i^I(M)$ is finitely generated for all $i > n$. □

Corollary 2.15. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring,* $I \subseteq \mathfrak{m}$ *be an ideal of* R *and* M *an artinian* R*-module. Then we have*

i) $\sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M) \neq 0\} = \sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M)$ *is not artinian*$\} = \sup\{R^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \mathrm{Spec}(R)\}$,
ii) $\sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M)$ *is not finitely generated*$\} = \sup\{i \in \mathbb{N}_0 : \mathrm{H}_i^I(M)$ *is not minimax*$\} = \sup\{R^{IR_{\mathfrak{p}}}({}_{\mathfrak{p}}M) : \mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}\}$.

Proof. It follows by Theorems 2.13 and 2.14. □

Acknowledgment. The author would like to thank the referee for the careful reading of this paper.

References

[1] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, **60**, Cambridge University Press, Cambridge, 1998. https://doi.org/10.1017/CBO9780511629204

[2] N. T. Cuong and T. T. Nam, *The I-adic completion and local homology for Artinian modules*, Math. Proc. Cambridge Philos. Soc. **131** (2001), no. 1, 61–75. https://doi.org/10.1017/S0305004100004771

[3] ______, *On the co-localization, co-support and co-associated primes of local homology modules*, Vietnam J. Math. **29** (2001), no. 4, 359–368.

[4] ______, *A local homology theory for linearly compact modules*, J. Algebra **319** (2008), no. 11, 4712–4737. https://doi.org/10.1016/j.jalgebra.2007.11.030

[5] J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453. https://doi.org/10.1016/0021-8693(92)90026-I

[6] D. Kirby, *Dimension and length for Artinian modules*, Quart. J. Math. Oxford Ser. (2) **41** (1990), no. 164, 419–429. https://doi.org/10.1093/qmath/41.4.419

[7] I. G. Macdonald, *Duality over complete local rings*, Topology **1** (1962), 213–235. https://doi.org/10.1016/0040-9383(62)90104-0

[8] ______, *Secondary representation of modules over a commutative ring*, in Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), 23–43, Academic Press, London, 1973.

[9] L. Melkersson and P. Schenzel, *The co-localization of an Artinian module*, Proc. Edinburgh Math. Soc. (2) **38** (1995), no. 1, 121–131. https://doi.org/10.1017/S0013091500006258

[10] T. T. Nam, *Co-support and coartinian modules*, Algebra Colloq. **15** (2008), no. 1, 83–96. https://doi.org/10.1142/S1005386708000084

[11] ______, *Minimax modules, local homology and local cohomology*, Internat. J. Math. **26** (2015), no. 12, 1550102, 16 pp. https://doi.org/10.1142/S0129167X15501025

[12] S. Rezaei, *Some results on local homology and local cohomology modules*, Illinois J. Math. **57** (2013), no. 1, 17–23. http://projecteuclid.org/euclid.ijm/1403534483

[13] Z. M. Tang, *Local homology theory for Artinian modules*, Comm. Algebra **22** (1994), no. 5, 1675–1684. https://doi.org/10.1080/00927879408824928

[14] S. Yassemi, *Coassociated primes*, Comm. Algebra **23** (1995), no. 4, 1473–1498. https://doi.org/10.1080/00927879508825288

[15] D. Zelinsky, *Linearly compact modules and rings*, Amer. J. Math. **75** (1953), 79–90. https://doi.org/10.2307/2372616

[16] H. Zöschinger, *Koatomare Moduln*, Math. Z. **170** (1980), no. 3, 221–232.

[17] ______, *Linear-kompakte Moduln über noetherschen Ringen*, Arch. Math. (Basel) **41** (1983), no. 2, 121–130.

[18] ______, *Über koassoziierte Primideale*, Math. Scand. **63** (1988), no. 2, 196–211.

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